2017 Summer Course

Optical Oceanography and Ocean Color Remote Sensing

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Radiative Transfer Theory

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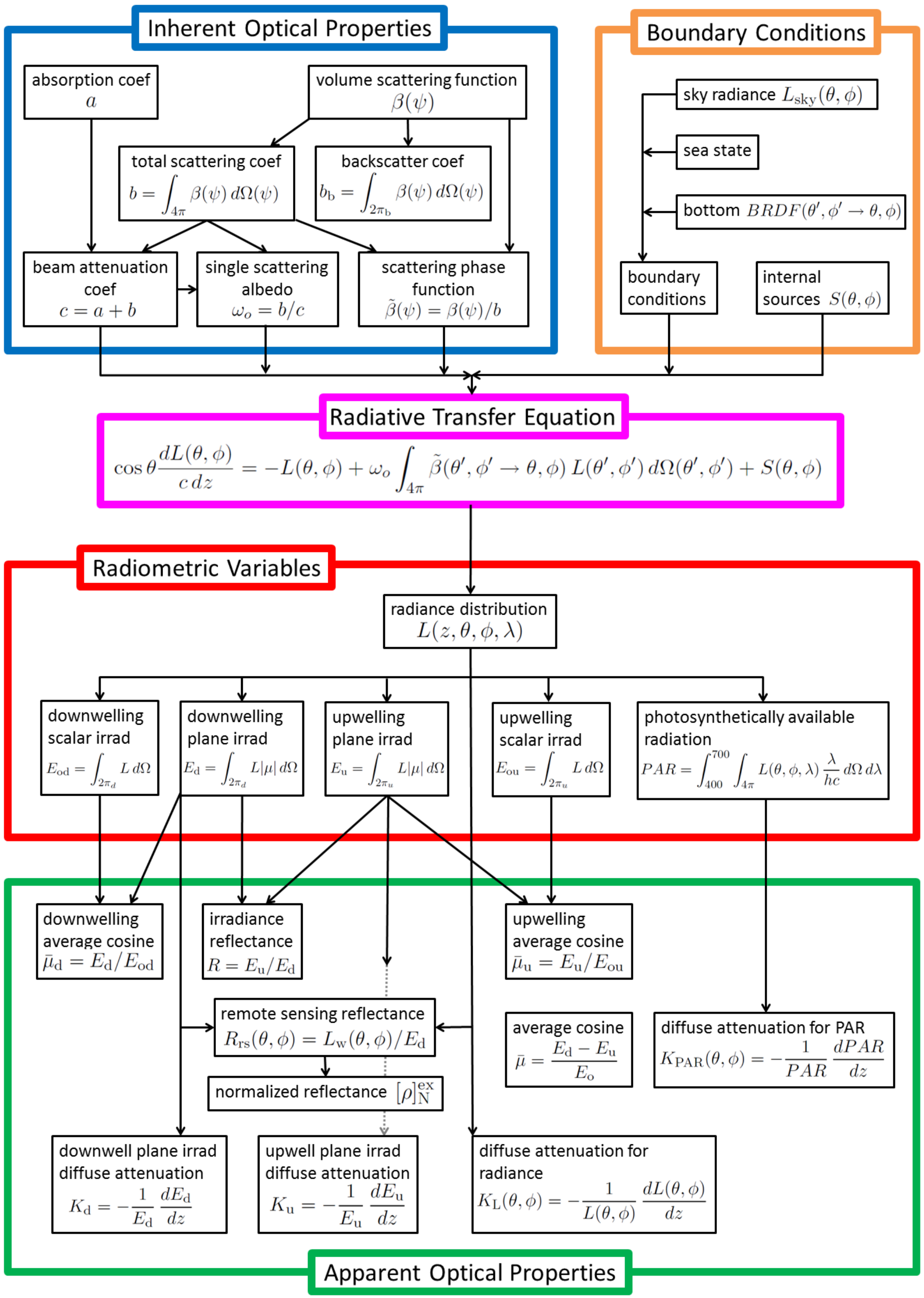
The Radiative Transfer Equation (RTE)

Connects the IOPs, boundary conditions, and light sources to the radiance

All other radiometric variables (irradiances) and AOPs can be derived from the radiance.

If you know the radiance (technically, the Stokes vector), you know everything there is to know about the light field.



Radiative Transfer Theory

Radiative transfer theory is the physical and mathematical framework for all of optical oceanography and ocean color remote sensing.

Today:

- A quick summary of Ken's lecture on polarization
- A qualitative outline of how you properly derive the equation solved by HydroLight
  - Radiative transfer equations for various levels of approximation
  - The assumptions made to go from one step to the next
- A few words about solving radiative transfer equations

Polarization

Light consists of propagating electric and magnetic fields, which are described by Maxwell's equations.

If the time- and space-dependent electric field vector **E**(**x**,t) is known, then the magnetic field vector **B**(**x**,t) can be computed from Maxwell's equations, and vice versa. It is thus sufficient to discuss just one of these fields, which is usually chosen to be the electric field vector.

“Polarization” refers to the plane in which the electric field vector is oscillating.

For more on polarization see the Web Book page
Polarization: Stokes Vectors

Linear Polarization

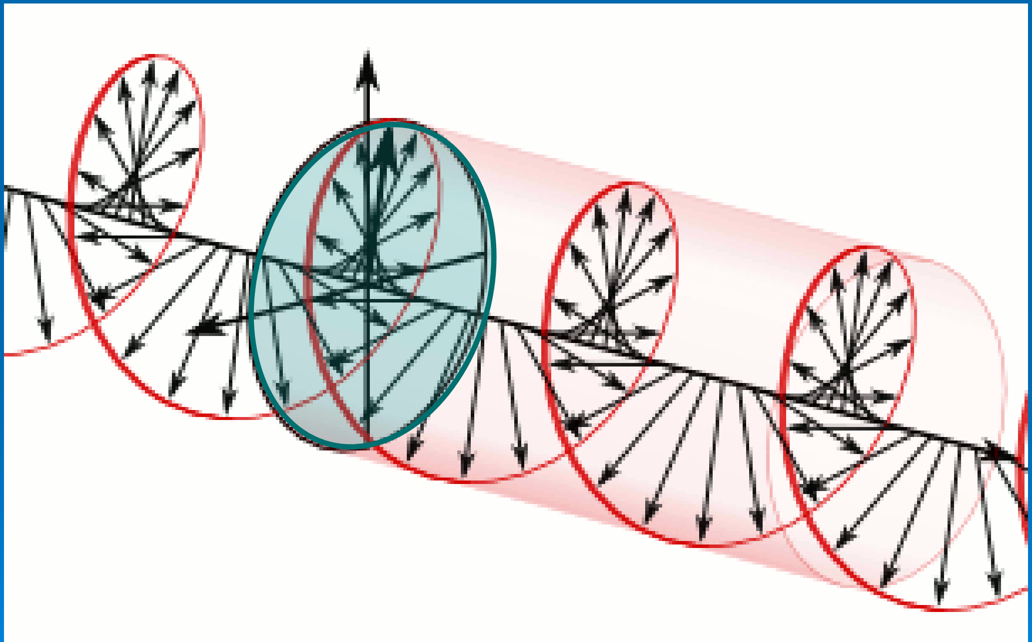
Suppose you are looking toward a light source, or "into the beam." In the simplest case, called linear polarization, the electric field **E**(**x**,t) lies in, or oscillates in, a fixed plane.

For visible wavelengths, these oscillations are at a frequency of around 10^{15} times per second and cannot be directly measured because of instrumentation limits.

https://en.wikipedia.org/wiki/Polarization_(waves)

Circular Polarization

The plane in which the electric field lies may also rotate with time as the beam of light passes. This is called circular polarization if the maximum value of **E**(**x**,t) is independent of time but the orientation of the plane rotates. If the plane rotates and the amplitude of **E**(**x**,t) also changes as the plane rotates, so that the maximum value of **E**(**x**,t) traces out an ellipse, it is called elliptical polarization.

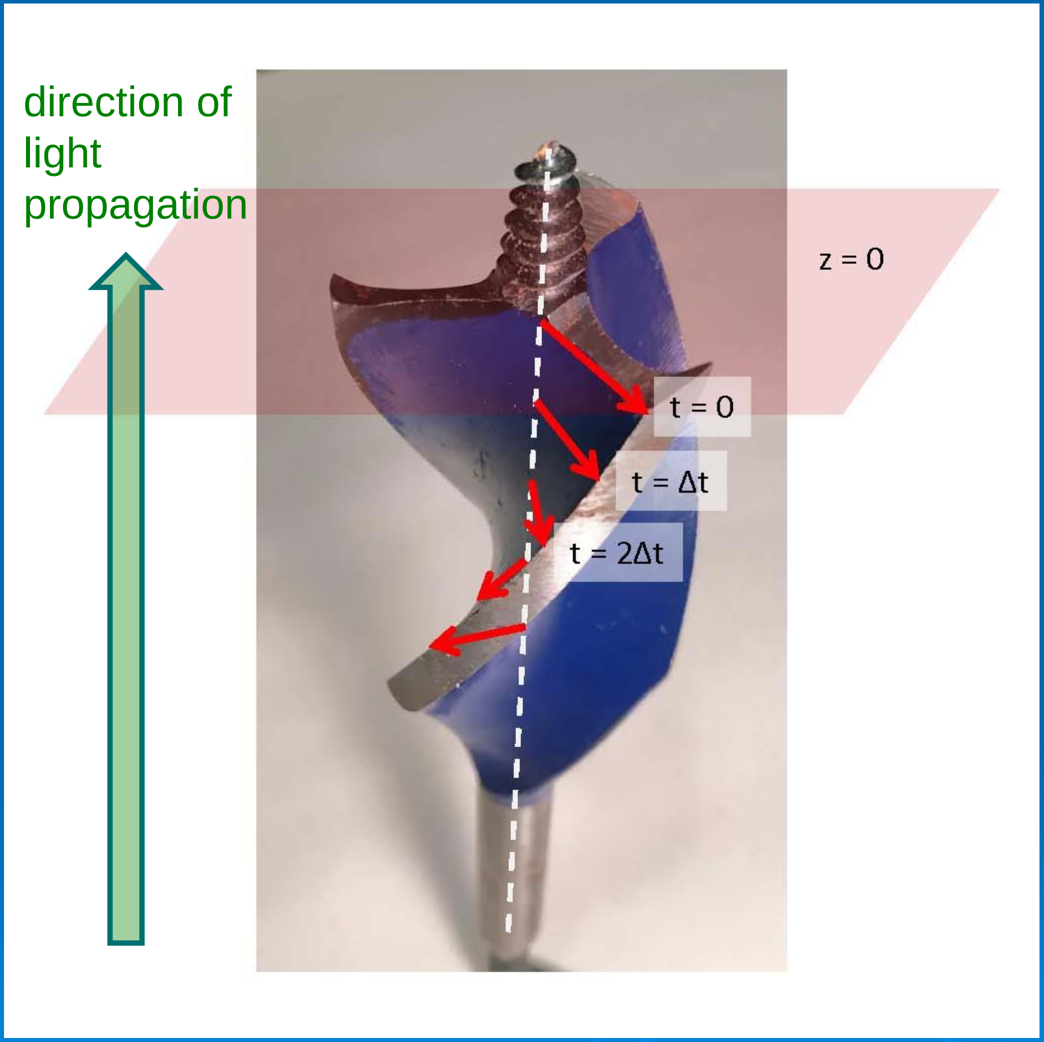


https://en.wikipedia.org/wiki/Circular_polarization

Circular/elliptical polarization can be “right-handed” or “left-handed”

Note in this animation that the ***E*** vector appears to rotate clockwise in the green plane when looking into the beam.

Right Circular Polarization (RCP)



This drill edge is a right-handed helix.

As the drill moves upward without rotating, the ***E*** field (red arrows) appear to rotate clockwise when looking into the beam.

This is the convention for RCP used by Bohren.

WARNING: others may reverse RCP and LCP conventions.

Either is OK. Just pick a convention and stay with it for your calculations.

Stokes Vectors

The Stokes vector is an array of four real numbers (not a true vector in the geometric sense) used to fully specify the state of polarization.

$$\underline{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}$$

To define these quantities, first pick a coordinate system that is convenient for your problem.

In the lab: Maybe choose x parallel to an optical bench top, y perpendicular to the bench top, and z in the direction of propagation. In this lab setting, x might then be called the "parallel" (to the bench top) direction, and y would then be the "perpendicular" direction. Or x and y might be called "horizontal" and "vertical", respectively.

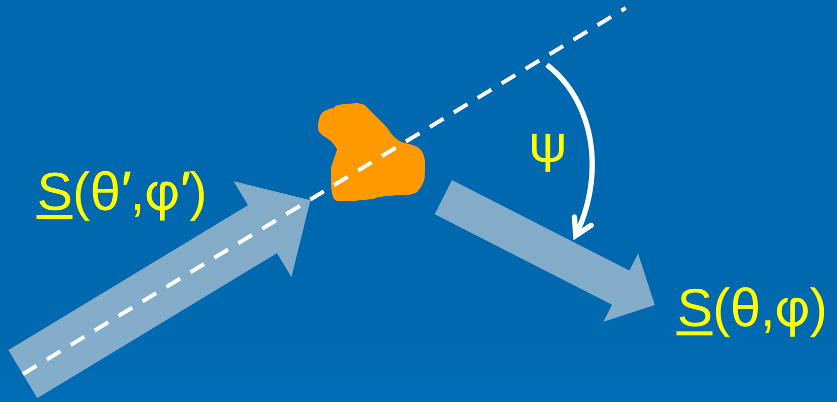
For atmospheric and oceanic work, we refer $\underline{S}$ to meridional planes (more in a minute); see the Web Book page
light and radiometry/level 2/polarization scattering geometry

Scattering of Polarized Light

Absorption does not change the state of polarization. Absorption just removes light from the beam.

Scattering can change one state of polarization into another.

Scattering is specified by a 4 x 4 matrix $\underline{Z}$, called the phase matrix.



$$\underline{S}(\theta,\varphi) = \underline{Z}(\theta',\varphi' \rightarrow \theta,\varphi)\ \underline{S}(\theta',\varphi')$$

$$\underline{S}(\theta,\varphi) = \underline{Z}(\psi)\ \underline{S}(\theta',\varphi')$$

4x1 4x4 4x1

The scattering angle is given by

$$\cos\psi = \boldsymbol{\xi}' \cdot \boldsymbol{\xi} = \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi - \phi')$$

Radiative Transfer Theory

Radiative transfer theory is now a well established branch of physics (thanks to M. Mishchenko and a few others)

Quantum electrodynamics (QED)

↓

Maxwell's equations

↓

The general vector radiative transfer equation (VRTE)

↓

The VRTE for particles with mirror symmetry

↓

The scalar RTE for the first component of the Stoke's vector

(what HydroLight solves)

QED

Quantum electrodynamics (QED) is the fundamental theory that explains with total accuracy (as far as we know) the interactions of light and matter (or of charged particles). Example accuracy: electron spin g factor:

Measured value of g/2 = 1.001 159 652 180 73 (±28)
QED computed value = 1.001 159 652 181 64 (±67)

QED views light as consisting of elementary "excitations" called photons. These are the quanta of the electromagnetic field.

QED is extremely mathematical and abstract. Can do the calculations only for interactions between elementary particles.

Recommended reading: *QED: The Strange Story of Light and Matter* by Feynman

Maxwell's Equations

"The classical limit of the quantum theory of radiation is achieved when the number of photons becomes so large that the occupation number may as well be regarded as a continuous variable. The space-time development of the classical electromagnetic wave approximates the dynamical behavior of trillions of photons.“ –J. J. Sakurai, *Advanced Quantum Mechanics* (1967)

This limit leads to Maxwell's equations, which govern electric and magnetic fields.

Maxwell's equations view light as propagating electric and magnetic fields.

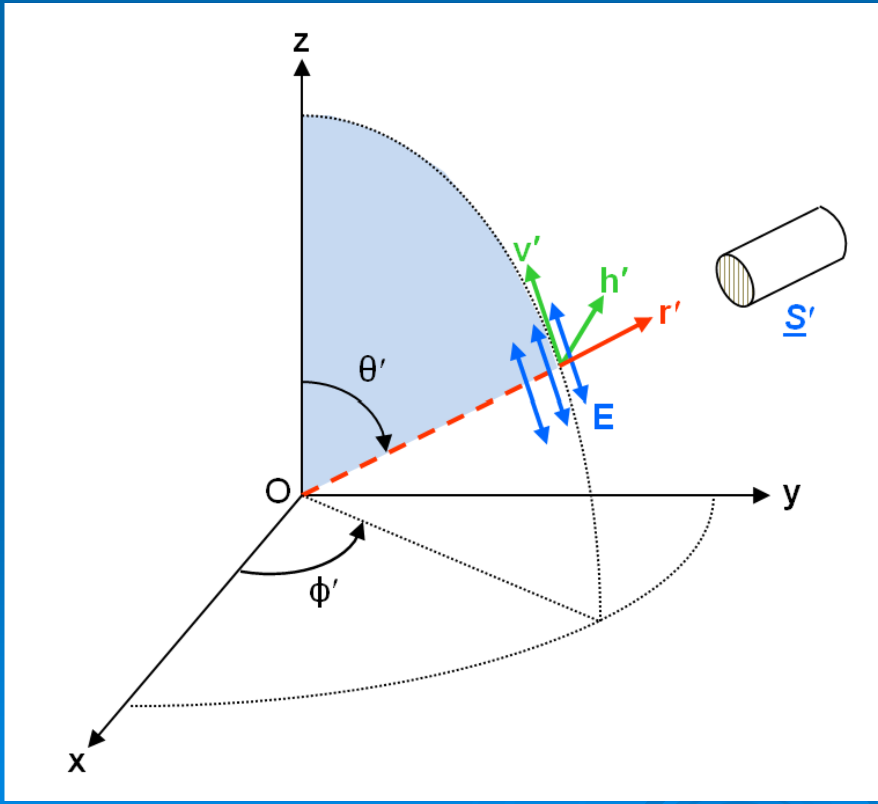
These equations are accurate for everyday problems: e.g., generation, propagation, and detection of radio waves; the generation of electrical power; the refraction of light at an air-water surface; and the scattering of light by phytoplankton. They break down for atomic-scale processes, very high energies, very low temperatures, or when individual photons become important ("quantum optics").

Correct, but still too complicated to use for oceanographic calculations.

Meridian Planes

Maxwell: a propagating electric field has the form $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_o \exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$

In oceanography, use meridian planes to resolve the electric field into horizontal (perpendicular) and vertical (parallel) compoents, and to express the Stokes vectors of incident and scattered light.



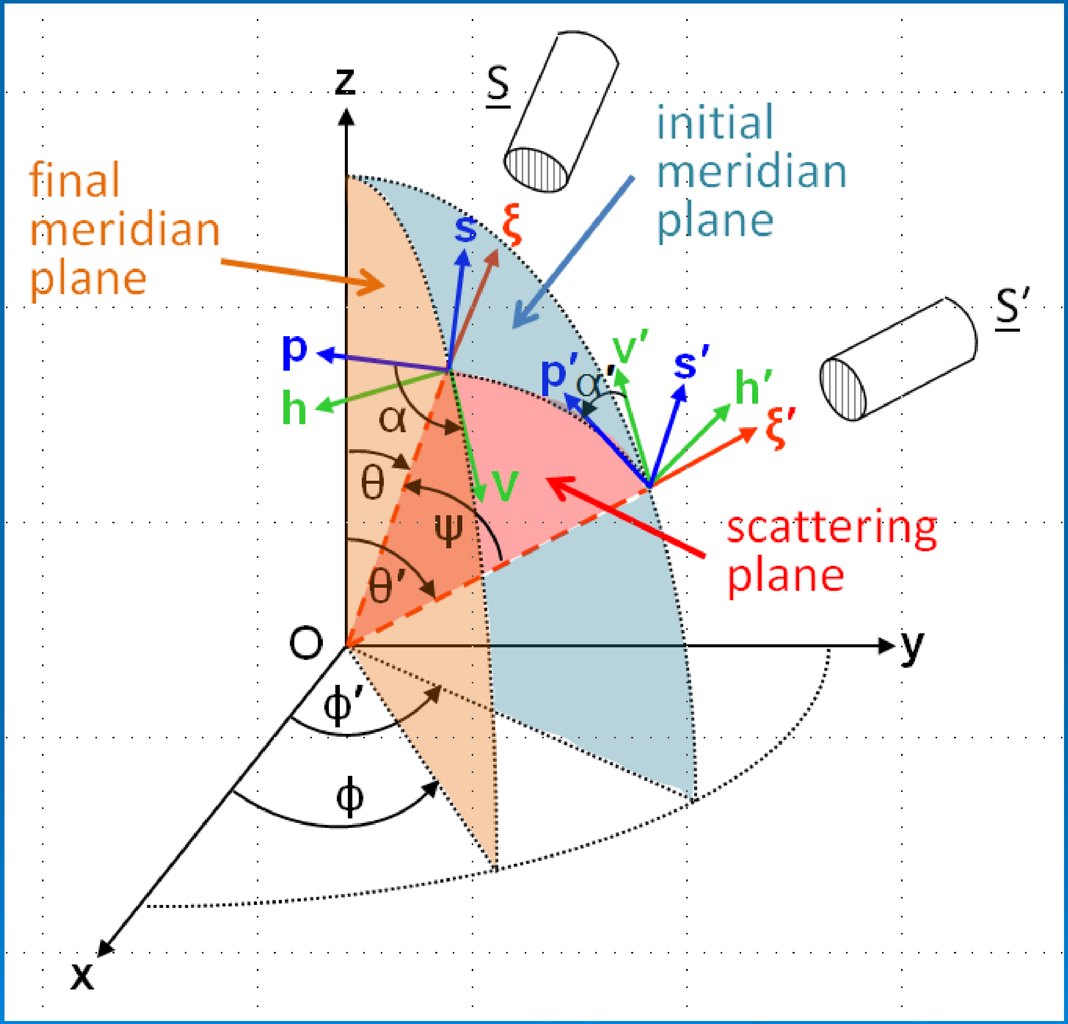
The meridian plane is the plane containing the normal to the mean sea surface and the direction of light propagation.

S′ is defined in the local (h′, v′) (horizontal or vertical; perpendicular or parallel) system

The meridian plane changes if the light changes direction

Must translate Stokes vectors between these coordinate systems

Phase, Scattering, and Rotation Matrices



$\underline{S}(\theta,\varphi) = \underline{Z}(\psi)\ \underline{S}'(\theta',\varphi')$

$= \underline{R}(\alpha)\ \underline{M}(\psi)\ \underline{R}(\alpha')\ \underline{S}'(\theta',\varphi')$

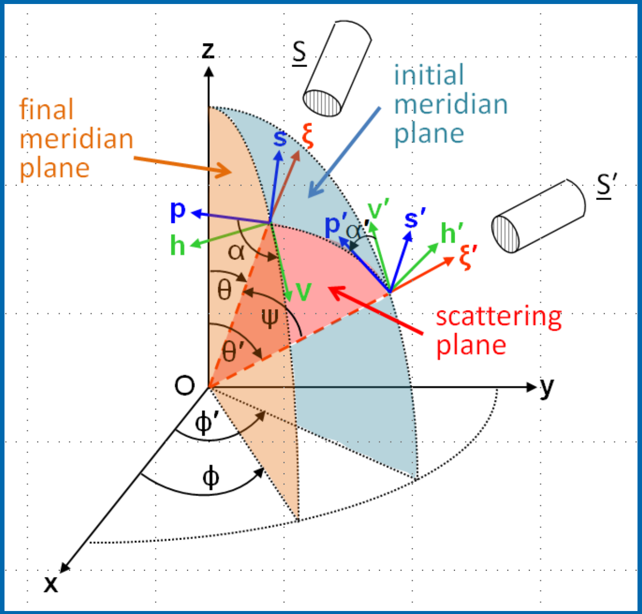
Start with $\underline{S}'$ in the initial meridian plane

“Rotate” $\underline{S}'$ into the scattering plane

Compute the scattering in the scattering plane

Rotate the final $\underline{S}$ from the scattering plane to the final meridian plane

Phase, Scattering, and Rotation Matrices



$\underline{S}(\theta,\varphi) = \underline{Z}(\psi)\ \underline{S}'(\theta',\varphi')$

$= \underline{R}(\alpha)\ \underline{M}(\psi)\ \underline{R}(\alpha')\ \underline{S}'(\theta',\varphi')$

The phase matrix $\underline{Z}(\psi)$ transforms $\underline{S}'$ to $\underline{S}$, with both expressed in the meridian planes

The scattering matrix $\underline{M}(\psi)$ transforms $\underline{S}'$ to $\underline{S}$, with both expressed in the scattering plane

Rotation matrices $\underline{R}$ transform $\underline{S}$ from one plane to another

The decomposition $\underline{Z}(\psi) = \underline{R}(\alpha)\ \underline{M}(\psi)\ \underline{R}(\alpha')$ separates the physics of the scattering process (described by $\underline{M}$) from the bookkeeping related to the different coordinate systems (described by $\underline{R}$) . $\underline{M}$ is often called the Mueller matrix.

For the equations, see the Web Book page Polarization: Scattering Geometry

The General VRTE

A propagating-wave solution to Maxwell's equations leads to a general vector radiative transfer equation (VRTE) for the Stokes vector (e.g., Mishchenko's books and papers):

$$\boldsymbol{\xi} \cdot \nabla \underline{S}(\mathrm{x}, \boldsymbol{\xi}) = -\underline{K}(\mathrm{x}, \boldsymbol{\xi})\, \underline{S}(\mathrm{x}, \boldsymbol{\xi}) + \iint_{4\pi} \underline{Z}(\mathrm{x}, \boldsymbol{\xi}' \to \boldsymbol{\xi})\, \underline{S}(\mathrm{x}, \boldsymbol{\xi}')\, d\Omega(\boldsymbol{\xi}') + \underline{\Sigma}(\mathrm{x}, \boldsymbol{\xi})$$

where

$$\boldsymbol{\xi} \cdot \nabla = \xi_x \frac{\partial}{\partial x} + \xi_y \frac{\partial}{\partial y} + \xi_z \frac{\partial}{\partial z} = \sin\theta\, \cos\phi\, \frac{\partial}{\partial x} + \sin\theta\, \sin\phi\, \frac{\partial}{\partial y} + \cos\theta\, \frac{\partial}{\partial z}$$

and

- $\underline{K}(\mathrm{x}, \boldsymbol{\xi})$ is a 4×4 *extinction matrix*, which describes the attenuation (by the background medium and any particles imbedded in the medium) of the light propagating in direction $\boldsymbol{\xi}$.
- $\underline{Z}(\mathrm{x}, \boldsymbol{\xi}' \to \boldsymbol{\xi})$ is a 4×4 *phase matrix*, which describes how light in an initial state of polarization and direction $\boldsymbol{\xi}'$ in the incident meridian plane is scattered to a different state of polarization and direction $\boldsymbol{\xi}$ in the final meridian plane.
- $\underline{\Sigma}(\mathrm{x}, \boldsymbol{\xi})$ is a 4×1 internal source term, which specifies the Stokes vector of any emitted light such as bioluminescence or light at the wavelength of interest that comes from other wavelengths via inelastic scattering.

In general, all 16 elements of $\underline{K}$ and $\underline{Z}$ are non-zero.

The General VRTE

$$\xi \cdot \nabla \underline{S}(\mathrm{x}, \xi) = -\underline{K}(\mathrm{x}, \xi)\, \underline{S}(\mathrm{x}, \xi) + \iint_{4\pi} \underline{Z}(\mathrm{x}, \xi' \to \xi)\, \underline{S}(\mathrm{x}, \xi')\, d\Omega(\xi') + \underline{\Sigma}(\mathrm{x}, \xi)$$

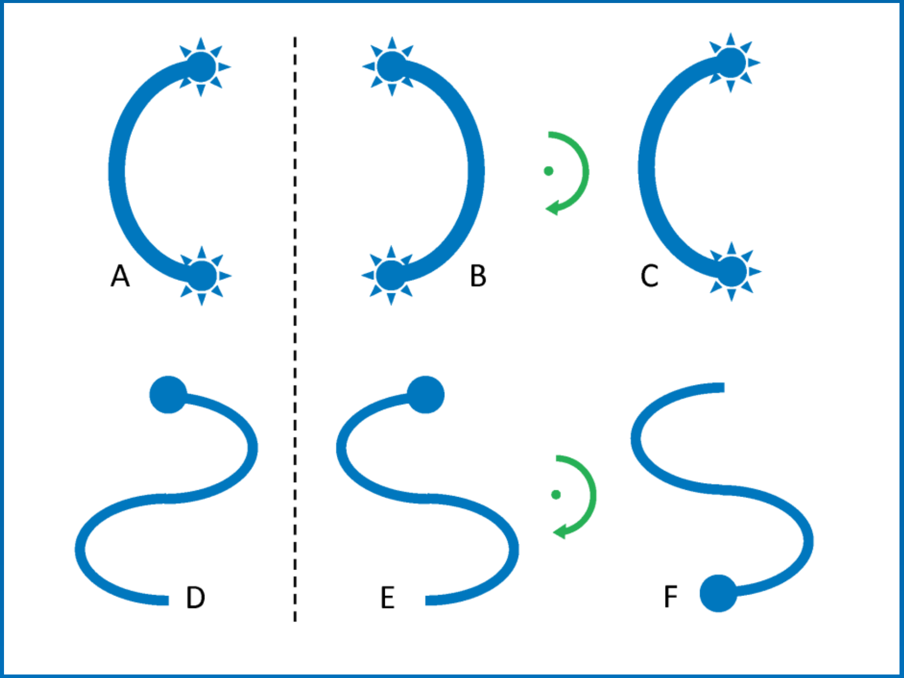
This equation can describe polarized light propagation in matter that is directionally non-isotropic (e.g., in a crystal), that can absorb light differently for different states of polarization (dichroism), and that contains scattering particles of any shape and random or non-random orientation.

Can solve the equation numerically, but we almost never have the needed inputs: the 16 elements of $\underline{K}$ and 16 of $\underline{Z}$.

Still need to simplify for oceanographic applications

Mirror-symmetric Particles

The general VRTE becomes much simpler if the scattering particles are mirror-symmetric and randomly oriented.



mirror-symmetric particle

not mirror-symmetric

Are oceanic particles like phytoplankton or mineral particles mirror-symmetric?

Mirror-symmetric Phytoplankton

http://www.mikroskopie-ph.de/Kreispraeparat-25-G.jpg

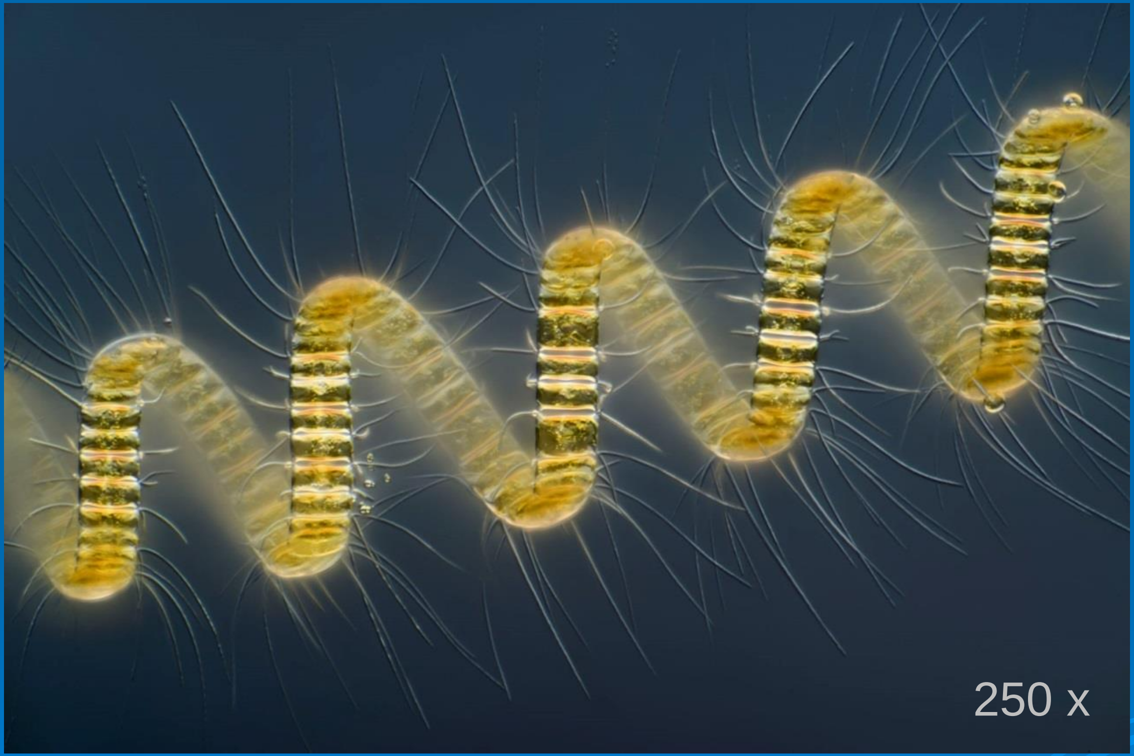
https://media1.britannica.com/eb-media/93/184793-004-11DAC31B.jpg

The assumption that phytoplankton are mirror-symmetric is reasonable and usually correct.

The assumption that they are spherical is not.

Non-mirror-symmetric Phytoplankton

This chain-forming diatom (*Chaetoceros debilis*) is a left-handed helix (but maybe the photo was flipped). It is not mirror symmetric. A bloom of these would require the general VRTE for Stokes vector calculations.



 From https://www.wired.com/2013/10/nikon-small-world-2013/

(If you have equal numbers of randomly oriented left- and right-handed helices, then the bulk medium is mirror symmetric, and you can use the simpler VRTE.)

The VRTE for a Mirror-symmetric Medium

If the particles are mirror-symmetric and randomly oriented:

- The attenuation matrix becomes diagonal and the attenuation does not depend on direction or state of polarization:

$$\underline{K}(\mathbf{x}, \boldsymbol{\xi}) = c(\mathbf{x}) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c is the "beam attenuation coefficient"

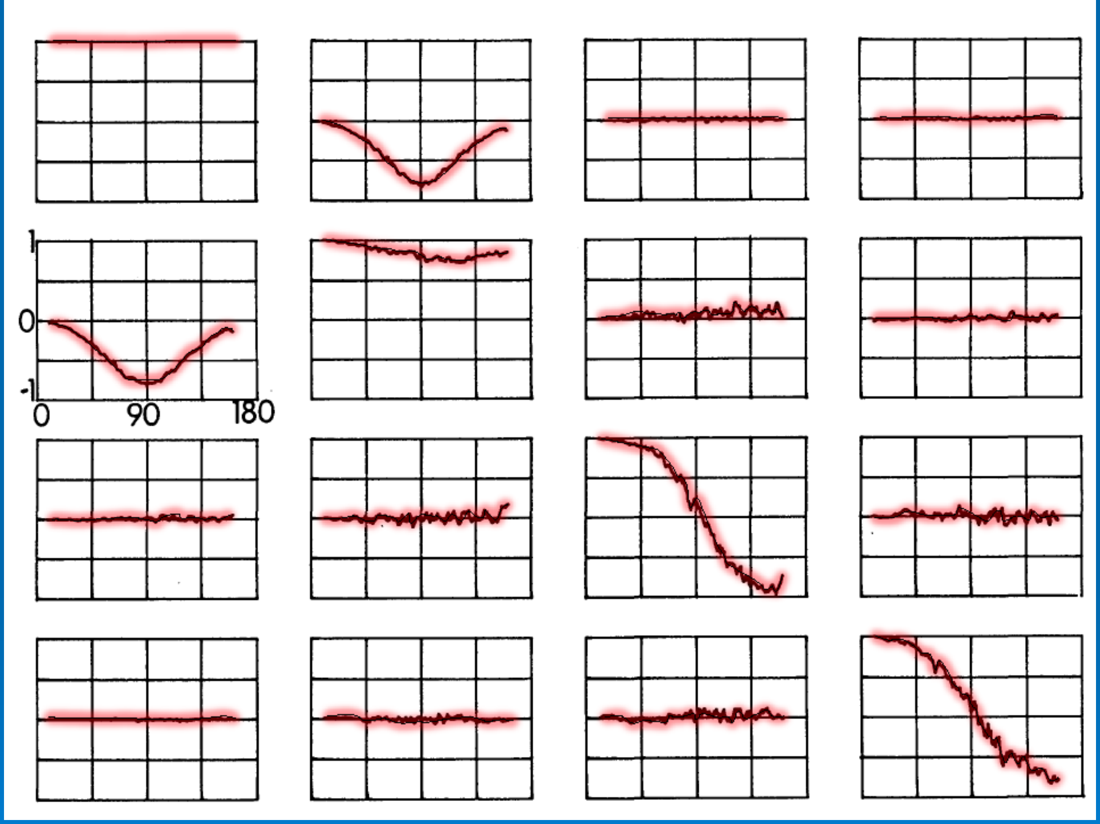
- The scattering matrix becomes block diagonal with only 6 independent elements:

$$\underline{M} = \begin{bmatrix} M_{11}(\psi) & M_{12}(\psi) & 0 & 0) \\ M_{12}(\psi) & M_{22}(\psi) & 0 & 0 \\ 0 & 0 & M_{33}(\psi) & M_{34}(\psi) \\ 0 & 0) & -M_{34}(\psi) & M_{44}(\psi) \end{bmatrix}$$

Do measured oceanic scattering matrices look like this?

Measured Scattering Matrices

The reduced scattering matrix is $\tilde{M}_{ij}(\psi) = M_{ij}(\psi)/M_{11}(\psi)$



Redrawn from Fig. 3(a) of Voss and Fry (1984)

$M_{11} = \beta(\psi)$ is the volume scattering function

$M_{22} \neq M_{11}$ indicates non-spherical particles

To within the measurement error, $M_{12} = M_{21}$ and $M_{33} = M_{44}$ and others are 0, so really only 4 independent elements

The assumption of randomly oriented, mirror-symmetric particles is justified.

The VRTE for Mirror-symmetric Particles

If the IOPs and boundary conditions are horizontally homogeneous, depth *z* is the only spatial variable and

$$\boldsymbol{\xi} \cdot \nabla \underline{S}(\mathbf{x}, \boldsymbol{\xi}) = \cos\theta \frac{d}{dz} \underline{S}(z, \theta, \phi)$$

The one-dimensional (1D) VRTE for a mirror-symmetric medium then becomes

$$\begin{aligned} \cos\theta \frac{d}{dz} \underline{S}(z, \theta, \phi) = & -c(z)\, \underline{S}(z, \theta, \phi) \\ & + \iint_{4\pi} \underline{R}(\alpha) \underline{M}(z, \psi) \underline{R}(\alpha') \underline{S}(z, \theta', \phi')\, d\Omega(\theta', \phi') + \underline{\Sigma}(z, \theta, \phi) \end{aligned}$$

where

$$\underline{M} = \begin{bmatrix} M_{11}(\psi) & M_{12}(\psi) & 0 & 0) \\ M_{12}(\psi) & M_{22}(\psi) & 0 & 0 \\ 0 & 0 & M_{33}(\psi) & M_{34}(\psi) \\ 0 & 0) & -M_{34}(\psi) & M_{44}(\psi) \end{bmatrix}$$

$$\underline{R}(\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & -\sin 2\gamma & 0 \\ 0 & \sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The RTE for the total radiance *I*

The first element of the Stokes vector is the total radiance, *I*, without regard for the state of polarization. In oceanography, this is usually called the radiance *L*.

$$
\begin{aligned}
\cos\theta \frac{d}{dz} I(z,\theta,\phi) = & -c(z)\, I(z,\theta,\phi) \\
& + \iint_{4\pi} M_{11}(z,\psi)\, I(z,\theta',\phi')\, d\Omega(\theta',\phi') + \Sigma_I(z,\theta,\phi) \\
& + \iint_{4\pi} \cos\alpha' M_{12}(z,\psi)\, Q(z,\theta',\phi')\, d\Omega(\theta',\phi') + \Sigma_Q(z,\theta,\phi) \\
& - \iint_{4\pi} \sin\alpha' M_{12}(z,\psi)\, U(z,\theta',\phi')\, d\Omega(\theta',\phi') + \Sigma_U(z,\theta,\phi).
\end{aligned}
$$

Note: we cannot solve this equation for *I* because we do not know *Q* and *U* unless we simultaneously solve the VRTE for all of *I, Q, U, V*

The RTE for the Total Radiance

Some modern instruments exist for measureing the VSF M_{11} = β(ψ) [e.g., Lee and Lewis (2003), Harmel et al. (2016), Li et al. (2012), Tan et al. (2013)]. However, β(ψ) is seldom measured during field experiments.

There are only a few instruments for measuring some of the other M_{ij}(ψ) elements of the scattering matrix [see Chami et al. (2014), Twardowski et al. (2012), Slade et al. (2013)].

If the VRTE is solved, the M_{ij}(ψ) are usually modeled (often using Mie theory, which assumes spherical particles).

Because of the lack of data or good models for M_{ij}(ψ) for different water types, modelers (including me) often just drop the terms involving $M_{12}Q$ and $M_{12}U$ and hope for the best.

This gives the scalar radiance transfer equation (SRTE) for the total radiance.

The SRTE for the Total Radiance *I*

Dropping the polarization-dependent terms gives the SRTE:

$$\begin{aligned}\cos\theta\frac{dL(z,\theta,\phi,\lambda)}{dz} &= -\,c(z,\lambda)L(z,\theta,\phi,\lambda)\\ &+ \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\beta(z;\theta',\phi'\rightarrow\theta,\phi;\lambda)\sin\theta'\,d\theta'\,d\phi'\\ &+ \Sigma(z,\theta,\phi,\lambda)\ .\end{aligned}$$

This is the equation HydroLight solves.

SRTE Error Estimate

The degree of linear polarization in the ocean is typically 10-30%; so *Q/I* or *U/I* < 0.3. For $\psi \approx 90$ deg, $M_{12} \approx 0.8\, M_{11}$. Then the error in ignoring the *Q* or *U* terms can be as large as

$$\frac{M_{12}Q}{M_{11}I} = 0.8 \times 0.3 \approx 0.25$$

However, comparison with *L* computed by the VRTE and the SRTE shows that the difference is typically ~10%. The error in radiance is positive in some directions and negative in others.

Irradiances are integrals of *L* over direction, so the errors in *L* tend to cancel, and irradiances are then good to a few percent.

Even though the SRTE is somewhat inaccurate, it is commonly used because

- The inputs *c* and β(ψ) are better known.
- The math needed to solve the SRTE is much easier that for the VRTE.
- The output is accurate enough for many (but not all) applications.
- Commercial software (HydroLight) is available.

RTE Summary

Maxwell's equations are correct but too complicated to solve in the oceanographic setting. Also, they give electric and magnetic fields, which is more and different information than we need or want.

The general VRTE gives us what we want (Stokes vectors), but we don't have all of the needed inputs (extinction $\underline{K}$ and phase $\underline{Z}$ matrices).

The VRTE for mirror-symmetric media is usually applicable to the ocean, but we still don't have all of the inputs needed for routine usage. This VRTE is commonly solved for the atmosphere, but not yet for the ocean. The output is accurate, but there is no user-friendly public or commercial software for solving this VRTE in the ocean.

The SRTE gives output that is accurate enough for many (but not all) applications. There are more data and bio-geo-optical models for defining the inputs. There is commercial software (HydroLight) for solving this equation.

A Way to Think About the SRTE

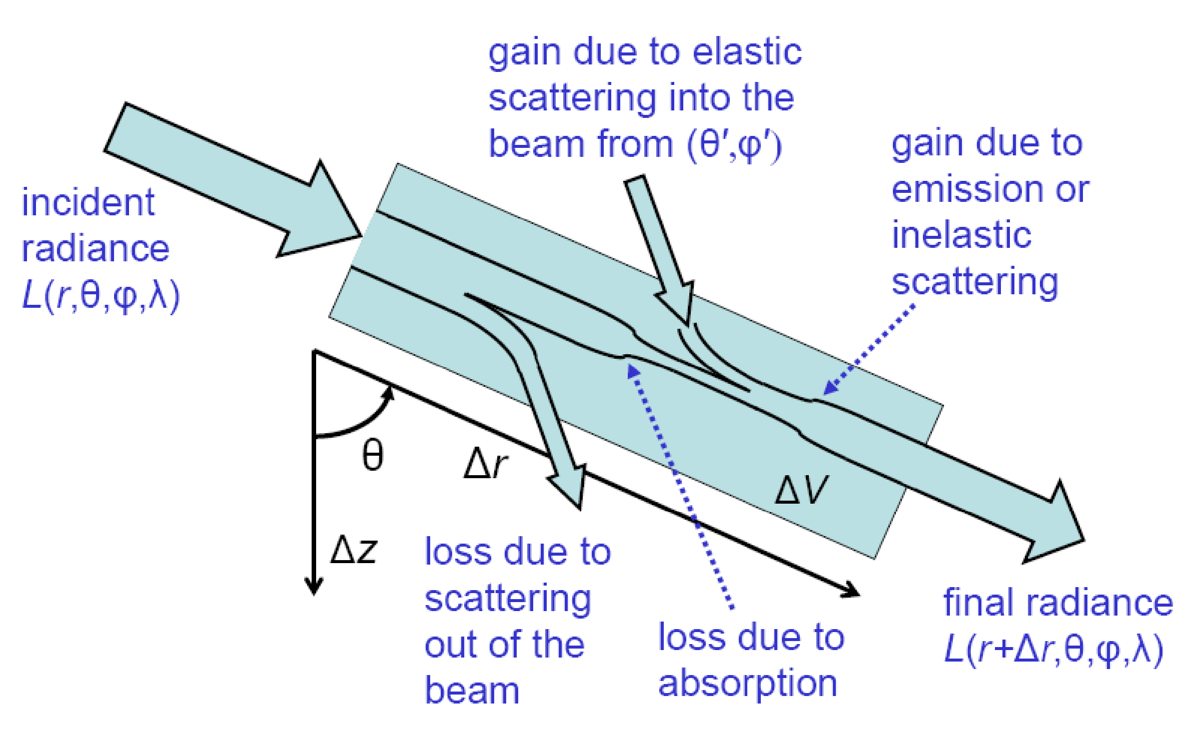
The previous discussion derived the SRTE from the general VRTE. This is the proper way to think about the SRTE because the various steps showed all of the assumptions made along the way and gave an estimate of the errors made if you use the SRTE rather than the full VRTE.

However, you will often see the SRTE “derived” using arguments about conservation of energy. This development would be correct if light were unpolarized, so that there is no VRTE.

I think of this “scalar derivation” of the SRTE as more of an aid to remembering the equation than as a derivation.

"Derivation" of the SRTE

To "derive" the time-independent SRTE for horizontally homogeneous water, we assume that light is not polarized (always wrong). Then consider the total radiance at a given depth *z*, traveling in a given direction (θ, ϕ), at a given wavelength λ. We then add up the various ways the radiance $L(z,\theta,\phi,\lambda)$ can be created or lost in a distance Δr along direction (θ, ϕ), going from depth *z* to $z+\Delta z$



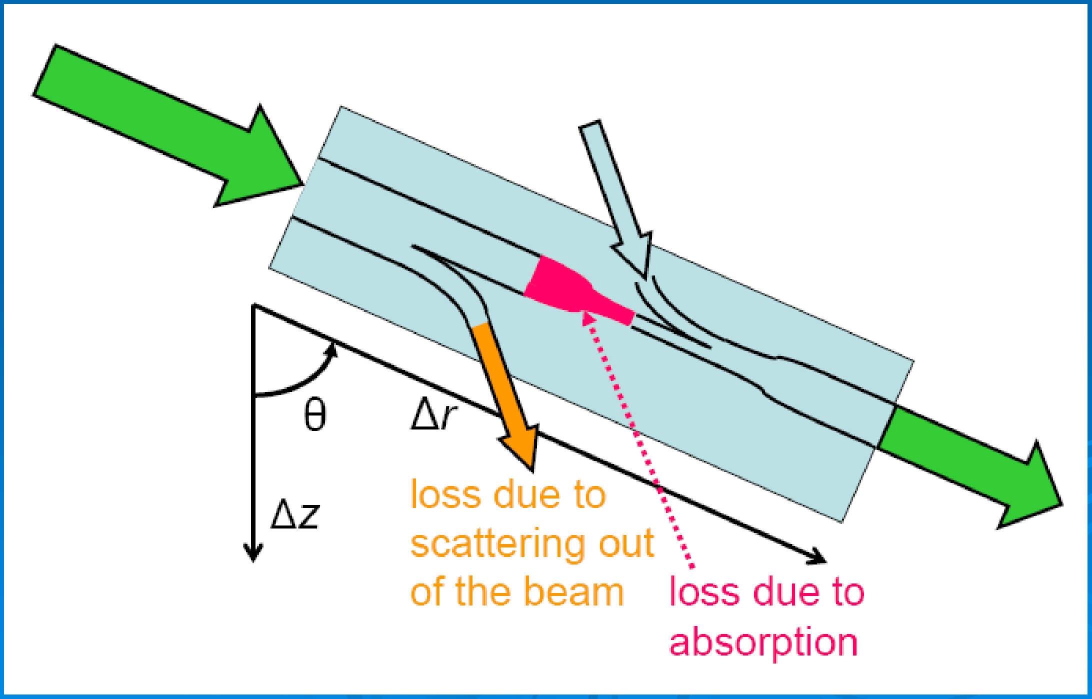
Losses of Radiance

The loss due to absorption is proportional to how much radiance there is:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\, a(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

Likewise for loss of radiance due to scattering out of the beam:

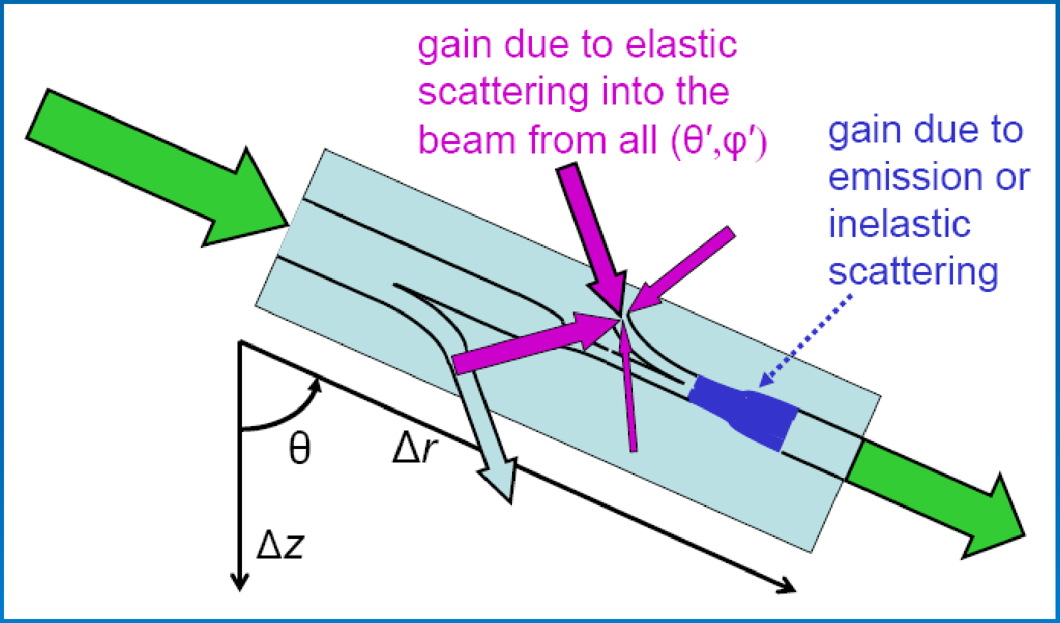
$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\, b(z,\lambda)\, L(z,\theta,\phi,\lambda)$$



Sources of Radiance

Scattering into the beam from all other directions increases the radiance:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(z;\, \theta',\phi' \rightarrow \theta,\phi\, ;\lambda)\, d\Omega'$$



There can be internal sources of radiance $\Sigma(z,\theta,\phi,\lambda)$, such as bioluminescence

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = \Sigma(z,\theta,\phi,\lambda)$$

Add up the Losses and Sources

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\,a(z,\lambda)\,L(z,\theta,\phi,\lambda)$$

$$-\,b(z,\lambda)\,L(z,\theta,\phi,\lambda)$$

$$+\int_{4\pi} L(z,\theta',\phi',\lambda)\,\beta(\mathrm{z};\,\theta',\phi'\rightarrow\theta,\phi;\,\lambda)\,d\Omega'$$

$$+\,\Sigma(z,\theta,\phi,\lambda)$$

Finally, note that $a + b = c$ and that $dz = dr\cos\theta$ to get

The 1D SRTE, Geometric-depth Form

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = - c(z,\lambda)\, L(z,\theta,\phi,\lambda) + \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(\mathrm{z};\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega' + \Sigma(z,\theta,\phi,\lambda)$$

This is the same equation we got from the VRTE, but without the rigor and understanding.

The VSF $\beta(\mathrm{z};\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)$ is usually written as $\beta(\mathrm{z}, \psi, \lambda)$ in terms of the scattering angle ψ, where

$$\cos\psi = \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi' - \phi)$$

The 1D SRTE, Optical-depth Form

Define the increment of dimensionless optical depth ζ as $d\zeta = c\, dz$ and write the VSF as *b* times the phase function, $\tilde{\beta}$, and recall that $\omega_o = b/c$ to get

$$\cos\theta \, \frac{dL(\zeta,\theta,\phi,\lambda)}{d\zeta} = -\, L(\zeta,\theta,\phi,\lambda)$$

$$+\, \omega_o \int_{4\pi} L(\zeta,\theta',\phi',\lambda)\, \tilde{\beta}(\zeta;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega'$$

$$+\, \Sigma(\zeta,\theta,\phi,\lambda)/c(\zeta,\lambda)$$

Can specify the IOPs by *c* and the VSF β, or by ω_o and the phase function $\tilde{\beta}$ (and also *c*, if there are internal sources)

Note that a given geometric depth *z* corresponds to a different optical depth $\zeta(\lambda) = \int_0^z c(z',\lambda)\, dz'$ at each wavelength

The 1D SRTE, Geometric-depth Form

$$\cos\theta \, \frac{\underline{dL(z,\theta,\phi,\lambda)}}{dz} = -\,\underline{c(z,\lambda)}\;\underline{L(z,\theta,\phi,\lambda)}$$

$$+ \int_{4\pi} \underline{L(z,\theta',\phi',\lambda)}\;\underline{\beta(z;\,\theta',\phi'\rightarrow\theta,\phi;\,\lambda)}\; d\Omega'$$

$$+ \underline{\Sigma(z,\theta,\phi,\lambda)}$$

NOTE: The SRTE has the TOTAL *c* and TOTAL VSF. Only oceanographers (not light) care how much of the total absorption and scattering are due to water, phytoplankton, CDOM, minerals, etc.

The SRTE is a linear (in the unknown radiance), first-order (only a first derivitive) integro-differential equation. Given the green (plus boundary conditions), solve for the red. This is a two-point (surface and bottom) boundary value problem.

Solving the RTE: The Lambert-Beer Law

A trivial solution:

• homogeneous water (IOPs do not depend on z)
• no scattering (VSF β = 0, so c = a + b = a)
• no internal sources (Σ = 0)
• infinitely deep water (no radiance coming from the bottom boundary, so L → 0 as z → ∞)
• incident radiance *L*(z=0) is known just below the sea surface

$$\cos\theta \, \frac{dL(z,\theta,\phi,\lambda)}{dz} \; = \; -\,a(\lambda)L(z,\theta,\phi,\lambda)$$

$$\int_{L(z=0,\theta,\phi,\lambda)}^{L(z,\theta,\phi,\lambda)} \frac{dL}{L} \; = \; -\int_0^z \frac{a\,dz}{\cos\theta}$$

$$L(z,\theta,\phi,\lambda) \; = \; L(z=0,\theta,\phi,\lambda)\, e^{-az/\cos\theta}$$

Note that this *L* satisfies the SRTE, the surface boundary condition, and the bottom boundary condition *L*(z=∞) = 0.

Solving the SRTE: Gershun's Law

Start with the 1D, source-free, SRTE.

$$\begin{aligned}\cos\theta \frac{dL(z,\theta,\phi,\lambda)}{dz} &= -\,c(z,\lambda)\,L(z,\theta,\phi,\lambda)\\ &+ \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\,\beta(z,\theta',\phi'\to\theta,\phi,\lambda)\sin\theta'\,d\theta'\,d\phi'\end{aligned}$$

Integrate over all directions. The left-hand-side becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi}\cos\theta\frac{dL(z,\theta,\phi)}{dz}d\Omega(\theta,\phi) &= \frac{d}{dz}\int_0^{2\pi}\int_0^{\pi} L(z,\theta,\phi)\cos\theta\, d\Omega(\theta,\phi)\\ &= \frac{d}{dz}[E_{\rm d}(z) - E_{\rm u}(z)]\end{aligned}$$

Solving the SRTE: Gershun's Law

The – *cL* term becomes

$$\begin{aligned}\iint -c(z)L(z,\theta,\phi)d\Omega(\theta,\phi) &= -c(z)\iint L(z,\theta,\phi)d\Omega(\theta,\phi)\\ &= -c(z)E_{\mathrm{o}}(z)\end{aligned}$$

The elastic-scatter path function becomes

$$\begin{aligned}&\iint\left[\iint L(z,\theta',\phi')\,\beta(z,\theta',\phi'\to\theta,\phi)d\Omega(\theta',\phi')\right]d\Omega(\theta,\phi)\\ =&\iint L(z,\theta',\phi')\left[\iint \beta(z,\theta',\phi'\to\theta,\phi)d\Omega(\theta,\phi)\right]d\Omega(\theta',\phi')\\ =&\,b(z)\iint L(z,\theta',\phi')d\Omega(\theta',\phi')\\ =&\,b(z)E_{\mathrm{o}}(z)\end{aligned}$$

Solving the SRTE: Gershun's Law

Collecting terms,

$$\frac{d}{dz}\left[E_{\mathrm{d}} - E_{\mathrm{u}}\right] \;=\; -\,cE_{\mathrm{o}} + bE_{\mathrm{o}}$$

or

$$a(z,\lambda) \;=\; -\,\frac{1}{E_{\mathrm{o}}(z,\lambda)}\,\frac{d}{dz}\left[E_{\mathrm{d}}(z,\lambda) - E_{\mathrm{u}}(z,\lambda)\right]$$

Gershun's law can be used to retrieve the absorption coefficient from measured in-water irradiances (at wavelengths where inelastic scattering effects are negligible).

This is an example of an explicit inverse model that recovers an IOP from measured light variables.

Water Heating and Gershun's Law

The rate of heating of water depends on how much irradiance there is and on how much is absorbed:

$$\frac{\partial T}{\partial t} = \frac{1}{c_v \rho} a_o E_o = -\frac{1}{c_v \rho} \frac{\partial (E_d - E_u)}{\partial z} \qquad \left[\frac{\deg\ C}{\sec}\right]$$

c_v = 3900 J (kg deg C)$^{-1}$ is the specific heat of sea water
ρ = 1025 kg m^{-3} is the water density

This is how irradiance is used in a coupled physical-biological-optical ecosystem model to couple the biological variables (which, with water, determine the absorption coefficient and the irradiance) to the hydrodynamics (heating of the upper ocean water)

Solving the SRTE

Exact analytical (i.e., pencil and paper) solutions of the SRTE can be obtained only for very simple situations, such as no scattering. There is no function (that anyone has ever found) that gives

$L(z,\theta,\phi,\lambda) = f(a, VSF,$ sun angle, bottom reflectance, etc.)

even for very simple situations such as homogenous water with isotropic scattering. Even the extremely simple geometry of an isotropic point light source in an infinite homogeneous ocean is unsolved (a very complicated solution for $E_o(r)$ around a point source with isotropic scattering does exist). This is because of the complications of scattering (which don't exist for problems like the gravitational field around a point mass).

Solving the SRTE: Approximate Methods

Approximate analytical solutions can be obtained for idealized situations such as single scattering in a homogeneous ocean. These solutions were important in the early (pencil and paper) days of remote sensing (used by Gordon in many CZCS-era papers).

They are seldom used today because we have computers for numerical solutions of the RTE.

See the Web Book pages on the single-scattering approximation (SSA) and the Quasi-single scattering approximation (QSSA) starting at

radiative transfer theory/level 2/the single scattering approximation

Solving the RTE: Numerical Methods

The solution of the RTE for any realistic conditions of scattering or geometry must be done numerically. Three widely used exact numerical methods are seen in the literature (in RT theory, “exact” means that we don’t make approximations such as single scattering. Given accurate inputs and enough computer time, you can get the correct answer as closely as you wish.)

- Discrete ordinates: often used in atmospheric optics
  - highly mathematical
  - difficult to program
  - doesn’t handle highly peaked phase functions well
  - some codes need a level sea surface
  - models the medium as homogeneous layers
  - fast for irradiances and homogeneous systems
  - slow for radiances and inhomogeneous systems
  - therefore, not much used in oceanography

Solving the RTE: Numerical Methods

- Invariant Imbedding: what Hydrolight uses
    - highly mathematical (see *Light and Water*, Chaps 7 and 8)
    - difficult to program
    - 1D (depth dependence) problems only
    - run time increases linearly with optical depth
    - computes radiances accurately (no statistical noise)
    - extremely fast and accurate even for radiances and large depths

- Monte Carlo: widely used
    - simple math, easy to program
    - can solve 3D problems
    - easy to implement polarization
    - run time increases exponentially with optical depth
    - have to trace many photons to get accurate radiance estimates (solutions have statistical noise)
    - very long run times for radiances and/or great depths
    - more useful for irradiance computations and/or shallow depths

Sea Kayaking in SE Greenland, 2005

photo by Curtis Mobley