Reporting and Propagating Error

Practicing science well requires comfort with error and good judgment of its magnitude. Otherwise, there is no way to tell whether observations fall outside predictions from alternative hypotheses. Science makes the most progress when ideas that seem reasonable are discarded in favor of better ideas on the basis of data. If measurement error is too large and sample size too small to distinguish which idea is right, then science can't advance. In science, error is a necessary fact of working with physical evidence rather than something to be avoided. It is uncertainty in measurement. Here we will assume that error and imprecision are identical. The only way to know about inaccuracy is to have some alternative measure, which is not universally the case. Moreover, when two methods disagree it is not always clear which is the more accurate. Our best measures of inaccuracy are against known standards because those standards have been tested thoroughly.

Significant figures

In everyday life, writing the number 1.3 means just that. In science, it means that you are reasonably sure that the true value (precise value, assuming that your measurement tool is working accurately) lies between 1.25 and 1.349999... When you publish any number, be sure that you can support the precision stated; don't simply copy all the digits from your calculator or instrument. Many pieces of softwware spit out a fixed number of significant figures, whether they can be justified or not. Do not add extra zeros after the decimal point: 4.60 implies that the true value lies between 4.595 and 4.604999… Think about what the number of significant figures implies. Reporting one significant figure implies a fractional uncertainty (the ratio between your uncertainty, δx, and your best estimate, x_b, $|\delta x|/x_b$) of 10-100% (roughly 50%), reporting two significant figures implies a fractional uncertainty of roughly 5%, and three, roughly 0.5%. Avoid arbitrary rounding by quoting a rational number, for example 45/87 eggs proved infertile [instead of 0.52 of the individuals (n = 87) proved infertile].

Both measurement and error should have the same dimensions AND units. Always include the leading zero (± 0.4 and not ± .4). The following are all acceptable: 55.9 ± 0.4; 56 ± 4; 50 ± 40. If you get disparate precision in calculations of the best estimate and its imprecision, round the more precise number to be compatible with the less precise one, after you have made sure that you can support even that number of significant figures. In scientific notation, place the uncertainty before and not after the power of 10, *e.g.,* $(3.43 \pm 0.02) \times 10^{-7}$, not $3.43 \times 10^{-7} \pm 2 \times 10^{-9}$. The latter expression is ungainly. Even though you would end up without any ambiguity if you followed the rules (multiplication before addition or subtraction), the meaning is much less quickly apparent than with the uncertainty out front, and it is much harder to see whether the precision of the measurement and the error match. Some situations warrant mild exceptions. For example, if your best estimate of the mean length of adult shrimp in a population is 38.6 mm, and you measured to the nearest whole millimeter, it is arguably less misleading to write 38.6 ± 1 mm than to write 39 ± 1 mm. Similarly, if you are doing a string of calculations, you should carry extra digits to the extent that rounding before calculation could introduce unnecessary errors, but at the end you should report the final result with the appropriate number of significant figures.

General style rules for reporting of measurements

Most science style rules are logical. Some are being broken routinely in some journals to save "print" space. That tide will reverse, since clarity trumps space in a virtual world. You have been exposed to one rule for reporting, *i.e.,* to always put a figure in front of a decimal point. That rule came about because decimal points on lead type were relatively fragile compared with letters and numbers and could easily change shape. Is it a decimal point or a smudge? If there is a zero or other number out front, you don't have to guess. Another rule is to space around equals signs,

inequalities and operators. That rule also dates back to lead type, when >4 or <7 could easily become hard to read; printing used to be more smudgy than even the cheapest inkjet printing. If a whole number is ≤ 10, general practice is to write it out in a paragraph unless the paragraph has other Arabic numerals, in which case it is acceptable or even preferred to write all numerals within the paragraph in their Arabic form. This rule came from convenience (needing to get out the box of Arabic numerals or not), but retains some aesthetic value. For large numbers, numbers that contain a decimal point, as well as rational numbers, Arabic numerals are both clearer and shorter. Good scientific practice is to use *standard* abbreviations in Roman font after Arabic numbers (with a single space between) and avoid them after numbers written as text. Thus either one centimeter or 1 cm is good practice in the proper context, but one cm or 1 centimeter is bad practice in any context. Notice the lack of a period after standard abbreviations for units. Don't make up abbreviations. If you don't know for sure, look them up http://physics.nist.gov/cuu/Units/units.html. For hour, use h not hr; for seconds use s not sec. Rules evolve, but a primary reason for them is to avoid ambiguity. It is now standard to use italics for quantities and variables but Roman type for units, Arabic numerals and descriptive terms. If I see *cm,* I think $c \times m$. If I see cm, I read it as centimeters. If you want to do it correctly, NIST http://physics.nist.gov/cuu/Units/rules.html is a good place to visit.

Propagating errors

Consider two variables, x and y, and a third variable q such that $q = f(x,y)$ (f means "a function of"). Let's denote by subscript b the "best" estimate of a variable and by δ its uncertainty. Here we look at a function of 2 variables. All formulas can be directly generalized to more variables. Thus $x = x_b \pm \delta x$. The fractional uncertainty is given as $\frac{|\delta x|}{x_b}$, implying $x = x_b\left(1 \pm \frac{|\delta x|}{x_b}\right)$. A small number of rules will serve you well for most purposes:

1. In addition and subtraction ($q = x + y$ or $q = x - y$), uncertainties add: $\delta q \le \delta x + \delta y$. If x and y are independent and random, then $\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}$.
2. In multiplication or division ($q = x \times y$, $q = x/y$), *fractional* uncertainties add: $\frac{|\delta q|}{q} \le \frac{|\delta x|}{x_b} + \frac{|\delta y|}{y_b}$. If x and y are independent and random, $\frac{|\delta q|}{q} = \sqrt{\left(\frac{|\delta x|}{x_b}\right)^2 + \left(\frac{|\delta y|}{y_b}\right)^2}$.
3. In multiplication by a constant, k ($\delta k = 0$, *i.e,* the value of k lacks any uncertainty or its uncertainty is orders of magnitude smaller than that in any other variable), uncertainty varies proportionally: $q = kx$, $\delta q = k|\delta x|$.
4. In a power function ($q = x^n$, $\delta n = 0$), error multiplies with the power: $\frac{|\delta q|}{q} = n\frac{|\delta x|}{x_b}$.
5. For known functional dependence $q = f(x,y)$, the chain rule gives: $\delta q = \left|\frac{\partial q}{\partial x}\right|\delta x + \left|\frac{\partial q}{\partial y}\right|\delta y$.

6. When the relationship between dependent (q) and independent variables (x,y) is very complicated or unknown (for example, the dependence of the weather prediction of a general circulation model on the uncertainty in the temperature at a given location) a probabilistic approach is often adopted (sometimes called a Monte Carlo approach based on the reputation of MC for its casinos). Variables are sampled randomly, and their effect on the outcome is monitored until the statistics of the outcome's dependence on their variation is obtained. For example, let $q = f(x,y)$, $x = x_b \pm \delta x$, $y = y_b \pm \delta y$, with δx and δy being the standard deviation of many observation of x and y and x_b and y_b being the mean (here we assume both x and y to be normally, or Gaussian). Using a random number generator we sample randomly from the specified Gaussian distributions, getting a series of numbers $\{x_1, x_2, \ldots x_n\}$ and $\{y_1, y_2, \ldots y_n\}$ that we substitute into our function (or complex model) and generate a series of outcome: $\{q_1, q_2, \ldots q_n\}$. With enough replication we can generate a representative distribution of q, from which we can compute q_b and δq. In this example, q is also normally distributed.

Useful references

Taylor, J. R. 1997. *An Introduction to Error Analysis,* 2nd edition, University Science Books, Sausalito, California. Ch:1-3.

<http://physics.nist.gov/cuu/Uncertainty/index.html>