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Sound Scattering from a Fluid Sphere*

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The scattering of sound from a spherical fluid obstacle of size comparable to a wave-length is considered, neglecting dissipation. Calculations of the acoustic pressure and the total energy in the scattered wave are presented graphically; sound velocities and densities of the sphere lie between 0.5 and 2.0 times that of the external medium. The limiting cases of Rayleigh scattering and scattering from a fixed rigid sphere are also shown for comparison. In the region where the diameter of the sphere is comparable to a wave-length, the scattering is a complicated function of frequency, showing in some cases large maxima and minima. The amplitude of the scattered wave in the backward direction from a fluid sphere a few wave-lengths in diameter exceeds twice that from a rigid sphere of the same size for the case of the sound velocity 0.8 and density equal to that of the surrounding medium.

I. INTRODUCTION

THE scattering of sound by marine life is an important factor in the problem of sound ranging in the ocean. The presence of these biological scatterers in the sound beam will cause some of the sound to be deflected from its original direction. A portion of this sound will be returned to the source giving rise to volume reverberation. The attenuation of the transmitted sound by the scattering of energy out of the sound beam will also occur, but is of less importance.

The exact theoretical study of the acoustic effect of the individual scatterers is prohibitive in its complexity since no simple geometric form can be attributed to marine life, and further, the material of which the scatterers are composed is in general not homogeneous. However, the kindred problem of the acoustic scattering from a homogeneous fluid sphere does lend itself to a mathematical analysis and may give an insight into the real problems.

The effect of a fluid sphere in a sound field was first investigated by Rayleigh[1] who considered a sphere of dimensions small compared to a wave-length. For scattering from a sphere of these dimensions (termed Rayleigh scattering), both the zero-order and the first-order terms of the series solution are important; the zero order being dependent on the compressibility of the sphere, and the first order dependent on the density of the sphere. Both the diameter and the wave-length enter into the scattering in a simple power relationship.

A large portion of the marine life found in the ocean is covered by the theory of Rayleigh scattering. However, it is probable that the more important contribution to the total acoustic scattering arises from those scatterers which are comparable to a wave-length. The scattering from spheres of diameter comparable to a wave-length has also been treated by Rayleigh for the special case of a fixed rigid sphere.

The present work is concerned with scattering from spheres whose acoustic properties are near those of the medium, and of diameters up to several wave-lengths.† To simplify the theory, a fluid spherical scatterer has been considered and the effects of viscosity and heat conduction neglected.

II. GENERAL SOLUTION

The problem to be solved may be set forth in the following manner: Consider a sphere of radius a composed of fluid I which has a density ρ' and a sound velocity c' whose center is located at the origin of the polar coordinate system (r, θ, ϕ). Surrounding this sphere is an infinite fluid II whose acoustical properties differ from those of the sphere in general and will be designated by ρ and c. A plane acoustic wave of angular frequency ω, and pressure amplitude $\mathcal{P}$, traveling parallel to the polar axis in the $-z$ direction, impinges upon the sphere. (Choosing the incident wave in such a manner eliminates the dependence on ϕ so that only the variables r and θ need be considered.) This wave gives rise to an internal wave p' and an external spherical wave, p. The amplitude of p at large distances from the sphere is to be determined.

At the boundary of the sphere, $r=a$, both the pressure and the normal component of the particle velocity u, must be continuous. This condition gives rise to the following equations: For the acoustic pressure, p,

$$p(a)+p_0(a)=p'(a), \tag{1}$$

and for the radial component of the particle velocity, u_r,

$$u_r(a)+u_{0,r}(a)=u_r'(a). \tag{2}$$

In addition the acoustic pressure, p, must satisfy the three-dimensional wave equation

$$\nabla^2 p=(1/c^2)(\partial^2 p/\partial t^2).$$

* This work represents one of the results of research carried out under contract with the Bureau of Ships, Navy Department.

[1] Lord Rayleigh, *Theory of Sound* (Dover Publications, New York), Vol. II, p. 282.

† A solution to the general problem of absorption and scattering by a sphere in terms of its boundary impedance is given by Lax and Feshbach (J. Acous. Soc. Am. 20, 108 (1948)) using the phase shift analysis which is well suited to problems involving spheres with acoustic coatings of known impedance. The problem of a fluid sphere is included in the theory, but, in view of the complicated dependence of the boundary impedance on frequency for this case, the elegance of the phase shift method for computation is nullified.

As in most physical problems, the solutions of this differential equation which are of interest are those in which the time dependence is sinusoidal of the form $\exp(-i\omega t)$. In the polar coordinate system chosen the general solution for axial symmetry is given in terms of spherical harmonics:[2]

$$p=\sum_{m=0}^{\infty} A_m P_m(\mu)\begin{bmatrix} j_m(kr) \\ n_m(kr) \end{bmatrix} \exp(-i\omega t),$$

where P_m is the Legendre function, $\mu=\cos\theta$, j_m is the spherical Bessel function and n_m is the spherical Neumann function.

Choosing the most general solution finite in the region $r<a$, the internal wave may be expressed by the series:

$$p'=\sum_{m=0}^{\infty} B_m P_m(\mu) j_m(k'r) \exp(-i\omega t). \quad (3)$$

Similarly for the region $r>a$ the scattered wave will be:

$$p=\sum_{m=0}^{\infty} A_m P_m(\mu)[j_m(kr)+in_m(kr)] \exp(-i\omega t). \quad (4)$$

The spherical Neumann function is included here since the origin where n_m becomes infinite lies outside the region. The incident acoustic wave may also be expanded in a series of spherical harmonics,[3]

$$p_0=\mathcal{P}_0 \sum_{m=0}^{\infty} (-i)^m(2m+1)P_m(\mu) j_m(kr) \exp(-i\omega t). \quad (5)$$

Using the relation for the radial component of the particle velocity,[4]

$$u_r=(-i/\rho c)[\partial(p)/\partial(kr)]$$

the expressions for the particle velocities of the three waves are obtained:

$$u_r=(-i/\rho c)\sum_{m=0}^{\infty}[A_m/(2m+1)] \times P_m(\mu)[\alpha_m(kr)+i\beta_m(kr)] \exp(-i\omega t). \quad (6)$$

$$u_{0,r}=(-i/\rho c)\mathcal{P}_0\sum_{m=0}^{\infty}(-i)^m P_m(\mu)\alpha_m(kr) \exp(-i\omega t), \quad (7)$$

and

$$u_r'=(-i/\rho'c')\sum_{m=0}^{\infty}[B_m/(2m+1)] \times P_m(\mu)\alpha_m(k'r) \exp(-i\omega t), \quad (8)$$

where

$$\alpha_m(kr)\equiv(2m+1)\partial[j_m(kr)]/\partial(kr) = mj_{m-1}(kr)-(m+1)j_{m+1}(kr)$$

and

$$\beta_m(kr)\equiv(2m+1)\partial[n_m(kr)]/\partial(kr) = mn_{n-1}(kr)-(m+1)n_{m+1}(kr).$$

Substituting Eqs. (3)–(5) in (1), and (6)–(8) in (2), two equations in the unknown, $_sA_m$ and B_m are obtained. Solving these equations simultaneously for A_m,

$$A_m=-\mathcal{P}_0(-i)^m(2m+1)/(1+iC_m),$$

where

$$\ddagger C_m\equiv\frac{[\alpha_m(k'a)/\alpha_m(ka)][n_m(ka)/j_m(k'a)] -[\beta_m(ka)/\alpha_m(ka)]gh}{[\alpha_m(k'a)/\alpha_m(ka)][j_m(ka)/j_m(k'a)]-gh} \quad (9)$$

and

$$g=\rho'/\rho, \quad h=c'/c.$$

Using this expression for A_m, the pressure of the scattered wave at any point outside the sphere is given by

$$p=-\mathcal{P}_0\sum_{m=0}^{\infty}[(-i)^m(2m+1)/(1+iC_m)] \times P_m(\mu)[j_m(kr)+in_m(kr)] \exp(-i\omega t),$$

which is the general solution for the scattered wave.

At large distances from the sphere a simpler expression for the pressure of the scattered wave may be obtained by replacing $j_m(kr)$ and $n_m(kr)$ by the limiting expressions:[5]

$$j_m(kr)\xrightarrow[kr\to\infty]{}(1/kr)\cos[kr-(m+1)\pi/2],$$

$$n_m(kr)\xrightarrow[kr\to\infty]{}(1/kr)\sin[kr-(m+1)\pi/2].$$

The following approximation is then valid for $kr\gg1$,

$$(-i)^m[j_m(kr)+in_m(kr)]\xrightarrow[kr\to\infty]{}(-i/kr)\exp[+ikr](-1)^m,$$

and the pressure at a point far removed from the sphere is given by

$$p=\mathcal{P}_0(+i/kr)\exp(+ikr-i\omega t) \times\sum_{m=0}^{\infty}P_m(\mu)(-1)^m(2m+1)/(1+iC_m). \quad (10)$$

This expression for p may be considered the applicable solution to the problem. In the following sections, different forms of this solution will be set up to facilitate computation.

III. REFLECTIVITY AND SCATTERING CROSS SECTION

A convenient yardstick for investigating the results of the previous section may be found in the idealized case of uniform scattering from a perfectly reflecting sphere (which is commonly called geometrical scattering). In this case, the total energy intercepted by the sphere of radius a from the incident plane wave of pressure

[2] P. M. Morse, *Vibration and Sound* (McGraw-Hill Book Company, Inc., 1948), second edition, p. 319.
[3] Lord Rayleigh, see reference 1, p. 334.
[4] P. M. Morse, see reference 2, p. 295.

‡ It has been pointed out that the Lax-Feshbach phase shifts (see † p. 2), η_m are related to the C_m by the equation:

$$\eta_m=\pi/2+\arctan(-C_m).$$

[5] P. M. Morse, see reference 2, p. 317.

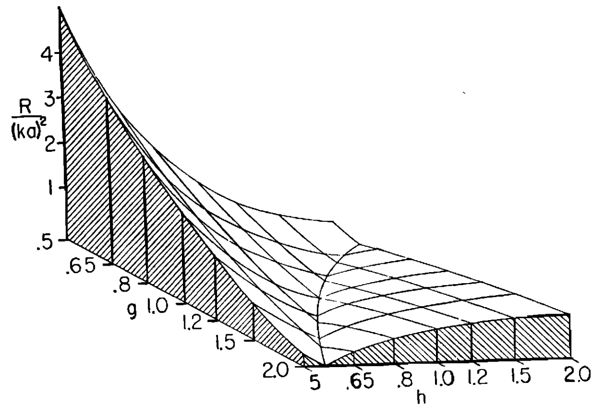


FIG. 1. Rayleigh scattering from small fluid spheres in the backward direction i.e., $\theta=0$. The quantity $R/(ka)^2$ is presented as a function of the relative density g, and the relative sound velocity h of the scattering sphere.

amplitude $\mathcal{P}_0$ is considered to be scattered uniformly through a solid angle of 4π steradians. The pressure amplitude of the scattered wave at a distance r from the sphere may be written as

$$|p_{\rm geom}|=\mathcal{P}_0(\pi a^2/4\pi r^2)^{\frac{1}{2}}=\mathcal{P}_0 a/2r.$$

Using this geometrical scattering for comparison, a reflectivity factor is defined:

$$R_\theta=|p|/|p_{\rm geom}|$$
$$=2/ka\left|\sum_{m=0}^{\infty}P_m(\mu)(-1)^m(2m+1)/(1+iC_m)\right|.$$

In the special case of scattering directed back toward the source of the incident wave, (i.e. $\theta=0$), which is of special interest here, the expression for R_θ becomes

$$R=2/ka\left|\sum_{m=0}^{\infty}(-1)^m(2m+1)/(1+iC_m)\right|. \quad (11)$$

It may be pointed out here that since the acoustic cross section of a scatterer is determined by the relative strength of the echo returned from it, it is evident that the reflectivity, R_θ, is related to the acoustic cross section. For example, the acoustic cross section for back scattering of a sphere whose geometric cross section is S units will be the product of the geometric cross section and the square of the reflectivity, R^2S.

The total power scattered by the sphere is also of interest. Again the result is compared to geometrical scattering and, since the total power in a geometrically scattered wave is just that arriving at the sphere through a cylinder of radius a, the ratio of total scattered power to geometrically scattered power may be written,

$$\Pi=\int_s|p|^2ds/\mathcal{P}_0{}^2\pi a^2,$$

where s is a spherical surface of radius r_0 surrounding the scatterer and $r_0\gg a$. From Eq. (10)

$$\Pi=\int_0^\pi\left|\sum_{m=0}^{\infty}P_m(\mu)(-1)^m(2m+1)/(1+iC_m)\right|^2$$
$$\times 2\pi r_0{}^2\sin\theta d\theta/\pi a^2k^2r_0{}^2,$$

making use of the orthogonality of the Legendre function $P_m(\mu)$, over the surface s,

$$\Pi=(4/k^2a^2)\sum_{m=0}^{\infty}(2m+1)/(1+C_m{}^2).$$

This quantity Π is a measure of the amount of power the scatterer diverts from the original wave and as such is related to the total scattering cross section of the sphere (a measure of the energy removed by the scatterer); the product ΠS will be equal to the total scattering cross section of a sphere whose geometric cross section is S units.

In the process of computing R, the terms $(-1)^m(2m+1)/(1+C_m{}^2)$ occurred, making it a simple operation to obtain Π. Since the curve for Π is much smoother than that for R, it serves as a useful check on possible computational errors of individual points.

IV. LIMITING CASES

A. Rayleigh Scattering

As the radius of the sphere becomes much less than a wave-length, and ka approaches zero, only the first terms in the series for the spherical Bessel and Neumann function need be considered. The following approximations are then valid:[5]

$$j_m(ka)\xrightarrow[ka\to 0]{}(ka)^m/1\cdot 3\cdots(2m+1),$$
$$n_m(ka)\xrightarrow[ka\to 0]{}-1\cdot 3\cdots(2m-1)/(ka)^{(m+1)}.$$

Using these limiting values in Eq. (11) and neglecting higher order terms in ka, the reflectivity coefficient becomes,

$$R_\theta=2(ka)^2|(1-gh^2)/(3gh^2)+\cos\theta(1-g)/(1+2g)|,$$

which is essentially the expression obtained by Rayleigh for this case.

B. Scattering from a Fixed Rigid Sphere

It is of interest to obtain as a limiting case the scattering for a fixed rigid sphere which has been studied in great detail by Stenzel.[6] Allowing g and h to approach infinity, the term C_m from Eq. (9) has a limiting value,

$$C_m\xrightarrow[g=h\to\infty]{}\beta_m(ka)/\alpha_m(ka).$$

In this case, the reflectivity from Eq. (11) becomes

$$R_\theta=2/ka\left|\sum_{m=0}^{\infty}P_m(\mu)(-1)^m(2m+1)/\right.$$
$$\left.[1+i\beta_m(ka)/\alpha_m(ka)]\right|,$$

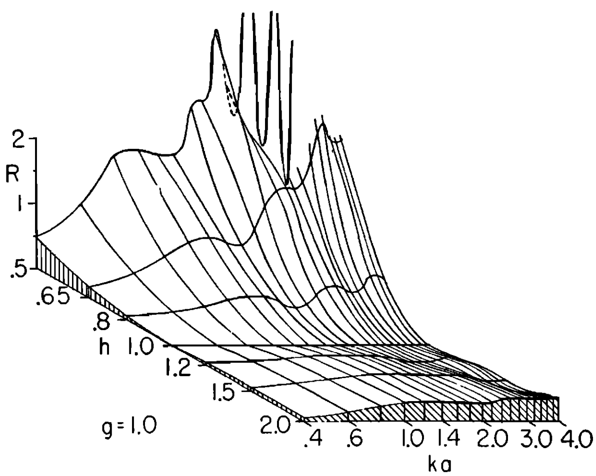
[6] H. Stenzel, E. N. T. **15**, 71 (1938).

which has the same form as that given by Stenzel.** For the total reflectivity, the expression is,

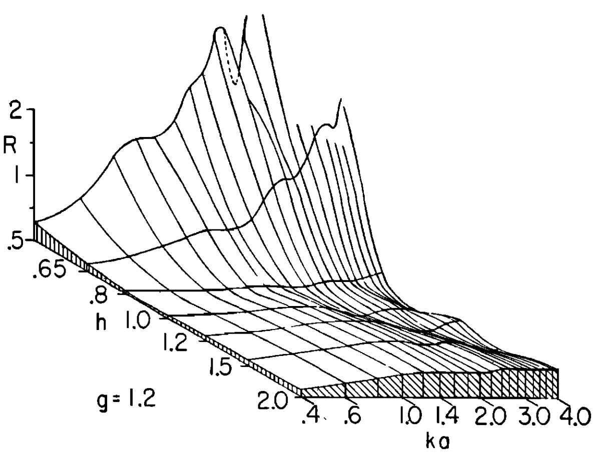
$$\Pi=4/(k^2a^2)\sum_{m=0}^{\infty}(2m+1)/[1+\beta_m^2(ka)/\alpha_m^2(ka)].$$

V. NUMERICAL EVALUATION

The numerical evaluation of the preceding equations giving the scattering from a fluid sphere whose dimensions are comparable to a wave-length would not have been feasible except for the recent publication of a table of spherical Bessel Functions.[7] In spite of the aid of these tables considerable computation was still required. Exclusive of checking, the numerical examination presented in the accompanying graphs, consisting of more than 600 points, required the full time services of two computers for a period of about two months. The possibility of using one of the large automatic computing machines was investigated but it was soon discovered that the problem did not lend itself feasibly to automatic computation.



(a)



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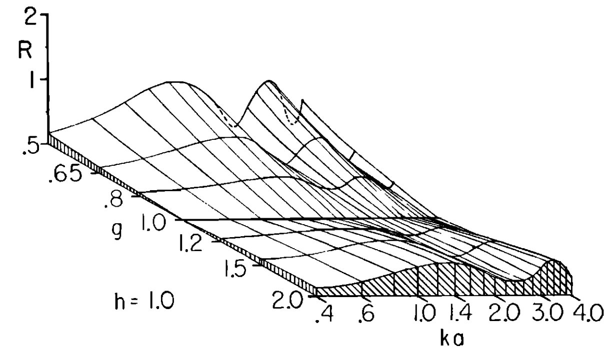

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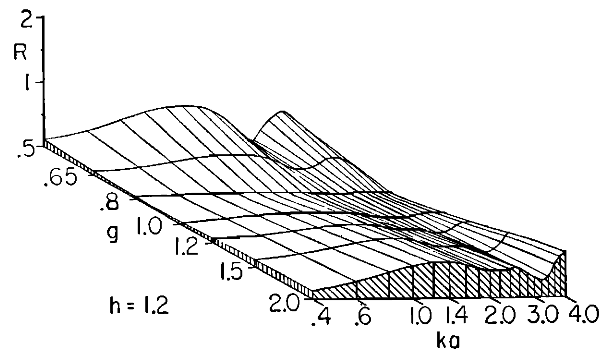

(d)

FIG. 2. Reflectivity, R, for direct backward scattering from fluid spheres of dimensions comparable to a wave-length. Figure 2a and 2b show the reflectivity as a function of acoustic radius ka and relative sound velocity, h, for two values of relative density g. Figures 2c and 2d give the reflectivity in terms of ka and g for two values of h.

A. Investigation of the Reflectivity

An insight into the general nature of the scattering from a fluid sphere may be gained by investigating first the scattering in the Rayleigh region. Figure 1 is a graphic presentation of Rayleigh scattering from fluid spheres whose acoustic properties do not differ greatly from those of the surrounding medium. It is apparent that to this approximation the reflectivity becomes large without limit as the relative sound velocity (h) and the relative density (g) both become small; of the two factors, the dependence on the relative sound velocity h is more pronounced. For very small values of g and h, R increases as $1/gh^2$. This quantity, $1/gh^2$ is simply the ratio of the compressibility inside the sphere to that of the surrounding medium. On the other hand, for g and h large (an incompressible fixed particle), the reflectivity approaches the finite limit, $R\rightarrow 5(ka)^2/6$. In Fig. 1, a curve of zero reflection for a small fluid sphere may be observed. It is interesting to note that this condition for zero reflection is not $gh=1$ as in the reflection from a plane boundary. The point $g=h=1$ is an exception, but this is a trivial case.

As the value of ka approaches 1.0, the simple Rayleigh solution no longer holds and more terms of the series solution must be considered. In this region, which may be termed the critical region, some of the general

** The definition of R differs by a factor of 2 from the reflexions faktor, $\mathfrak{Z}$, defined by Stenzel.

[7] *Mathematical Table Project*, Table of Spherical Bessel Functions, I–II (Columbia University Press, New York, 1947).

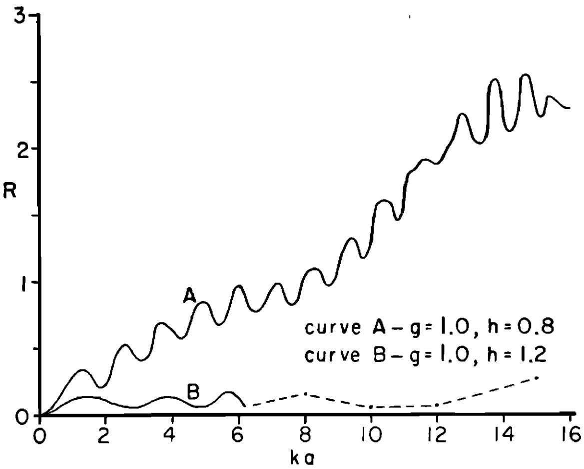

FIG. 3. Reflectivity R for direct backward scattering as a function of acoustic radius ka for two fluid spheres whose relative densities are 1.0 and whose relative sound velocities are 1.2 and 0.8.

characteristics of the scattering in the Rayleigh region are still apparent. Figures 2a and b show the dependence of R on the relative sound velocity of the sphere and the acoustic radius (ka) while Figs. 2c and d show the dependence of R on the relative density (g) and acoustic radius (ka). Comparing the figures, the most striking difference is the presence of large slopes for the small values of h compared with the gradual rate of change of R with the relative density g. The difference between the dependence of R on h and on g seems to become even more pronounced in this region than in the Rayleigh region, particularly for the larger values of (ka). Here

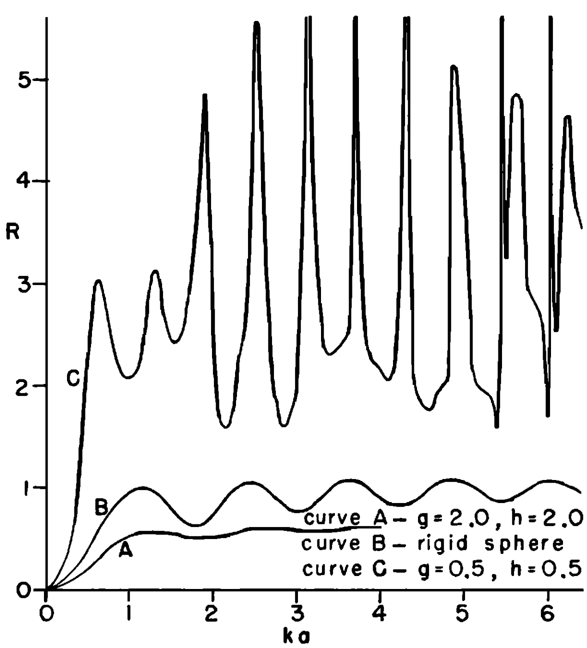

FIG. 4. Reflectivity R for direct backward scattering as a function of acoustic radius ka for various values of relative density g and relative velocity h.

again as for Rayleigh scattering, the solution approaches a finite limit as g and h become large, the reflectivity for $g=h=2.0$ being nearly that of the rigid sphere shown in Fig. 4.

The major difference between the Rayleigh region and the critical region lies in the dependence on acoustic radius (ka). In the critical region the reflectivity no longer varies as $(ka)^2$, but becomes a complicated function of (ka) having maxima and minima, indicating resonance phenomena. These resonances are well defined and particularly numerous in the case of $h=g=0.5$ which is shown in Fig. 4. The resonances of the rigid sphere and the case $h=g=2.0$ on the other hand are moderate fluctuations which gradually decrease in amplitude as (ka) increases.

Computations were carried out in detail to large values of ka for an arbitrary case for which the relative

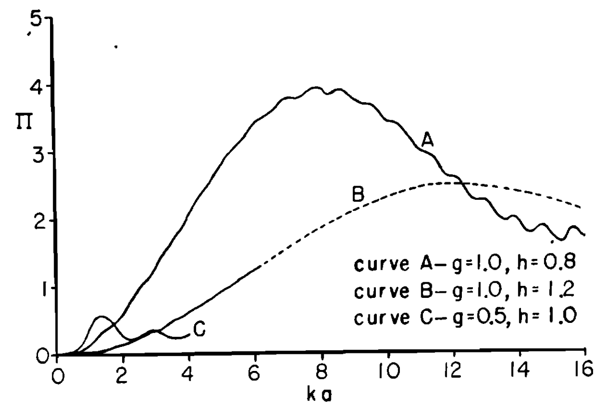

FIG. 5. Total scattering Π as a function of acoustic radius ka for the two spheres considered in Fig. 3 and also for a sphere for which $g=0.5$, $h=1.0$.

sound velocity $h=0.8$ and the relative density $g=1.0$. Figure 3 gives the results of these computations and also shows a few points which were computed for the case of $h=1.2$, $g=1.0$. It is interesting to note that the reflectivity for $h=0.8$ continues to rise through the range of ka considered while that of $h=1.2$ remains at a comparatively low value. For the large values of ka the reflectivity for $h=0.8$ becomes more than twice as great as that for a rigid sphere which approaches 1.0 as in Fig. 4. These large values of R bear out the statement made earlier that the dependence of R on relative velocity h for $h<1$ is more pronounced in this region than in the Rayleigh region.

The previous discussion has been concerned with scattering directed back toward the source of the incident sound. It is also of interest to briefly investigate the value of the reflectivity R_θ, at other angles. Figure 7 shows a polar plot of R_θ for two values of (ka) for each of the cases considered in Fig. 3. As would be expected these semi-transparent spheres scatter chiefly in the forward direction—for the larger spheres this might be attributed to the effect of refraction. In general these

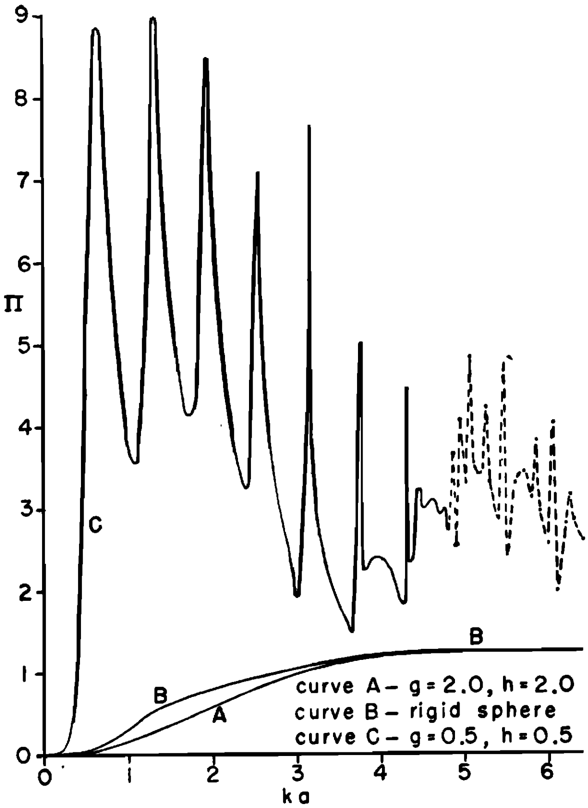

FIG. 6. Total scattering Π as a function of acoustic radius ka for the three cases shown in Fig. 4. The dashed line of curve C denotes a region where the number of points calculated was not sufficient to draw a reliable curve.

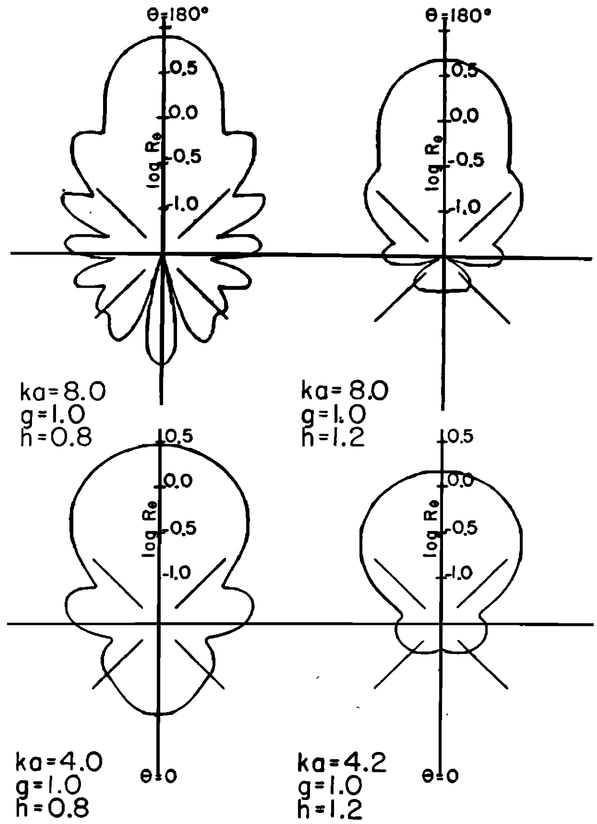

FIG. 7. Dependence of reflectivity R_θ as a function of azimuthal angle θ for two values of acoustic radius for the two spheres considered in Fig. 3.

patterns possess lobes, the number of lobes increasing with increasing ka and with decreasing h. The presence of a strong lobe for $h=0.8$ at $\theta=0$ accounts for the large values of R_θ for this case which occurred in Fig. 3. It should be pointed out that the quantity R_θ does not give directly the diffraction pattern due to a spherical obstacle; it represents only the pressure amplitude in the spherically scattered wave and must be combined with the incident plane wave to give the true diffraction pattern. The difference will be important only in the forward direction.

B. Total Scattering

The total scattering, Π, was defined neglecting the effect of the incident wave. Actually, energy in the scattered wave is cancelled by the incident wave, forming the shadow zone. Thus, Π if computed for geometrical scattering would yield a value of 2.0 rather than unity. Figure 5 shows the total scattering, Π, for the cases considered in Fig. 3. In contrast to the large difference in reflectivity, the value of Π for both of these cases approaches that of geometrical scattering, 2.0. These semi-transparent spheres, although transmitting a good portion of the incident sound, divert this sound from its original direction by the effect of refraction. This effect of refraction is emphasized by comparing these cases with curve C of Fig. 5 in which the relative velocity is unity so that no refraction occurs; here, the scattering for large ka can be attributed entirely to reflection due to the relative density, $g=0.5$. In this case, Π approaches a much lower value as ka increases; the majority of the sound being transmitted through the sphere with no change in direction.

ACKNOWLEDGMENTS

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