## Problem set III:

0. Provide the answer to the Mie-lab homework.

1. **b. Class example (and homework):** let's assume we are dealing with non-absorbing rain drops \(m=1.33\) and a wavelength in air of 550nm \((0.55\mu m)\). How does the mass-specific attenuation \(c^* = Q_{ext} \times \text{cross-section/volume/density}\) vary as function of size for drops varying in size \(D=0.01, 0.1, 1, 10, 100, 1000\). Use the Matlab routine callbh.m and the Mie solver (bhmie.m) to get \(Q_{ext} = Q_c\).

2. Given a water fraction of 1g/m³ calculate the transmission \(\left(\frac{L}{L_0} = e^{-cR}\right)\) through a cloud \(R=1\)km thick for water distributed uniformly with the above diameters.

<table>
<thead>
<tr>
<th>Diameter (μm)</th>
<th>Qc</th>
<th>G [m³]</th>
<th>volume [m³]</th>
<th>C_ext [m²]</th>
<th>(c^*) [m²/kg]</th>
<th>mass of a single drop [kg]</th>
<th># of drops/m³</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.00000118</td>
<td>7.85398E-17</td>
<td>5.23599E-25</td>
<td>9.2677E-23</td>
<td>0.177</td>
<td>5.23599E-22</td>
<td>1.90986E+18</td>
</tr>
<tr>
<td>0.1</td>
<td>0.0114</td>
<td>7.85398E-15</td>
<td>5.23599E-22</td>
<td>8.95354E-17</td>
<td>171</td>
<td>5.23599E-19</td>
<td>1.90986E+15</td>
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<td>3.93</td>
<td>7.85398E-13</td>
<td>5.23599E-19</td>
<td>3.08661E-12</td>
<td>5895</td>
<td>5.23599E-16</td>
<td>1.90986E+12</td>
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<tr>
<td>10</td>
<td>2.15</td>
<td>7.85398E-11</td>
<td>5.23599E-16</td>
<td>1.68861E-10</td>
<td>322.5</td>
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<td>100</td>
<td>2.03</td>
<td>7.85398E-09</td>
<td>5.23599E-13</td>
<td>1.59436E-08</td>
<td>30.45</td>
<td>5.23599E-10</td>
<td>1909859.317</td>
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<tr>
<td>1000</td>
<td>2.01</td>
<td>7.85398E-07</td>
<td>5.23599E-10</td>
<td>1.57865E-06</td>
<td>3.015</td>
<td>5.23599E-07</td>
<td>1909.859317</td>
</tr>
</tbody>
</table>

Beam-c (m⁻¹) transmission 1km:

\[0.000177 \quad 0.837779785\]
\[0.171 \quad 5.44056E-75\]
\[5.895 \quad 0\]
\[0.3225 \quad 8.7102E-141\]
\[0.03045 \quad 5.96668E-14\]
\[0.003015 \quad 0.049045835\]
The above results summarize the Mie calculations results for droplets with different diameters. $Q_c$ denotes the attenuation (= extinction) efficiency factor, $G$ the drop's cross-section, $C_{\text{ext}}$ its extinction cross section ($Q_c \times G$), $c^*$ the mass specific extinction coefficient ($C_{\text{ext}}$/volume/density), the number of drops are calculated ($N=\text{total mass/mass of single drop}$) to obtain the attenuation ($c = N \times C_{\text{ext}}$) and finally the transmission $\tau = \exp(-c \times 1000 \text{m})$.

3. How does it explain visibility differences between fog and rain?
   We observe very different attenuations for different size droplets and the same liquid content in the air. Fog droplet size varies from 2-60 μm encompassing a large variability in attenuation (smallest droplets have the highest attenuation per mass). Rain drop size are $\mathcal{O}(1 \text{mm})$, which have 2 order of magnitude less attenuation per mass than 10 μm drops.

2. Build a 2-stream radiative transfer model as follows:

   Input irradiance from sky: $E_d$

   A layer between sky and ground with transmission $T_1$ and reflectance $R_1$ and absorbance $A_1$ (their sum is 1, as they represent probabilities of being transmitted, reflected and absorbed).

   Light reaching the Earth: $T_1E_d$
   Light reflecting back to sky: $E_u=R_1E_d$
   Light absorbed: $A_1E_d$

   Assume the ground absorbs all the radiation hitting it. Add one more layer and calculate the properties of this new atmosphere ($R_2$, $T_2$, & $A_2$) as function of the properties of the one layer atmosphere. Note that light can bounce between layers, so it may be useful to first calculate the light field ($E_d$ & $E_u$) in the middle between the layers.

   Keep doubling the number of layers and find expressions for $R_N$, $T_N$, & $A_N$ as function of ($R_1$, $T_1$, & $A_1$) where $N$ is even.
Plot $R_N$ and $T_N$ for $N=1,2,4...1024$ (log the x-axis) for $A_1=0$, 0.01, 0.05 and $R_1=0.001$, 0.1 and 0.9 (total on 9 curves for each graph).

This problem was first published by Stokes in the Proceedings of the Royal Society and appears in volume IV of his Mathematical and Physical Papers. If you are having difficulties feel free to consult others for the solution!

First let's solve it for none absorbing layers. Starting with 2 layers we get:

$E_d(middle)=T_1E_d(0)+R_1E_u(middle)$

$E_u(middle)=R_1E_d(middle)$

$\Rightarrow E_d(middle)=T_1/(1-R_1^2) E_d(0)$, and $E_u(middle)=R_1T_1/(1-R_1^2)$

Light reflected from atmosphere $E_u(0)=R_2E_d(0)=R_1E_d(0)+T_1E_u(middle)= (R_1+R_1T_1^2/(1-R_1^2))E_d(0)$

$\Rightarrow R_2=(R_1+ R_1T_1^2/(1-R_1^2))$

Light transmitted through atmosphere $T_2E_d(0)= T_1E_d(middle)=T_1^2/(1-R_1^2) E_d(0)$

$\Rightarrow T_2= T_1^2/(1-R_1^2)$

Without absorption, $T_1+R_1=1 \Rightarrow R_2=2R_1/(1+R_1)$, $T_2=(1-R_1)/(1+R_1)$

Since the same argument works if we have two layers each with $R_2$ and $T_2$ (hence 4 layers), $R_4=2R_2/(1+R_2)=4R_1/(1+3R_1)$ and $T_4=(1-R_1)/(1+3R_1)$. Check: $T_4=1- R_4$.

By recursion, for an even $N$, $R_N=NR_1/(1+(N-1)R_1)$, $T_N=(1-R_1)/(1+(N-1)R_1)$.

As $N\to\infty R_N\sim NR_1/(1+NR_1) \to 1$, $T_N\sim (1-R_1)/(1+NR_1) \to 0$.

Now that we have some experience we can add absorption. The only difference with the above analysis is that we cannot assume that $T_1+R_1=1$, but rather $T_1+R_1+A_1=1$. Hence:

$R_2=(R_1+ R_1T_1^2/(1-R_1^2))$, $T_2= T_1^2/(1-R_1^2)$ and $A_2=1-R_2-T_2$. Using excel or Matlab we keep doubling the layers and obtain $A_N$, $R_N$, and $T_N$ as function of $A_1$, $R_1$, and $T_1$ (a closed form solution as we found above was not found (see Matlab program appended).

Note how a little bit of absorption affects the asymptotic values for reflectivity for $N$ layers and how, as $N$ increases, the transmission through $N$ layers always asymptote zero. Increase absorptivity also increases the rate by which the asymptotic state is achieved.
Figure 1. Reflectivity as function of number of layers for different values of absorptivity and reflectivity of a single layer.

Figure 2. Transmisivity as function of number of layers for different values of absorptivity and reflectivity of a single layer.
%program to solve question 3 in homework 3

A1=[0   0.01    0.05];
R1=[0.001 0.1   0.9];

for i=1:3
    for j=1:3
        Rn(1)=R1(i);
        An(1)=A1(j);
        Tn(1)=1-An(1)-Rn(1);
        n=1;
        for k=1:10
            n(k+1)=2^k;
            Rn(k+1)=Rn(k)+Rn(k)*Tn(k)^2/(1-Rn(k)^2);
            Tn(k+1)=Tn(k)^2/(1-Rn(k)^2);
            An(k+1)=1-Tn(k)-Rn(k);
        end
        figure(1)
        subplot(3,3,(j-1)*3+i)
        semilogx(n,Rn);
        figure(2)
        subplot(3,3,(j-1)*3+i)
        semilogx(n,Tn);
        figure(3)
        subplot(3,3,(j-1)*3+i)
        semilogx(n,An);
    end
end