Lectures on
Optical Oceanography and
Ocean Color Remote Sensing

Curtis Mobley

Radiative Transfer Theory

Schiller Coastal Studies Center

Radiative Transfer Theory

Radiative transfer theory is the physical and mathematical framework for all of optical oceanography and ocean color remote sensing.

Today:

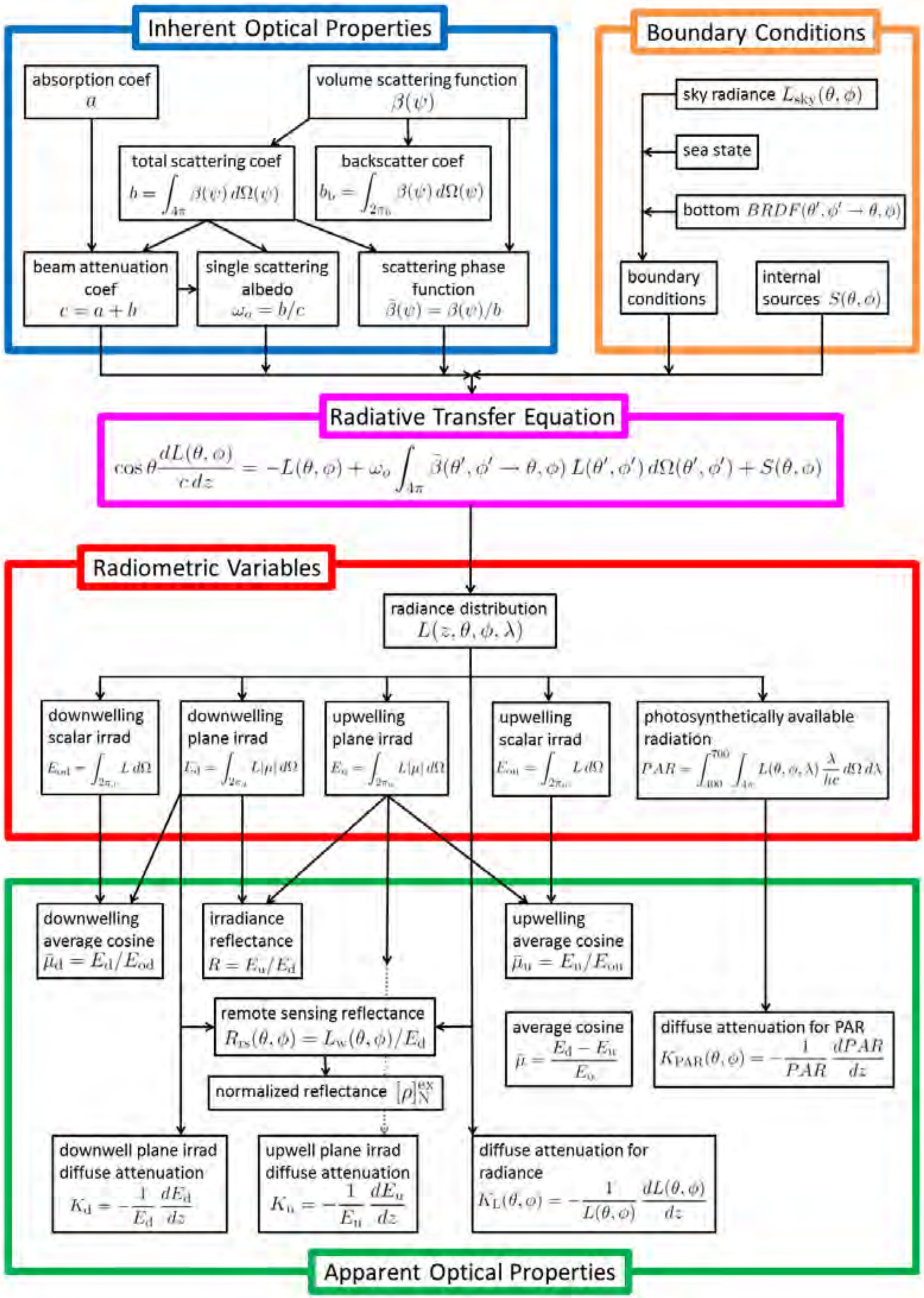
- A few words on scattering of polarized light
- Outline how you get from fundamental physics to RT theory
- Derive radiative transfer equations for various levels of approximation
- Three solutions to the scalar radiative transfer equation
    - The Lambert-Beer law
    - Gershun's equation
    - The Single-scattering approximation

The Radiative Transfer Equation (RTE)

Connects the IOPs, boundary conditions, and light sources to the radiance

All other radiometric variables (irradiances) and AOPs can be derived from the radiance.

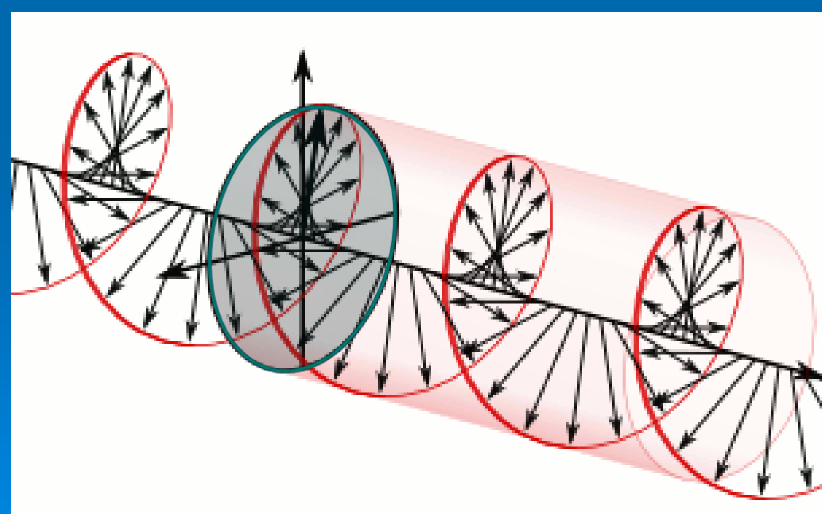
If you know the radiance (technically, the Stokes vector), you know everything there is to know about the light field.



Polarization

Light consists of propagating electric and magnetic fields, which are described by Maxwell's equations (see the Web Book pages at https://www.oceanopticsbook.info/view/radiative-transfer-theory/level-2/maxwells-equations-vacuo) and OOB Section 7.6)

"Polarization" refers to the plane in which the electric field vector is oscillating. This is given by the Stokes vector, $\underline{S} = [I, Q, U, V]^T$, which is an array of four real numbers (not a vector in the geometric sense).

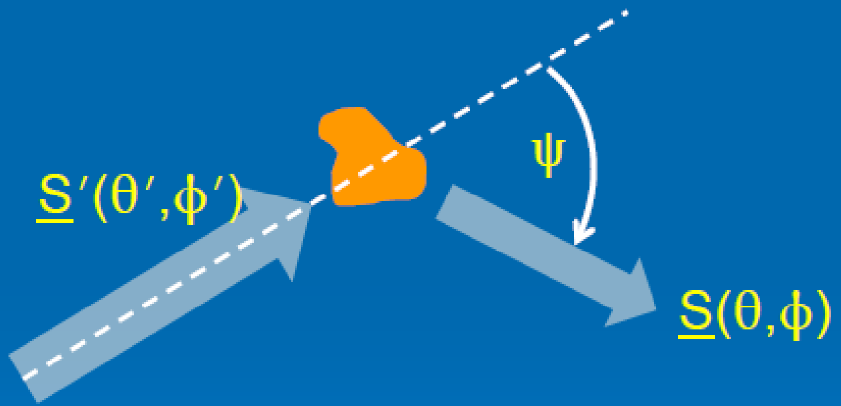


Scattering of Polarized Light

Absorption does not change the state of polarization. Absorption just removes light from the beam.

Scattering can change one state of polarization into another.

Scattering is specified by a 4 x 4 matrix $\underline{Z}$, called the phase matrix.



$$\underline{S}(\theta,\phi) = \underline{Z}(\theta',\phi' \rightarrow \theta,\phi)\, \underline{S}'(\theta',\phi')$$

$$\underline{S}(\theta,\phi) = \underline{Z}(\psi)\, \underline{S}'(\theta',\phi')$$

4x1 4x4 4x1

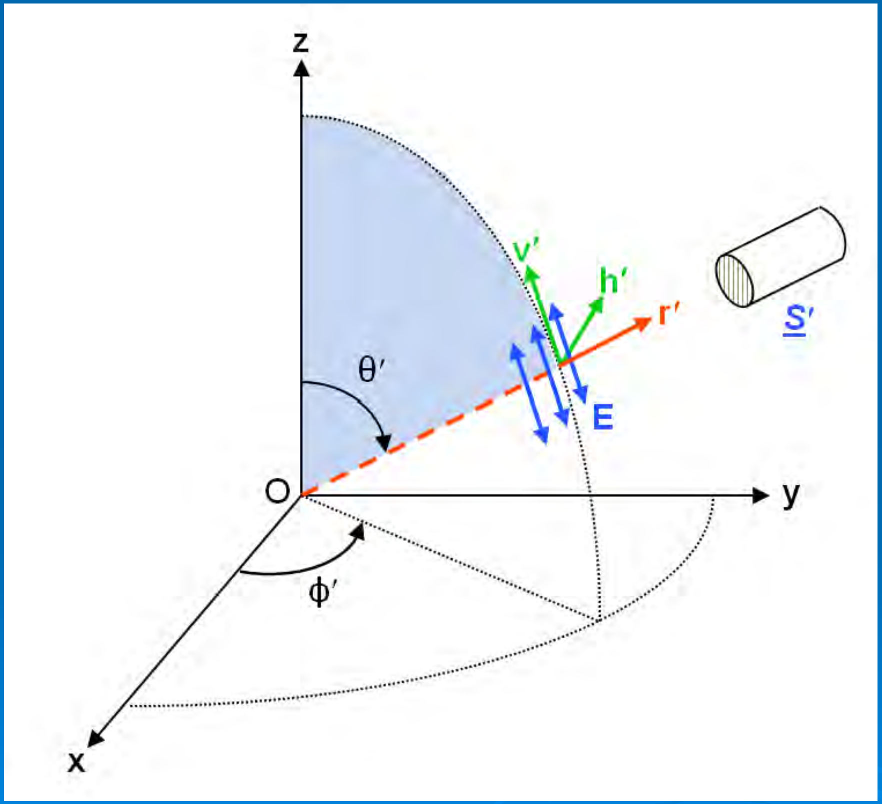
The scattering angle is given by

$$\cos\psi = \boldsymbol{\xi}' \cdot \boldsymbol{\xi} = \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi - \phi')$$

Meridian Planes

Stokes vector *Q* and *U* depend on the coordinate system.

In oceanography, we use meridian planes to express the Stokes vectors of incident and scattered light.



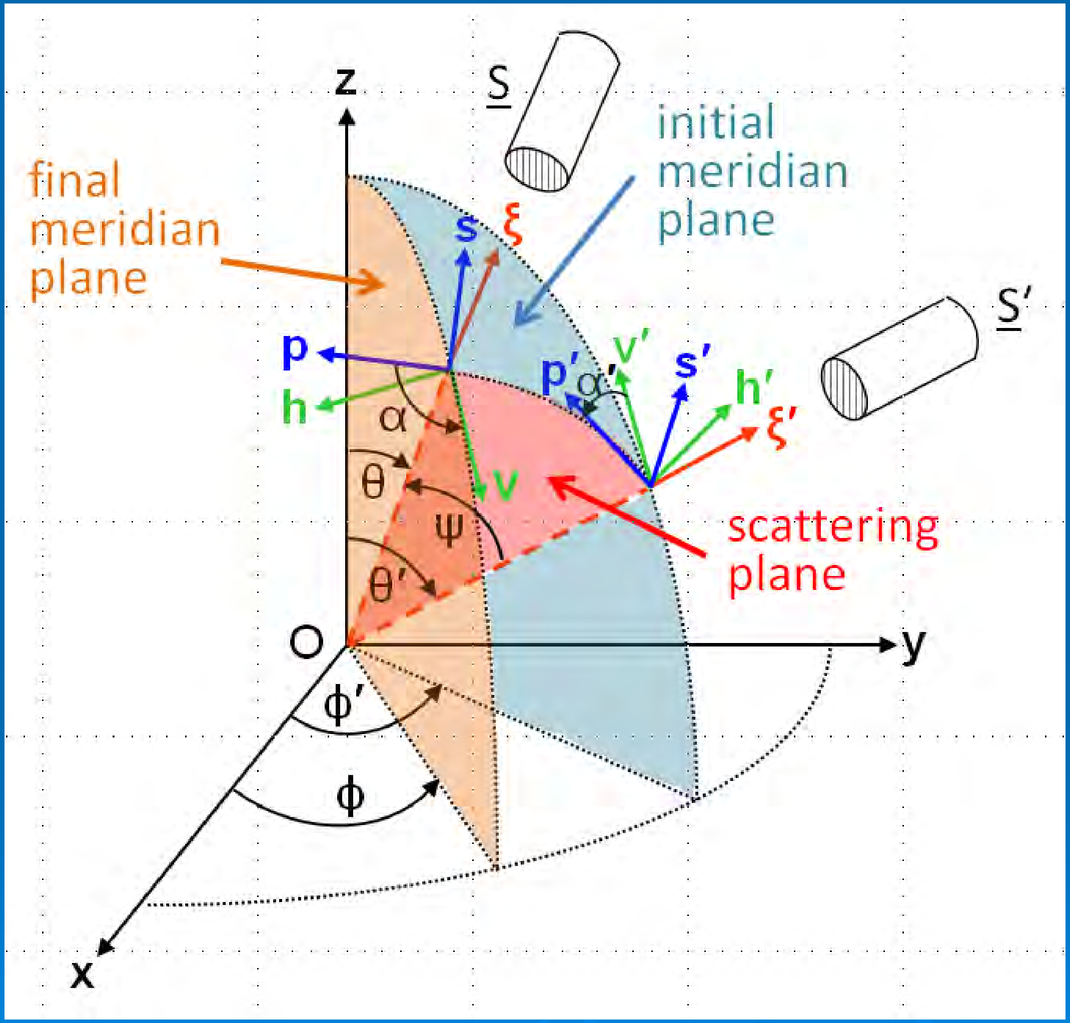
The meridian plane is the plane containing the normal to the mean sea surface and the direction of light propagation.

S′ is defined in the local (h′,v′) (horizontal or vertical; perpendicular or parallel) system

The meridian plane changes if the light changes direction

Must translate Stokes vectors between these coordinate systems

Phase, Scattering, and Rotation Matrices



$\underline{S}(\theta,\phi) = \underline{Z}(\psi)\ \underline{S}'(\theta',\phi')$

$= \underline{R}(\alpha)\ \underline{M}(\psi)\ \underline{R}(\alpha')\ \underline{S}'(\theta',\phi')$

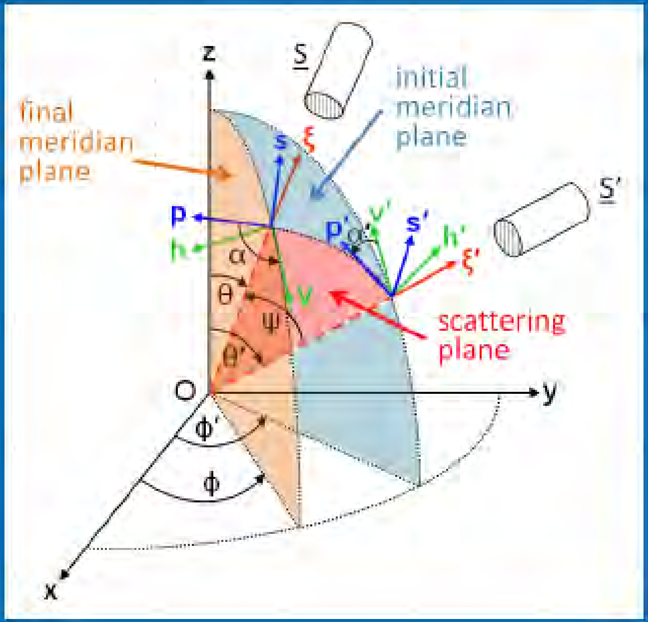
Start with $\underline{S}'$ in the initial meridian plane

"Rotate" $\underline{S}'$ into the scattering plane

Compute the scattering in the scattering plane (Mueller matrix)

Rotate the final $\underline{S}$ from the scattering plane to the final meridian plane

Phase, Scattering, and Rotation Matrices



$\underline{S}(\theta,\phi) = \underline{Z}(\psi)\, \underline{S}'(\theta',\phi')$

$\quad = \underline{R}(\alpha)\, \underline{M}(\psi)\, \underline{R}(\alpha')\, \underline{S}'(\theta',\phi')$

The phase matrix $\underline{Z}(\psi)$ transforms $\underline{S}'$ to $\underline{S}$, with both expressed in the meridian planes

The scattering matrix $\underline{M}(\psi)$ transforms $\underline{S}'$ to $\underline{S}$, with both expressed in the scattering plane

Rotation matrices $\underline{R}$ transform $\underline{S}$ from one plane to another

The decomposition $\underline{Z}(\psi) = \underline{R}(\alpha)\, \underline{M}(\psi)\, \underline{R}(\alpha')$ separates the physics of the scattering process (described by $\underline{M}$) from the bookkeeping related to the different coordinate systems (described by $\underline{R}$) . $\underline{M}$ is often called the Mueller matrix.

For the equations, see the Web Book

https://www.oceanopticsbook.info/view/light-and-radiometry/level-2/polarization-scattering-geometry

Section 1.10 of the OOB, or Mobley (2015) in the Library

The state of polarization of the radiance contains information about the environment (affected by the size distribution, shape, and index of refraction of particles in the water)

However, oceanographers usually measure only the total radiance because

- The 4x1 Stokes vector (and corresponding 4x4 scattering matrix, which describes scattering of polarized light) is much harder to measure than just *L*

- The state of polarization is believed to have little effect on processes like phytoplankton photosynthesis or water heating (but it can have large effects of other things like visibility and reflectance)

- The different polarizations of the radiance in different directions tend to average out when the radiance is integrated to get irradiance

- We do not have good models or data for the inputs needed to compute polarization in the ocean

Therefore, we will usually ignore polarization in this class.

Keep in mind, however, that ignoring polarization (e.g., in HydroLight) causes some error (~10% in radiance, ~1% in irradiance) and that use of polarization will become more important in future years, as instruments and models improve.

Two Paths to the Scalar Radiative Transfer Equation

but two levels of understanding and two different interpretations

Rigorous Derivation

Maxwell's Equations (light described by electric and magnetic fields)

very difficult physics and math, but <u>no energy arguments, and no mention of radiance</u>

The general vector RTE

various simplifications and approximations

Phenomenological Derivation

Define radiance

arguments about conservation of energy as expressed by the radiance

The scalar RTE as solved by HydroLight

Basic Physics to Radiative Transfer Theory

Radiative transfer theory is now a well established branch of physics.

Quantum electrodynamics (QED)

↓ ⟵ Noble prize level of difficulty

Maxwell's equations

↓ ⟵ Exceptionally difficult (only in the last 10 years)

The general vector radiative transfer equation (VRTE)

↓ ⟵ PhD in physics level of difficulty

The VRTE for particles with mirror symmetry

↓ ⟵ We can do this

The scalar RTE for the first component of the Stoke's vector (what HydroLight solves)

QED

Quantum electrodynamics (QED) is the fundamental theory that explains with total accuracy (as far as we know) the interactions of light and matter (or of charged particles).

Views light as consisting of elementary particles called photons. These are the quanta of the electromagnetic field.

Extremely mathematical and abstract. Can do the calculations only for interactions between elementary particles.

Recommended introduction: *QED: The Strange Story of Light and Matter* by Richard Feynman

Also, see
https://www.oceanopticsbook.info/view/radiative-transfer-theory/level-2/the-general-vector-radiative-transfer-equation
or OOB Section 7.3 for more discussion and links to other papers.

Maxwell’s Equations

"The classical limit of the quantum theory of radiation is achieved when the number of photons becomes so large that ... may as well be regarded as a continuous variable. The space-time development of the classical electromagnetic wave approximates the dynamical behavior of trillions of photons.“ –J. J. Sakurai, *Advanced Quantum Mechanics* (1967)

This limit leads to Maxwell’s equations, which govern electric and magnetic fields. For a discussion of these equations, see
https://www.oceanopticsbook.info/view/radiative-transfer-theory/level-2/maxwells-equations-vacuo or OOB Sec.7.6
Light is viewed classically as propagating electric and magnetic fields.

These equations are accurate for everyday problems: e.g., generation, propagation, and detection of radio waves; the generation of electrical power; the refraction of light at an air-water surface; and the scattering of light by phytoplankton. (They break down for atomic scale processes, very high energies, very low temperatures, or when individual photons become important (“quantum optics” or “photonics”).) So, correct, but still too complicated to use for oceanographic calculations.

The General VRTE

A propagating-wave solution to Maxwell's equations leads to a general vector radiative transfer equation (VRTE) for the Stokes vector (see Mishchenko's books and papers):

$$\xi \cdot \nabla \underline{S}(\mathrm{x}, \xi) = -\underline{K}(\mathrm{x}, \xi)\, \underline{S}(\mathrm{x}, \xi) + \iint_{4\pi} \underline{Z}(\mathrm{x}, \xi' \to \xi)\, \underline{S}(\mathrm{x}, \xi')\, d\Omega(\xi') + \underline{\Sigma}(\mathrm{x}, \xi)$$

where

- $\underline{K}(\mathrm{x}, \xi)$ is a 4×4 *extinction matrix*, which describes the attenuation (by the background medium and any particles imbedded in the medium) of the light propagating in direction ξ.
- $\underline{Z}(\mathrm{x}, \xi' \to \xi)$ is a 4×4 *phase matrix*, which describes how light in an initial state of polarization and direction ξ' in the incident meridian plane is scattered to a different state of polarization and direction ξ in the final meridian plane.
- $\underline{\Sigma}(\mathrm{x}, \xi)$ is a 4×1 internal source term, which specifies the Stokes vector of any emitted light such as bioluminescence or light at the wavelength of interest that comes from other wavelengths via inelastic scattering.

In general, all 16 elements of $\underline{K}$ and $\underline{Z}$ are non-zero.

The General VRTE

$$\xi \cdot \nabla \underline{S}(\mathrm{x}, \xi) = -\underline{K}(\mathrm{x}, \xi)\, \underline{S}(\mathrm{x}, \xi) + \iint_{4\pi} \underline{Z}(\mathrm{x}, \xi' \to \xi)\, \underline{S}(\mathrm{x}, \xi')\, d\Omega(\xi') + \underline{\Sigma}(\mathrm{x}, \xi)$$

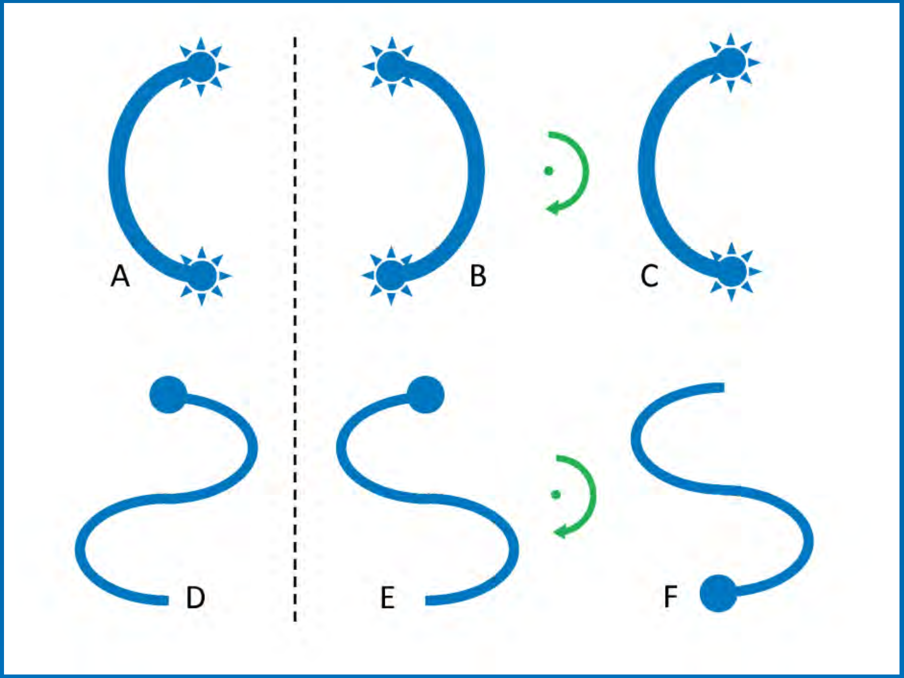
This equation can describe polarized light propagation in matter that is directionally non-isotropic (e.g., in a crystal), that can absorb light differently for different states of polarization (dichroism), and that contains scattering particles of any shape and random or non-random orientation.

Can solve the equation numerically, but we almost never have the needed inputs: the 16 elements of $\underline{K}$ and 16 of $\underline{Z}$.

Still need to simplify for oceanographic applications

Mirror-symmetric Particles

The general VRTE becomes much simpler if the scattering particles are mirror-symmetric and randomly oriented.



mirror-symmetric particle

not mirror-symmetric

Are oceanic particles like phytoplankton or mineral particles mirror-symmetric?

Mirror-symmetric Phytoplankton

http://www.mikroskopie-ph.de/Kreispraeparat-25-G.jpg

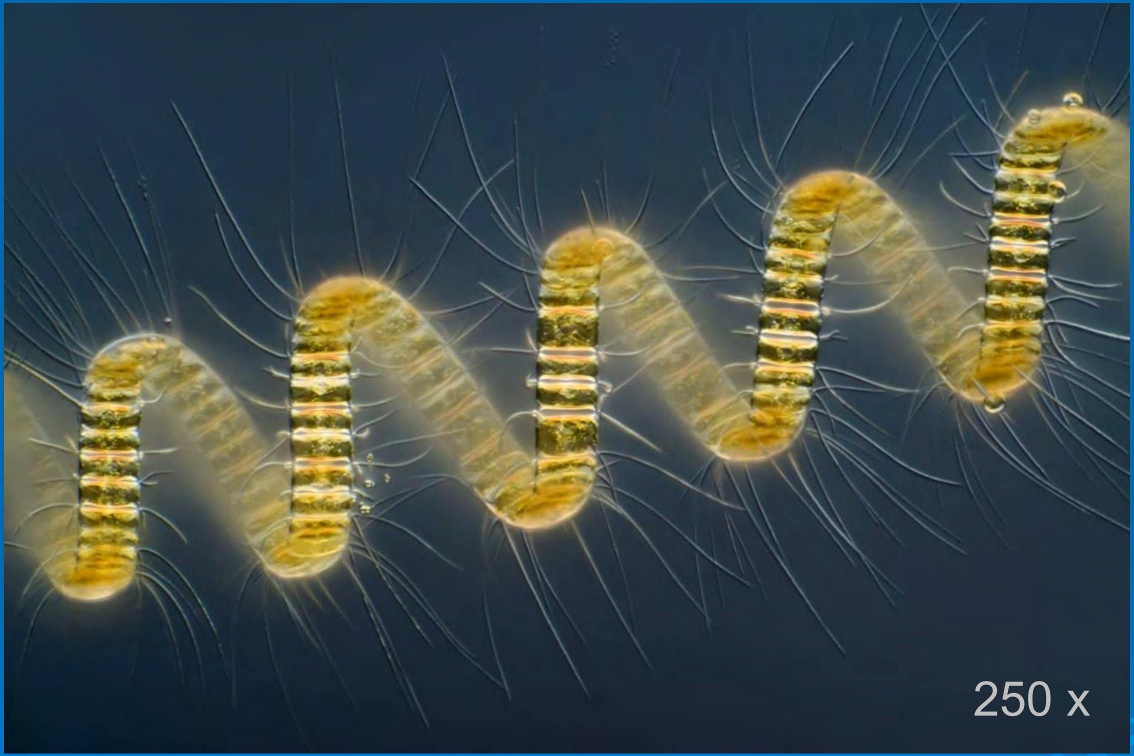
https://media1.britannica.com/eb-media/93/184793-004-11DAC31B.jpg

The assumption that phytoplankton are mirror-symmetric is reasonable and usually correct.

The assumption that they are spherical is not.

Non-mirror-symmetric Phytoplankton

This chain-forming diatom (*Chaetoceros debilis*) is a left-handed helix (but maybe the photo was flipped). It is not mirror symmetric. A bloom of these would require the general VRTE for Stokes vector calculations.



 From https://www.wired.com/2013/10/nikon-small-world-2013/

(If you have equal numbers of randomly oriented left- and right-handed helices, then the bulk medium is mirror symmetric, and you can use the simpler VRTE.)

The VRTE for a Mirror-symmetric Medium

If the particles are mirror-symmetric and randomly oriented:

- The attenuation matrix becomes diagonal and the attenuation does not depend on direction or state of polarization:

$$\underline{K}(\mathbf{x}, \boldsymbol{\xi}) = c(\mathbf{x}) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c is the "beam attenuation coefficient"

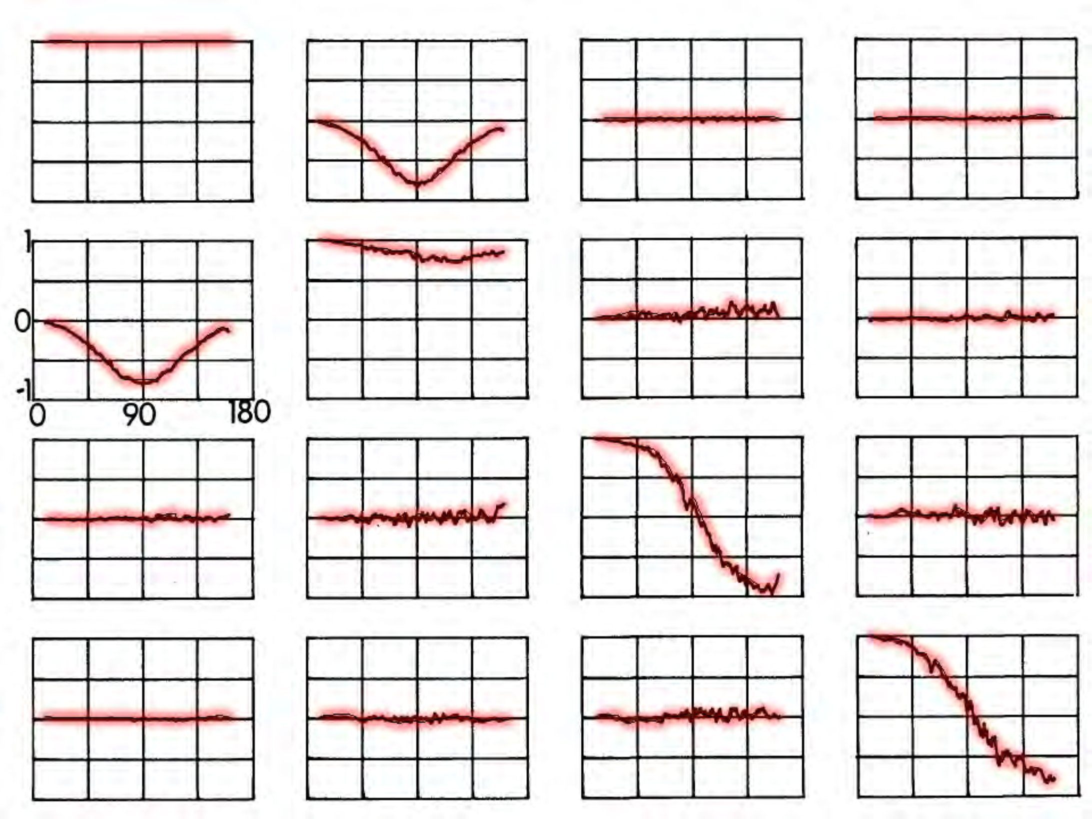
- The scattering matrix becomes block diagonal with only 6 independent elements:

$$\underline{M} = \begin{bmatrix} M_{11}(\psi) & M_{12}(\psi) & 0 & 0) \\ M_{12}(\psi) & M_{22}(\psi) & 0 & 0 \\ 0 & 0 & M_{33}(\psi) & M_{34}(\psi) \\ 0 & 0) & -M_{34}(\psi) & M_{44}(\psi) \end{bmatrix}$$

Do measured oceanic scattering matrices look like this?

Measured Scattering Matrices

The reduced scattering matrix is $\tilde{M}_{ij}(\psi) = M_{ij}(\psi)/M_{11}(\psi)$



Redrawn from Fig. 3(a) of Voss and Fry (1984)

$M_{11} = \beta(\psi)$ is the volume scattering function

$M_{22} \neq M_{11}$ indicates non-spherical particles

To within the measurement error, $M_{12} = M_{21}$ and $M_{33} = M_{44}$ and others are 0, so really only 4 independent elements

The assumption of randomly oriented, mirror-symmetric particles is justified.

The VRTE for Mirror-symmetric Particles

If the IOPs and boundary conditions are horizontally homogeneous, depth *z* is the only spatial variable and

$$\boldsymbol{\xi} \cdot \nabla \underline{S}(\mathbf{x}, \boldsymbol{\xi}) = \cos\theta \frac{d}{dz} \underline{S}(z, \theta, \phi)$$

The one-dimensional (1D) VRTE for a mirror-symmetric medium then becomes

$$\cos\theta \frac{d}{dz} \underline{S}(z, \theta, \phi) = -c(z)\, \underline{S}(z, \theta, \phi) + \iint_{4\pi} \underline{R}(\alpha) \underline{M}(z, \psi) \underline{R}(\alpha') \underline{S}(z, \theta', \phi')\, d\Omega(\theta', \phi') + \underline{\Sigma}(z, \theta, \phi)$$

where

$$\underline{M} = \begin{bmatrix} M_{11}(\psi) & M_{12}(\psi) & 0 & 0) \\ M_{12}(\psi) & M_{22}(\psi) & 0 & 0 \\ 0 & 0 & M_{33}(\psi) & M_{34}(\psi) \\ 0 & 0) & -M_{34}(\psi) & M_{44}(\psi) \end{bmatrix}$$

$$\underline{R}(\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\gamma & -\sin 2\gamma & 0 \\ 0 & \sin 2\gamma & \cos 2\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is the VRTE commonly used in atmospheric and oceanic optics

The RTE for the total radiance *I*

The first element of the Stokes vector is the total radiance, *I*, without regard for the state of polarization. In oceanography, *I* is usually called the radiance *L*.

$$\begin{aligned}
\cos\theta \frac{d}{dz} I(z,\theta,\phi) = & -c(z)\, I(z,\theta,\phi) \\
& + \iint_{4\pi} M_{11}(z,\psi)\, I(z,\theta',\phi')\, d\Omega(\theta',\phi') + \Sigma_I(z,\theta,\phi) \\
& + \iint_{4\pi} \cos\alpha' M_{12}(z,\psi)\, Q(z,\theta',\phi')\, d\Omega(\theta',\phi') + \Sigma_Q(z,\theta,\phi) \\
& - \iint_{4\pi} \sin\alpha' M_{12}(z,\psi)\, U(z,\theta',\phi')\, d\Omega(\theta',\phi') + \Sigma_U(z,\theta,\phi)\,.
\end{aligned}$$

Note: we cannot solve this equation for *I* because we do not know *Q* and *U* unless we simultaneously solve the VRTE for all of *I, Q, U, V*

The RTE for the Total Radiance

Some modern instruments exist for measuring the VSF $M_{11} = \beta(\psi)$ [e.g., Lee and Lewis (2003), Harmel et al. (2016), Li et al. (2012), Tan et al. (2013)]. However, $\beta(\psi)$ is seldom measured during field experiments.

There are only a few instruments for measuring some of the other $M_{ij}(\psi)$ elements of the scattering matrix [see Chami et al. (2014), Twardowski et al. (2012), Slade et al. (2013)].

If the VRTE is solved, the $M_{ij}(\psi)$ are usually modeled (often using Mie theory, which assumes spherical particles).

Because of the lack of data or good models for $M_{ij}(\psi)$ for different water types, modelers (including me) often just drop the terms involving $M_{12}Q$ and $M_{12}U$ and hope for the best.

This gives the scalar radiance transfer equation (SRTE) for the total radiance.

The SRTE for the Total Radiance *I* (or *L*)

Dropping the polarization-dependent terms gives the SRTE:

$$
\begin{aligned}
\cos\theta \frac{dL(z,\theta,\phi,\lambda)}{dz} &= -\,c(z,\lambda)L(z,\theta,\phi,\lambda) \\
&+ \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\beta(z;\theta',\phi' \to \theta,\phi;\lambda)\sin\theta' d\theta' d\phi' \\
&+ \Sigma(z,\theta,\phi,\lambda)\ .
\end{aligned}
$$

This is the equation HydroLight solves.

SRTE Error Estimate

The degree of linear polarization in the ocean is typically 10-30%; so *Q/I* or *U/I* < 0.3. For $\psi \approx 90$ deg, $M_{12} \approx 0.8\, M_{11}$. Then the error in ignoring the *Q* or *U* terms can be as large as

$$\frac{M_{12} Q}{M_{11} I} = 0.8 \times 0.3 \approx 0.25$$

However, comparison with *L* computed by the VRTE and the SRTE shows that the difference is typically ~10%. The error in radiance is positive in some directions and negative in others.

Irradiances are integrals of *L* over direction, so the errors in *L* tend to cancel, and irradiances are then good to a few percent.

Even though the SRTE is somewhat inaccurate, it is commonly used because

- The inputs *c* and $\beta(\psi)$ are better known.
- The math needed to solve the SRTE is much easier that for the VRTE.
- The output is accurate enough for many (but not all) applications.
- Commercial software (HydroLight) is available.

RTE Summary

Maxwell's equations are correct but too complicated to solve in the oceanographic setting. Also, they give electric and magnetic fields, which is more and different information than we need or want.

The general VRTE gives us what we want (Stokes vectors), but we don't have all of the needed inputs (extinction $\underline{K}$ and phase $\underline{Z}$ matrices).

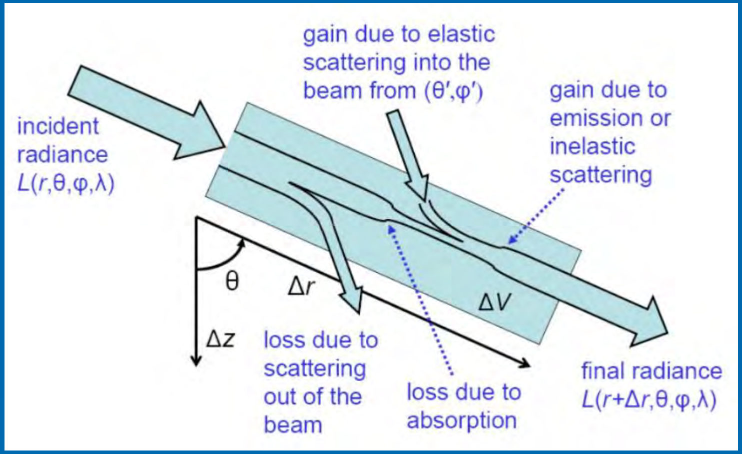
The VRTE for mirror-symmetric media is usually applicable to the ocean, but we still don't have all of the inputs needed for routine usage. This VRTE is commonly solved for the atmosphere, but not yet routinely solved for the ocean. The output is accurate, but there is no user-friendly public or commercial software for solving this VRTE in the ocean.

The SRTE gives output that is accurate enough for many (but not all) applications. There are more data and bio-geo-optical models for defining the inputs. There is commercial software (HydroLight) for solving this equation.

Another Way to Think About the SRTE

The previous discussion derived the SRTE from the general VRTE. This is the proper way to think about the SRTE because the various steps showed all of the assumptions made along the way and gave an estimate of the errors made if you use the SRTE rather than the full VRTE.

However, you will often see the SRTE "derived" using arguments about conservation of energy (e.g., Collin's lecture or *Light and Water*). This development would be correct if light were unpolarized, so that there is no VRTE. (This derivation is included for reference at the end of the Powerpoint.)



There are also conceptual problems with "phenomenological" RT theory; see the recent papers of Mischenko (2013, 2014) in the library.

The 1D SRTE, Geometric-depth Form

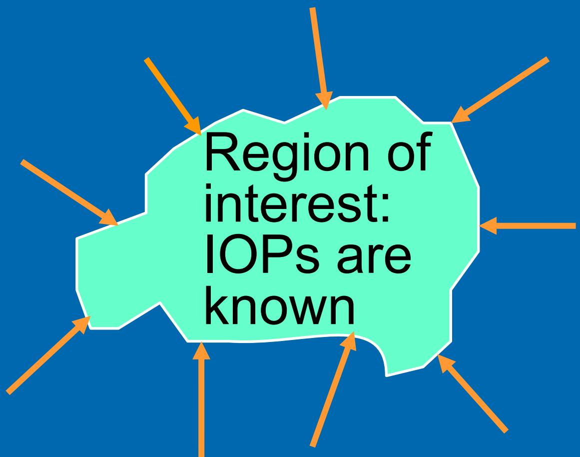
$$\cos\theta \frac{dL(z,\theta,\phi,\lambda)}{dz} = -c(z,\lambda)L(z,\theta,\phi,\lambda) + \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\beta(z;\theta',\phi' \to \theta,\phi;\lambda)\sin\theta' d\theta' d\phi' + \Sigma(z,\theta,\phi,\lambda) .$$

NOTE: The SRTE has the TOTAL *c* and TOTAL VSF. Only oceanographers (not light) care how much of the total absorption and scattering are due to water, phytoplankton, CDOM, minerals, etc.

The SRTE is a linear (in the unknown radiance), first-order (only a first derivative) integro-differential equation. Given the green (plus boundary conditions), solve for the red. This is a two-point (surface and bottom) boundary value problem. The math goes way beyond this course....

Solving the RTE

A unique solution of the RTE requires:

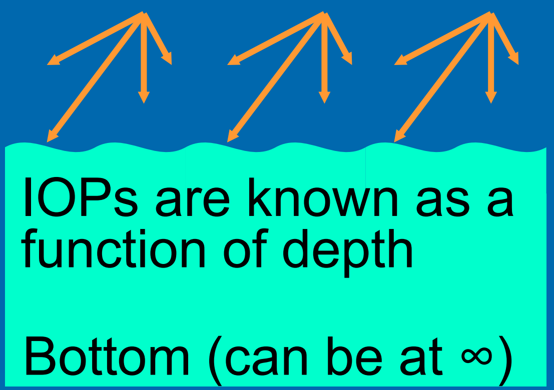


Radiance incident onto all boundaries from outside the region is known

A 3-D problem

Stretch out the region to make a horizontally homogeneous ocean

Radiance incident onto sea surface is known



A 1-D problem

Given the IOPs within the region and the incident radiances, we can solve for the radiance within and leaving the region

Solving the SRTE: The Lambert-Beer Law

A trivial solution:

• homogeneous water (IOPs do not depend on z)

• no scattering (VSF β = 0, so c = a + b = a)

• no internal sources (S = 0)

• infinitely deep water (no radiance coming from the bottom boundary, so L → 0 as z → ∞)

• incident radiance *L*(z=0) is known just below the sea surface

$$\cos\theta \frac{dL(z,\theta,\phi,\lambda)}{dz} = -a(\lambda)L(z,\theta,\phi,\lambda)$$

$$\int_{L(z=0,\theta,\phi,\lambda)}^{L(z,\theta,\phi,\lambda)} \frac{dL}{L} = -\int_0^z \frac{a\,dz}{\cos\theta}$$

$$L(z,\theta,\phi,\lambda) = L(z=0,\theta,\phi,\lambda)\,e^{-az/\cos\theta}$$

Note that this *L* satisfies the SRTE, the surface boundary condition, and the bottom boundary condition *L*(z→∞) = 0.

Solving the SRTE: Gershun's Law

Start with the 1D, source-free, SRTE.

$$\begin{aligned}\cos\theta \frac{dL(z,\theta,\phi,\lambda)}{dz} &= -\, c(z,\lambda)\, L(z,\theta,\phi,\lambda) \\ &+ \int_0^{2\pi}\int_0^{\pi} L(z,\theta',\phi',\lambda)\,\beta(z,\theta',\phi' \to \theta,\phi,\lambda)\sin\theta' d\theta' d\phi'\end{aligned}$$

Integrate over all directions. The left-hand-side becomes

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} \cos\theta \frac{dL(z,\theta,\phi)}{dz} d\Omega(\theta,\phi) &= \frac{d}{dz}\int_0^{2\pi}\int_0^{\pi} L(z,\theta,\phi)\cos\theta d\Omega(\theta,\phi) \\ &= \frac{d}{dz}[E_{\mathrm{d}}(z) - E_{\mathrm{u}}(z)]\end{aligned}$$

Solving the SRTE: Gershun's Law

The – *cL* term becomes

$$\iint -c(z)L(z,\theta,\phi)d\Omega(\theta,\phi) = -c(z)\iint L(z,\theta,\phi)d\Omega(\theta,\phi)$$
$$= -c(z)E_o(z)$$

The elastic-scatter path function becomes

$$\iint \left[\iint L(z,\theta',\phi')\,\beta(z,\theta',\phi' \to \theta,\phi)d\Omega(\theta',\phi')\right] d\Omega(\theta,\phi)$$
$$= \iint L(z,\theta',\phi')\left[\iint \beta(z,\theta',\phi' \to \theta,\phi)d\Omega(\theta,\phi)\right] d\Omega(\theta',\phi')$$
$$= b(z)\iint L(z,\theta',\phi')d\Omega(\theta',\phi')$$
$$= b(z)E_o(z)$$

Solving the SRTE: Gershun's Law

Collecting terms,

$$\frac{d}{dz}\left[E_{\mathrm{d}} - E_{\mathrm{u}}\right] \;=\; -\,cE_{\mathrm{o}} + bE_{\mathrm{o}}$$

or

$$a(z,\lambda) \;=\; -\frac{1}{E_{\mathrm{o}}(z,\lambda)}\,\frac{d}{dz}\left[E_{\mathrm{d}}(z,\lambda) - E_{\mathrm{u}}(z,\lambda)\right]$$

Gershun's law can be used to retrieve the absorption coefficient from measured in-water irradiances (at wavelengths where inelastic scattering effects are negligible). See Voss L&O (1989).

This is an example of an explicit inverse model that recovers an IOP from measured light variables.

Water Heating and Gershun's Law

The rate of heating of water depends on how much irradiance there is and on how much is absorbed:

$$\frac{\partial T}{\partial t}(z, \text{at } \lambda) = \frac{1}{c_v \rho} a(z,\lambda) E_{\text{o}}(z,\lambda) = -\frac{1}{c_v \rho} \frac{\partial [E_{\text{d}}(z,\lambda) - E_{\text{u}}(z,\lambda)]}{\partial z} \quad \left[\frac{\text{deg C}}{\text{sec}}\right]$$

c_v = 3900 J (kg deg C)$^{-1}$ is the specific heat of sea water
ρ = 1025 kg m^{-3} is the water density

$$E_{\text{d,u}}(z) = \int_{400}^{1000} E_{\text{d,u}}(z,\lambda) d\lambda$$

0

$$\frac{\partial T}{\partial t}(z, \text{short-wave}) = -\frac{1}{c_v \rho} \frac{\partial [E_{\text{d}}(z) - E_{\text{u}}(z)]}{\partial z}$$

This is how irradiance is used in a coupled physical-biological-optical ecosystem model to couple the biological variables (which, with water, determine the absorption coefficient and the irradiance) to the hydrodynamics (heating of the upper ocean water)

Solving the SRTE: Approximate Analytical Solutions

Approximate analytical solutions can be obtained for idealized situations such as single scattering in a homogeneous ocean. These solutions were very important in the early (pencil and paper) days of remote sensing (used by Howard Gordon in many CZCS-era papers). They are still useful for understanding first-order relations and basic ideas.

Quick outline: SOS → SSA → QSSA
(successive order of scattering → single-scattering approximation → quasi-single-scattering approximation)

Assume:
(A1): The water is homogeneous: the IOPs do not depend on depth
(A2): There are no internal sources or inelastic scattering
(A3): The sea surface is level (zero wind speed)
(A4): The sun is a point source is a black sky, so that the incident radiance onto the sea surface is collimated
(A5): The water is infinitely deep

Solving the SRTE: The SOS Approximation

Assumptions (A1) and (A2) reduce the RTE (using optical depth $\zeta = cz$) to

$$\mu \frac{dL(\zeta,\mu,\phi)}{d\zeta} = -L(\zeta,\mu,\phi) + \omega_o \int_0^{2\pi} \int_{-1}^{1} L(\zeta,\mu',\phi')\,\tilde{\beta}(\mu',\phi' \to \mu,\phi)\,d\mu'\,d\phi'$$

Now write radiance = unscattered + scattered once + scattered twice + …

$$L(\zeta,\mu,\phi) = L^{(0)}(\zeta,\mu,\phi) + \omega_o L^{(1)}(\zeta,\mu,\phi) + \omega_o^2 L^{(2)}(\zeta,\mu,\phi) + \cdots$$

This gives

$$\mu \left[\frac{dL^{(0)}}{d\zeta} + \omega_o \frac{dL^{(1)}}{d\zeta} + \omega_o^2 \frac{dL^{(2)}}{d\zeta} + \cdots \right] = -\left[L^{(0)} + \omega_o L^{(1)} + \omega_o^2 L^{(2)} + \cdots\right] + \omega_o \int_0^{2\pi} \int_{-1}^{1} \left[L^{(0)} + \omega_o L^{(1)} + \omega_o^2 L^{(2)} + \cdots\right] \tilde{\beta}(\mu',\phi' \to \mu,\phi)\,d\mu'\,d\phi'$$

Solving the RTE: The SOS Approximation

Regrouping the terms gives

$$
\begin{aligned}
& \left[\mu \frac{dL^{(0)}}{d\zeta} + L^{(0)} \right] \\
+ \omega_o & \left[\mu \frac{dL^{(1)}}{d\zeta} + L^{(1)} - \int_0^{2\pi} \int_{-1}^{1} L^{(0)} \, \tilde{\beta}(\mu', \phi' \to \mu, \phi) \, d\mu' \, d\phi' \right] \\
+ \omega_o^2 & \left[\mu \frac{dL^{(2)}}{d\zeta} + L^{(2)} - \int_0^{2\pi} \int_{-1}^{1} L^{(1)} \, \tilde{\beta}(\mu', \phi' \to \mu, \phi) \, d\mu' \, d\phi' \right] \\
+ \cdots & = 0 \, .
\end{aligned}
$$

This equation must hold true for any value of $0 \le \omega_o < 1$. Setting $\omega_o = 0$ would leave only the first line of the equation, whose terms must sum to 0. Similarly, when $\omega_o \neq 0$, each group of terms multiplying a given power of ω_o must equal zero in order for the entire left side of the equation to sum to zero. We can therefore equate to zero the groups of terms in brackets multiplying each power of ω_o. This gives a sequence of equations:

Solving the RTE: The SOS Approximation

$$\mu \frac{dL^{(0)}}{d\zeta} = -L^{(0)} \qquad \text{(S0)}$$

$$\mu \frac{dL^{(1)}}{d\zeta} = -L^{(1)} + \int_0^{2\pi}\int_{-1}^{1} L^{(0)}\,\tilde{\beta}(\mu', \phi' \to \mu, \phi)\,d\mu'\,d\phi' \qquad \text{(S1)}$$

$$\mu \frac{dL^{(2)}}{d\zeta} = -L^{(2)} + \int_0^{2\pi}\int_{-1}^{1} L^{(1)}\,\tilde{\beta}(\mu', \phi' \to \mu, \phi)\,d\mu'\,d\phi' \qquad \text{(S2)}$$

We first solve Eq. (S0), which governs the unscattered radiance. The solution for $L^{(0)}$ then can be used in Eq. (S1) to evaluate the path integral, which becomes a source function for singly scattered radiance. After solving Eq. (S1) for singly scattered radiance, $L^{(1)}$ can be used to evaluate the path function in Eq. (S2), and so on. This process constitutes the successive-order-of-scattering (SOS) solution technique.

Solving the RTE: The Single-Scattering Approximation

Next we must integrate these equations with the appropriate boundary conditions at the sea surface and the bottom. The boundary condition at the sea surface is given by assumptions (A3) and (A4). The boundary condition at the bottom is given by (A5).

Turning these assumptions into the proper mathematical form is tricky and requires use of the Dirac delta function. For that math, see the Web Book page at
https://www.oceanopticsbook.info/view/radiative-transfer-theory/level-2/the-single-scattering-approximation
OOB Section 7.8

The final result is just the Lambert-Beer law:

$$L^{(0)}(\zeta, \mu, \phi) \; = L^{(0)}(0, \mu, \phi)\, e^{-\zeta/\mu}$$

This equation describes the propagation of unscattered radiance with depth and direction.

Solving the RTE: The Single-Scattering Approximation

Next we insert the form of L(0) into Eq. (S1) and

$$L^{(0)}(\zeta,\mu,\phi) = L^{(0)}(0,\mu,\phi)\, e^{-\zeta/\mu}$$

$$\mu \frac{dL^{(1)}}{d\zeta} = -L^{(1)} + \int_0^{2\pi}\int_{-1}^{1} L^{(0)}\, \tilde{\beta}(\mu',\phi' \to \mu,\phi)\, d\mu'\, d\phi' \qquad \text{(S1)}$$

Solving Eq. (S1) is messy, as are the resulting equations for $L^{(1)}$.
Again, see the Web Book or OOB pages on the SSA

Solving the RTE: The Single-Scattering Approximation

The SSA gives good results for small values of ω_o, i.e. when there is very little multiple scattering, or when the medium in thin (optical depth < 1). For values of ω_o > 0.1 and large optical depths, the SSA solution becomes very inaccurate.

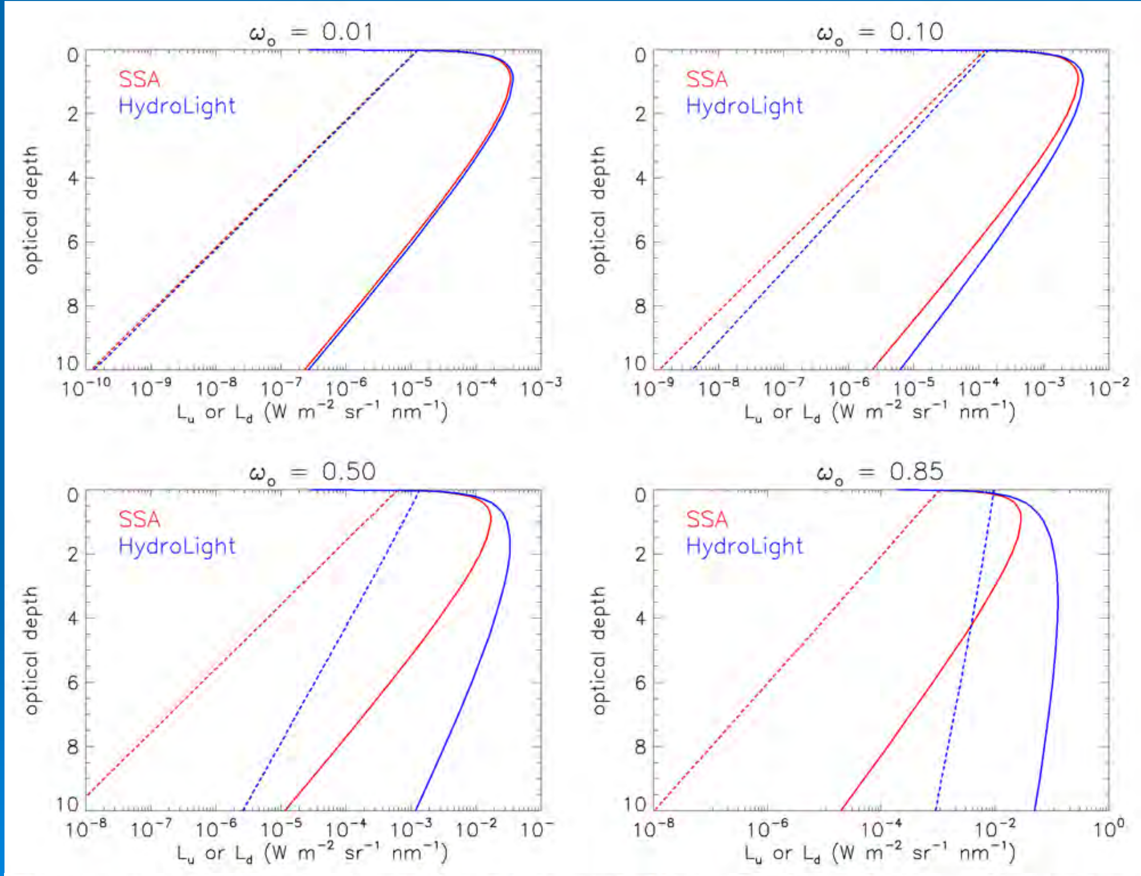
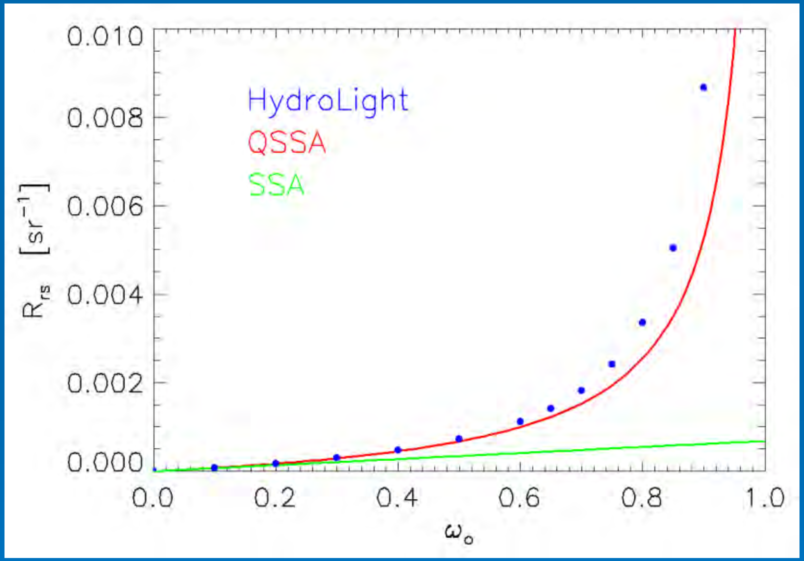


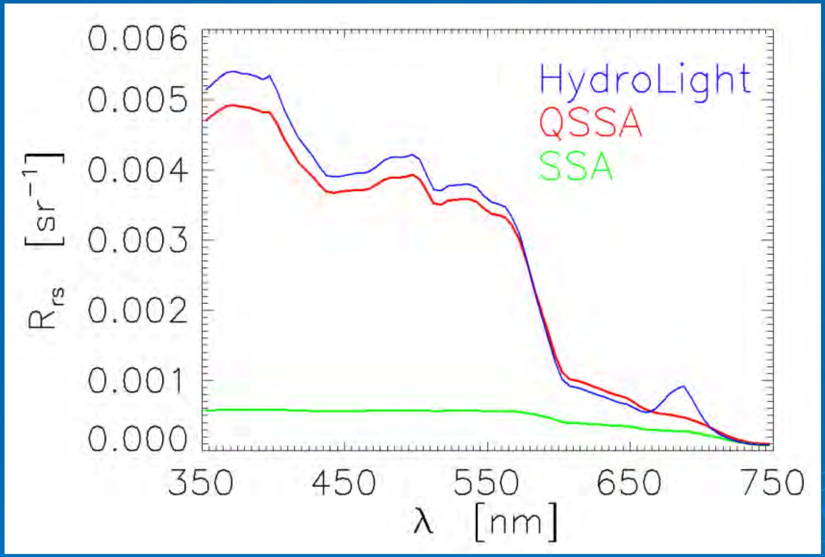
Figure 3: Comparison of the SSA and HydroLight solutions for nadir-viewing (L_u; dashed lines) and zenith-viewing (L_d; solid lines) radiances for four ω_o values; $E_d(0) = 1$ W m^{-2} sr^{-1}.

Solving the SRTE: The QSSA

The SSA can be improved by a clever approximation called the Quasi-Single-Scattering Approximation (QSSA).

However, this would be a full lecture.... For the math, see OOB Sec 7.9
https://www.oceanopticsbook.info/view/radiative-transfer-theory/level-2/the-quasi-single-scattering-approximation





These approximations are seldom used today, but you'll need to understand them before reading the foundational papers by Howard Gordon.

Solving the SRTE: Exact Numerical Methods

To get accurate solutions of the SRTE for realistic ocean conditions (depth-dependent IOPs, realistic sky radiances, large ω_o, large optical depth, etc.) you must use numerical solutions and a lot of computer power.

Tomorrow you will learn to run the HydroLight software, which solves the SRTE for any IOPs, bottom reflectance, incident sky radiance, etc.

In week 4 we'll discuss Monte Carlo methods for solving radiative transfer equations (Lecture 28, Lab 14).

Sea Kayaking in SE Greenland, 2005

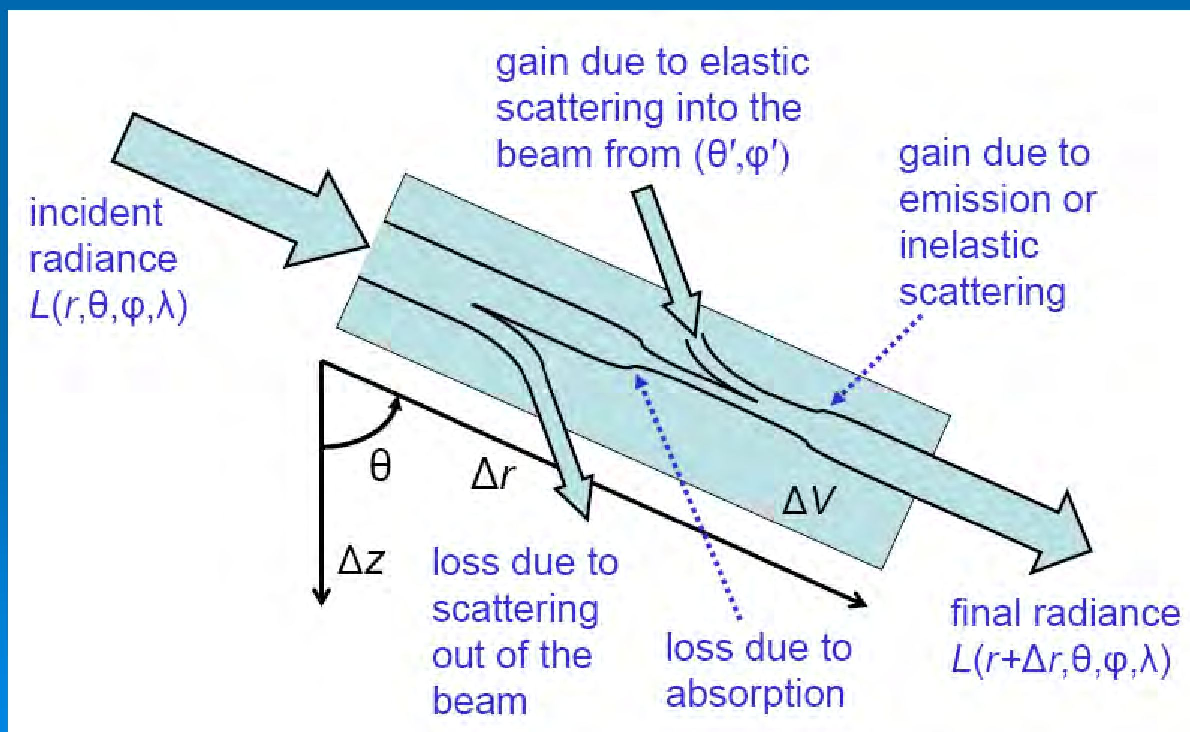
photo by Curtis Mobley

Sea Kayaking in SE Greenland, 2005

photos by Curtis Mobley

Phenomonological “Derivation” of the SRTE

To “derive” the time-independent SRTE for horizontally homogeneous water, we assume that light is not polarized (always wrong). Then consider the total radiance at a given depth *z*, traveling in a given direction (θ, ϕ), at a given wavelength λ. We then add up the various ways the radiance $L(z,\theta,\phi,\lambda)$ can be created or lost in a distance Δr along direction (θ, ϕ), going from depth *z* to $z + \Delta z$



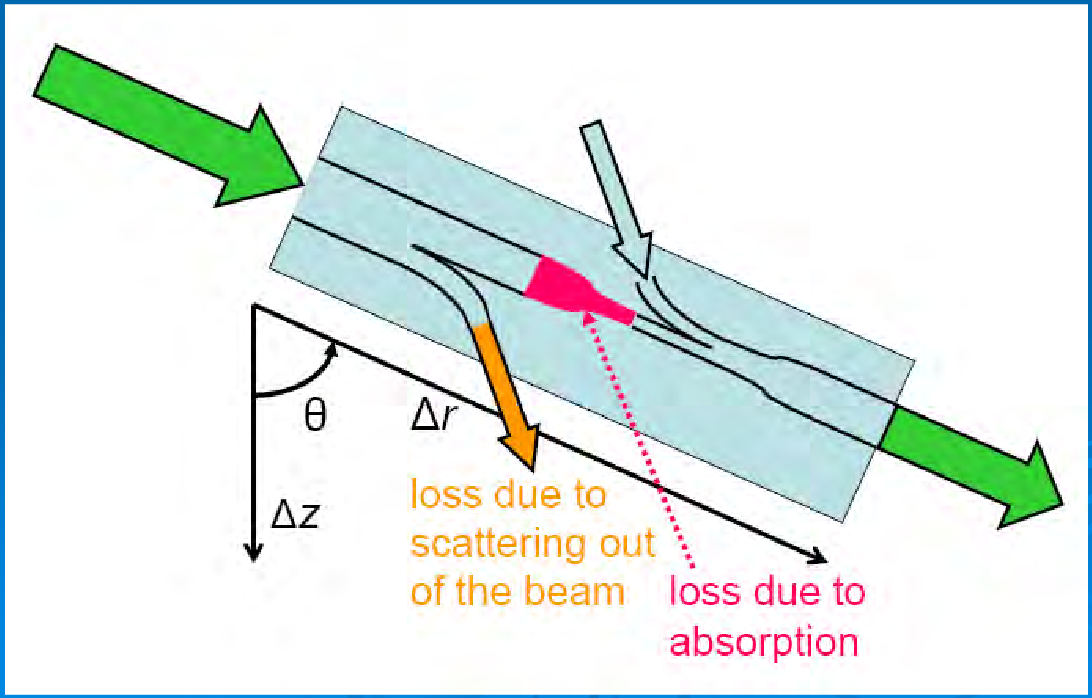
Losses of Radiance

The loss due to absorption is proportional to how much radiance there is:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\, a(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

Likewise for loss of radiance due to scattering out of the beam:

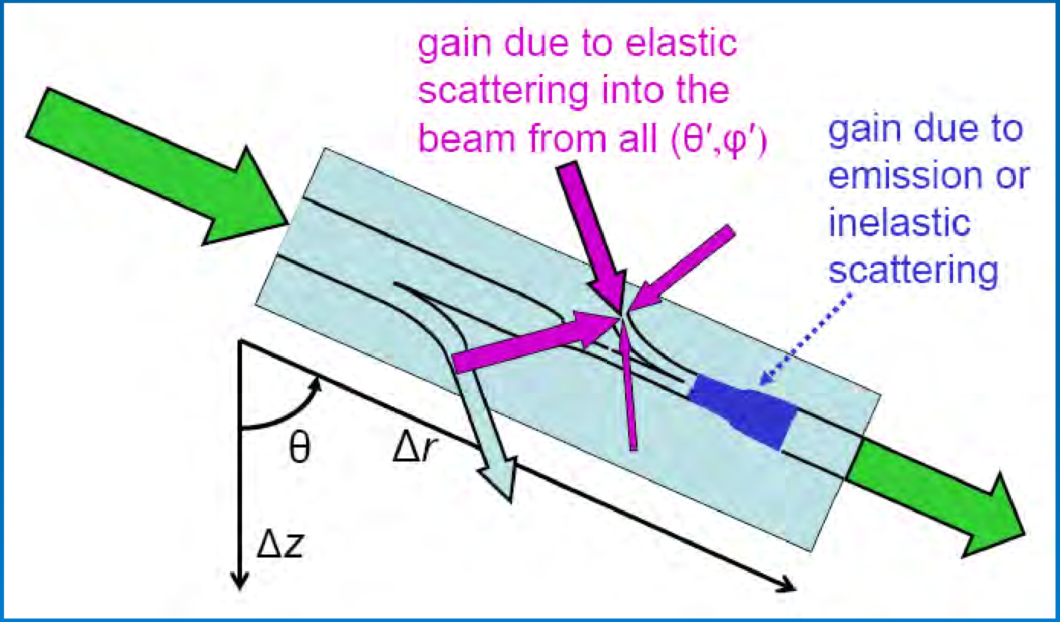
$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = -\, b(z,\lambda)\, L(z,\theta,\phi,\lambda)$$



Sources of Radiance

Scattering into the beam from all other directions increases the radiance:

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = \int_{4\pi} L(z,\theta',\phi',\lambda)\,\beta(z;\theta',\phi'\rightarrow\theta,\phi\,;\lambda)\,d\Omega'$$



There can be internal sources of radiance $S(z,\theta,\phi,\lambda)$, such as bioluminescence

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = S(z,\theta,\phi,\lambda)$$

Add up the Losses and Sources

$$\frac{dL(z,\theta,\phi,\lambda)}{dr} = - a(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

$$- b(z,\lambda)\, L(z,\theta,\phi,\lambda)$$

$$+ \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(z;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega'$$

$$+ S(z,\theta,\phi,\lambda)$$

Finally, note that $a + b = c$ and that $dz = dr \cos\theta$ to get

The 1D SRTE, Geometric-depth Form

$$\cos\square \, \frac{dL(z,\theta,\phi,\lambda)}{dz} = - c(z,\lambda)\, L(z,\theta,\phi,\lambda)$$
$$+ \int_{4\pi} L(z,\theta',\phi',\lambda)\, \beta(z;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)\, d\Omega'$$
$$+ S(z,\theta,\phi,\lambda)$$

This is the same equation we got from the VRTE, but without the rigor and understanding.

The VSF $\beta(z;\, \theta',\phi' \rightarrow \theta,\phi;\, \lambda)$ is usually written as $\beta(z,\, \square,\, \lambda)$ in terms of the scattering angle $\square$, where

$\cos\psi = \cos\theta' \cos\theta + \sin\theta' \sin\theta \cos(\phi'-\phi)$

The 1D SRTE, Optical-depth Form

Define the increment of dimensionless optical depth ζ as $d\zeta = c\,dz$ and write the VSF as *b* times the phase function, $\tilde{\beta}$, and recall that $\omega_o = b/c$ to get

$$\cos\theta \, \frac{dL(\zeta,\theta,\phi,\lambda)}{d\zeta} = - L(\zeta,\theta,\phi,\lambda)$$

$$+ \, \omega_o \int_{4\pi} L(\zeta,\theta',\phi',\lambda) \, \tilde{\beta}(\zeta; \theta',\phi' \rightarrow \theta,\phi; \lambda) \, d\Omega'$$

$$+ \, S(\zeta,\theta,\phi,\lambda)/c(\zeta,\lambda)$$

Can specify the IOPs by *c* and the VSF β, or by ω_o and the phase function $\tilde{\beta}$ (and also *c*, if there are internal sources)

Note that a given geometric depth *z* corresponds to a different optical depth $\zeta(\lambda) = \int_0^z c(z',\lambda)\, dz'$ at each wavelength