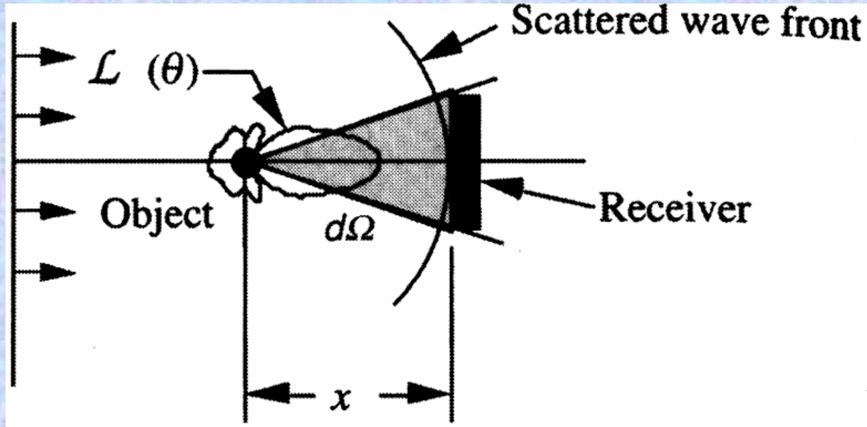
Scattering of sound - Part II- scattering from a sphere

The problem:

- An incident plane-parallel wave
- Possible absorption
- A scattered wave



Medwin et al., 2005

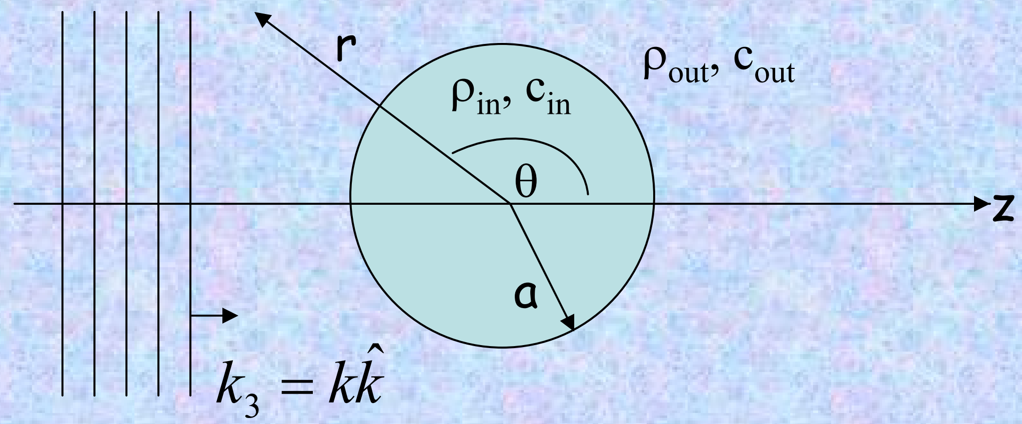
Scattering: combined action of reflection, refraction and diffraction from objects with differing sound speed.

References:
Anderson, V. C., 1950: Sound scattering from a fluid sphere, JASA, 22, 426-431.
Faran, J. J., 1951: Sound scattering by solid cylinders and spheres, JASA, 23, 405-418.
Medwin et al., 2005, Ch. 6.
Williams, E. G., 1999: Fourier Acoustics, Academic Press, 306pp.

Formulation of the problem:

Plane-wave insonification (neglect attenuation within the medium):

$$p_i(\vec{r},t) = A_0 \exp\{i(\vec{k}\cdot\vec{x} - \omega t)\} = A_0 \exp\{i(kr\cos\theta - \omega t)\}$$



The problem is azimuthally symmetric, which facilitate its formulation in spherical coordinates.

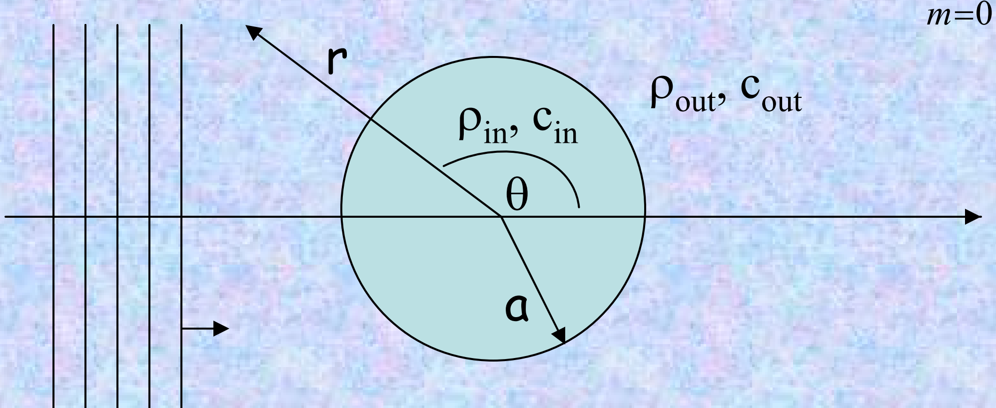
The scattered wave also obeys:

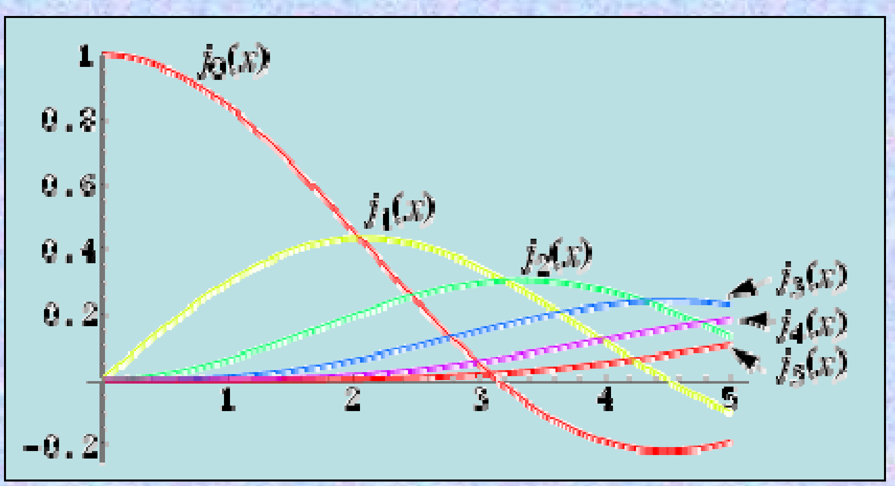
$$\nabla^2 p_s = \frac{1}{c^2}\frac{\partial^2 p_s}{\partial t^2}$$

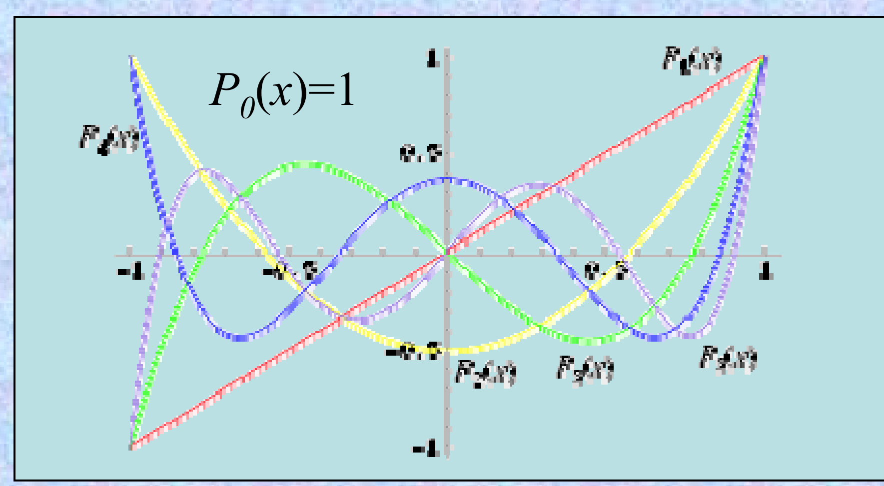
Formulation of the problem (scattering by a fluid sphere):

We need to express the incident field in spherical coordinates which for azimuthal symmetry is given by:

$$p_i(\vec{x},\theta,\omega) = A_0 \exp(ikr\cos\theta - \omega t) = A_0 \sum_{m=0}^{\infty} (-i)^m (2m+1)\, j_m(kr)\, P_m(\cos\theta)\, e^{-i\omega t}$$







http://mathworld.wolfram.com/

BCs: radial velocity and pressure are continuous across the boundary.

$$u_i\left(a^+,\theta\right)+u_s\left(a^+,\theta\right)=u_{in}\left(a^-,\theta\right)$$

$$p_i\left(a^+,\theta\right)+p_s\left(a^+,\theta\right)=p_{in}\left(a^-,\theta\right)$$

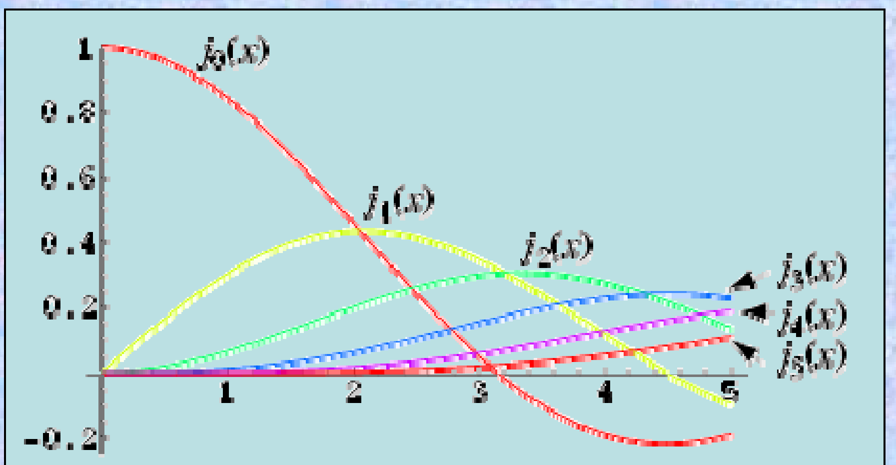
$$\omega_{in}=c_{in}\,k_{in}/2\pi=c_{out}\,k/2\pi=\omega$$

The scattered sound out of the sphere and the pressure within the sphere are given as the general axisymetric solution for the wave equation that vanishes at $r \rightarrow \infty$):

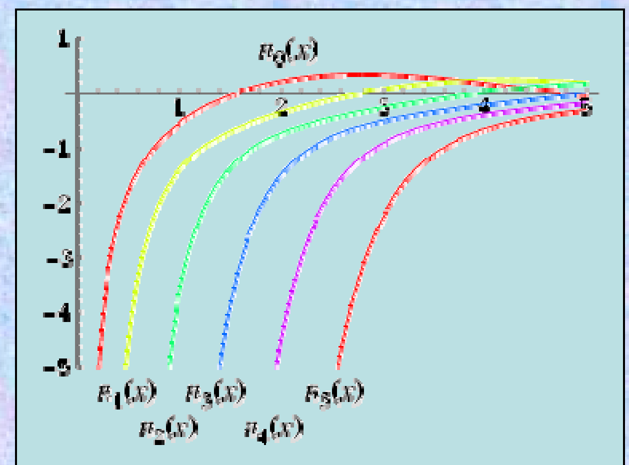
$$p_s\left(r,\theta\right)=\sum_{m=0}^{\infty}\boxed{C_m}h_m^{(1)}\left(kr\right)P_m\left(\cos\theta\right),\quad h_m^{(1)}\left(x\right)=j_m\left(x\right)+in_m\left(x\right)$$

$$p_{in}\left(r,\theta\right)=\sum_{m=0}^{\infty}\boxed{B_m}j_m\left(k_{in}r\right)P_m\left(\cos\theta\right)$$

$\boxed{}$ ⟵ Unknowns at this stage



$+i$



Euler equation:

$$\rho \frac{\partial u_r}{\partial t} = -\frac{\partial p}{\partial r} \Rightarrow u_r = -\frac{i}{\rho c}\frac{\partial p}{\partial (kr)}$$

➔ 2 more equations (in and out) for a total of 4 to solve for complex B_m and C_m.

$$C_m = \frac{-A_0(-i)^m(2m+1)}{(1+iD_m)}, \text{where:}$$

$$D_m \equiv \frac{\{mj_{m-1}(k'a)-(m+1)j_{m+1}(k'a)\}n_m(ka)-\{j_m(k'a)n_{m-1}(ka)-(m+1)n_{m+1}(ka)\}gh}{\{mj_{m-1}(k'a)-(m+1)j_{m+1}(k'a)\}j_m(ka)-\{mj_{m-1}(ka)-(m+1)j_{m+1}(ka)\}gh}$$

and $g=\rho_{in}/\rho_{out}$, $h=c_{in}/c_{out}$.

Thus the scattered wave at any point outside the sphere is given by:

$$p_s(r,\theta) = -A_0 \sum_{m=0}^{\infty} (-i)^m \frac{(2m+1)}{(1+iD_m)} h_m^{(1)}(kr) P_m(\cos\theta) e^{-i\omega t}$$

Far field: $r \to \infty$:

$$j_m(kr) \underset{kr\to\infty}{\to} \frac{1}{kr}\cos\left(kr-(m+1)\frac{\pi}{2}\right)$$

$$n_m(kr) \underset{kr\to\infty}{\to} \frac{1}{kr}\sin\left(kr-(m+1)\frac{\pi}{2}\right)$$

$$\Rightarrow (-i)^m h_m^{(1)} \underset{kr\to\infty}{\to} (-1)^m \frac{1}{kr} e^{ikr}$$

→ Far field solution for scattered wave:

$$p_s(r,\theta) = A_0 \frac{i\mathrm{e}^{-\mathrm{i}(\mathrm{kr}-\omega \mathrm{t})}}{kr} \sum_{m=0}^{\infty} (-1)^m \frac{(2m+1)}{(1+iD_m)} P_m(\cos\theta)$$

Normalizing factor- let assume the whole intensity impinging on the sphere is radiated out uniformly:

$$A_0^2 \pi a^2 = \left|p_{s,norm}\right|^2 4\pi r^2 \to \left|p_{s,norm}\right| = \frac{A_0 a}{2r}$$

Define 'reflectivity':

$$R(\theta) \equiv \frac{|p_s(r,\theta)|}{|p_{s,norm}(r,\theta)|} = \frac{2}{ka}\left|\sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)}{(1+iD_m)} P_m(\cos\theta)\right|$$

and for backscattering (sometimes called 'form function', f_∞):

$$R = R(\theta=0) = \frac{2}{ka}\left|\sum_{m=0}^{\infty}(-1)^m \frac{(2m+1)}{(1+iD_m)}\right|$$

with respect to 'differential scattering cross-section' and 'scattering length':

$$\Delta\sigma_s(\theta,k) \equiv |L(\theta,k)|^2 \equiv \frac{|p_s|^2 r^2}{|p_i|^2} = \frac{R(\theta)^2 a^2}{4}$$

Normalized power (the amount of power the scatterer diverts):

$$\Pi_{norm} = \frac{\int_s |p_s(r,\theta)|^2 ds}{A_0^2 \pi a^2} = \ldots = \frac{4}{(ka)^2}\left|\sum_{m=0}^{\infty}\frac{(2m+1)}{(1+D_m)}\right| = \frac{\sigma_s}{\pi a^2}$$

Scattering from a sphere:

Limiting cases: I. Rayleigh scattering (*ka<<1*)

$$j_m(x) \underset{x\to 0}{\rightarrow} x^m / 1\cdot 3\cdots(2m+1)$$

$$n_m(x) \underset{x\to 0}{\rightarrow} -1\cdot 3\cdots(2m-1)/x^{m+1}$$

$$R(\theta) \rightarrow 2(ka)^2 \left| \frac{1-gh^2}{3gh^2} + \cos\theta \, \frac{1-g}{1+2g} \right|$$

$gh^2 = E_{in}/E_{out}$

$g = \rho_{in}/\rho_{out}$, $h = c_{in}/c_{out}$.

$$\Rightarrow \Delta\sigma_s, \sigma_s, \Pi_s \propto (ka)^4$$

Very sensitive to frequency!

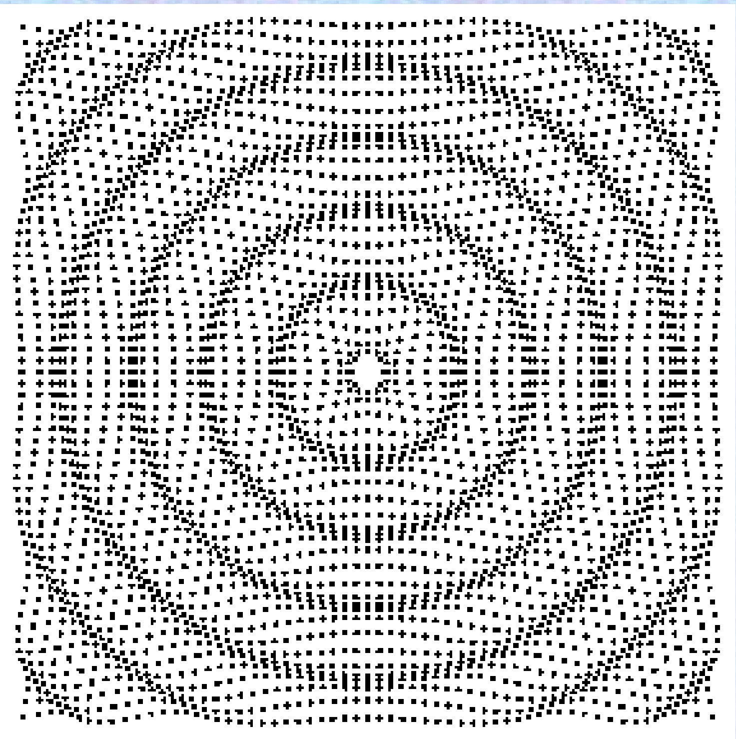
II. Geometric acoustics (*ka>>1*) :

???

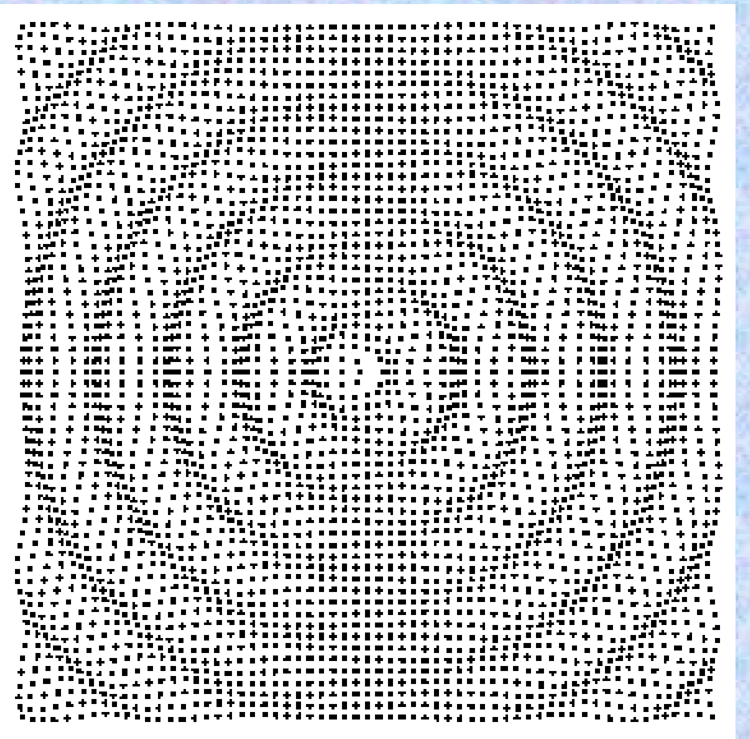
$$\Pi_{norm} = \frac{\int_s |p_s(r,\theta)|^2 \, ds}{A_0^2 \pi a^2} = \ldots = 2 = \frac{\sigma_s}{\pi a^2}$$

Independent of *ka* !!!

Monopole:



Dipole:



Animations courtesy of Dr. Dan Russell, Kettering University

Scattering from a sphere:

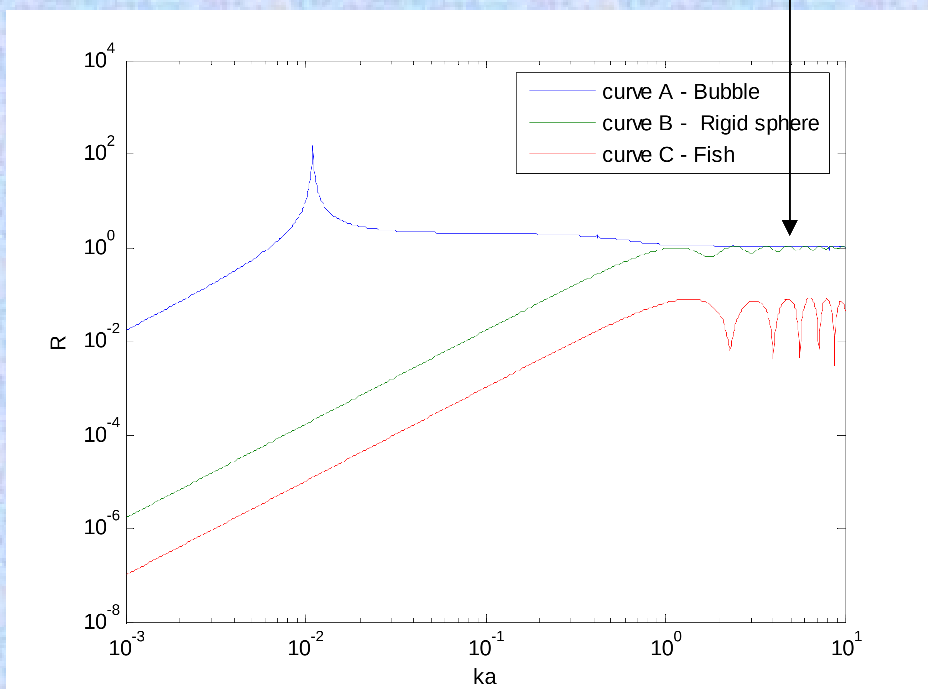
Three types of particles:

I. Rigid sphere, $c_{in} >> c_{out}$, $\rho_{in} >> \rho_{out}$. ← sediments, calibration targets.

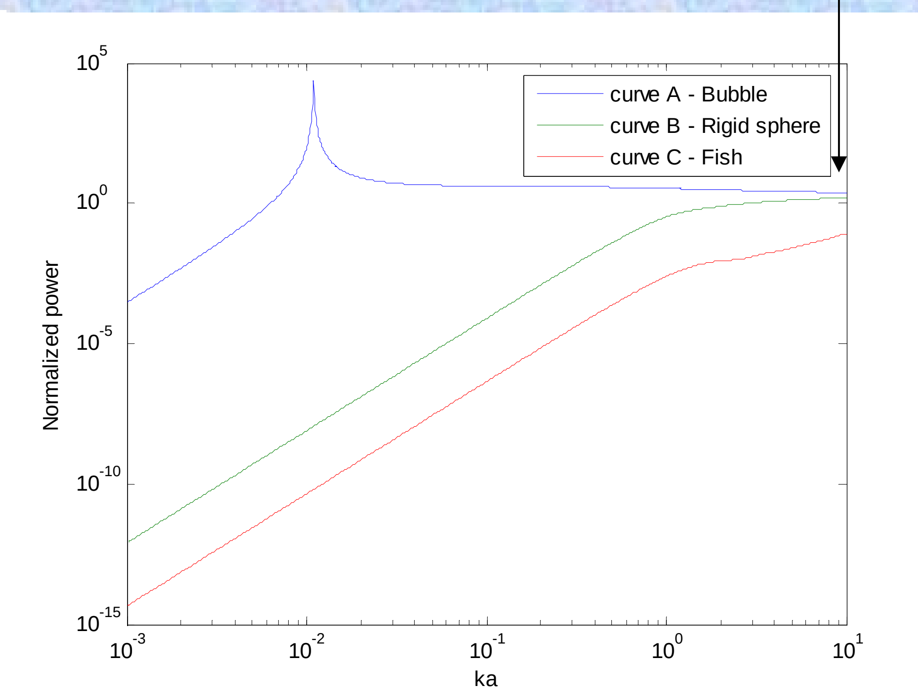
II. Generic 'Marine particle (fish, zooplankton)': $1.08 > g > 1.02$, $1.033 > h > 1.00$

III. Bubble ($\rho_{in}/\rho_{out} \sim 1/1000$, $c_{in}/c_{out} \sim 1/5$)

Form-function (reflectivity) R=1



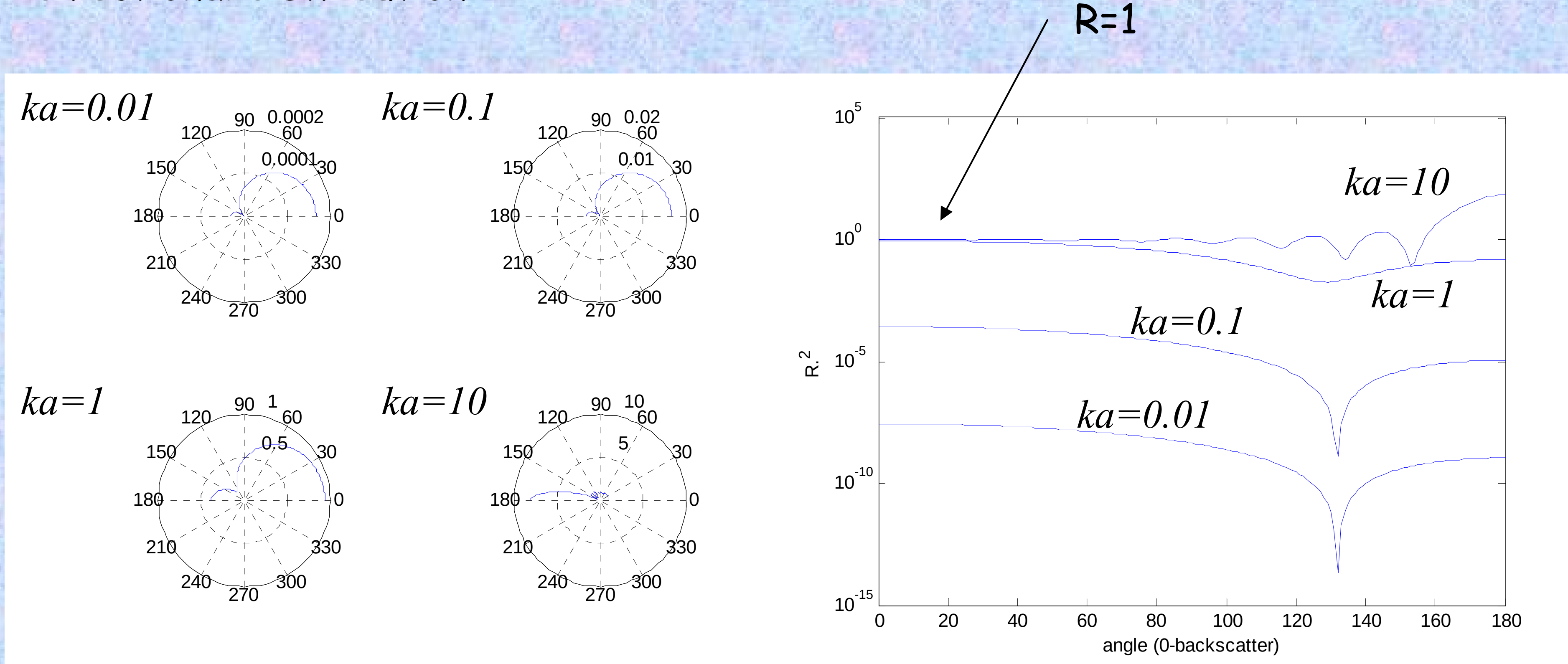
Normalized power (Π_{norm}) Π_{norm} =2



Scattering from a sphere:

I. Rigid sphere, $c_{in} >> c_{out}$, $\rho_{in} >> \rho_{out}$.

directional distribution:



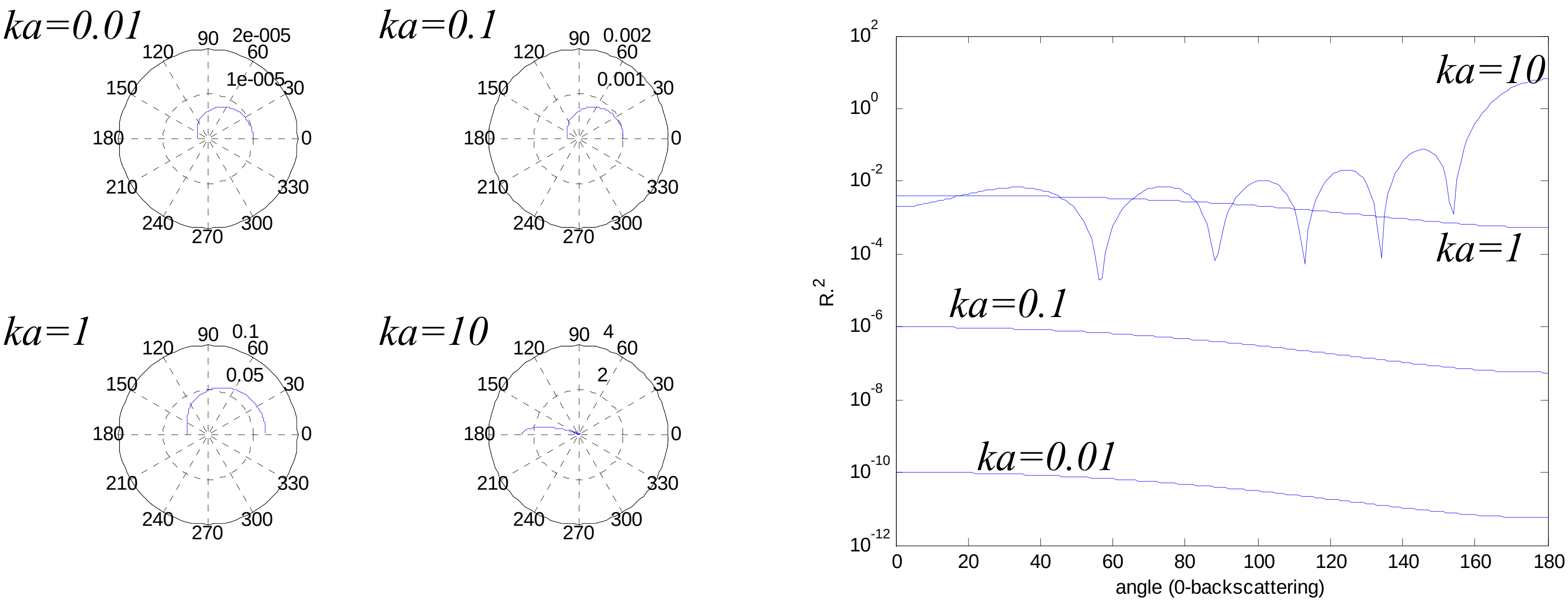
Note: our treatment has neglected shear waves which may exist (Faran, 1951).

Scattering from a sphere:

II. Generic 'Marine particle (fish, zooplankton)': *1.08>g>1.02*, *1.033>h>1.007*

directional distribution:

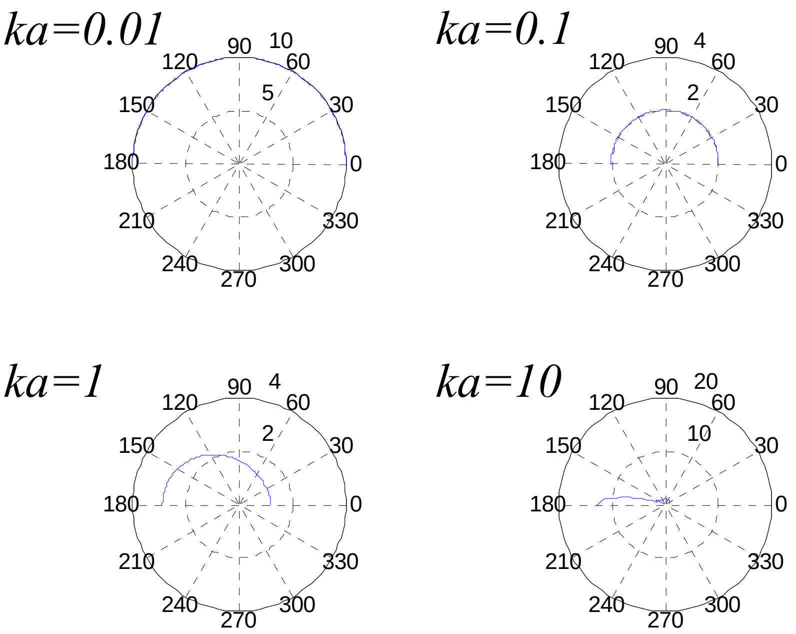
Fish

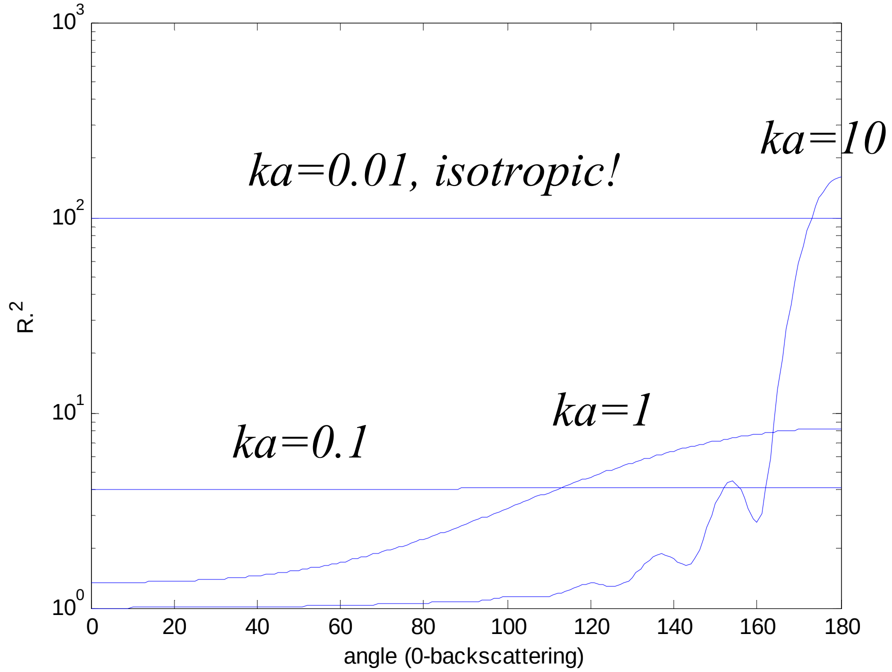


Scattering from a sphere:

III. Bubble (ρ_{in}/ρ_{out}~$1/1000$, c_{in}/c_{out}~$1/5$)

directional distribution:





How do we get an absorption, scattering and attenuation coefficient from all this?

$$\sigma_a \equiv \frac{\Pi_a}{I_i}, \sigma_s \equiv \frac{\Pi_s}{I_i}, \sigma_e = \sigma_a + \sigma_s$$

$$a(k) = \sum_i \sigma_a(k, a_i) N_i$$

$$b(k) = \sum_i \sigma_s(k, a_i) N_i$$

$$\gamma(k) = \sum_i \sigma_e(k, a_i) N_i$$

Beer-Lambert-Bouguer's law: in a dilute system with a variety of different substances, the absorption, scattering and attenuation of the different substances sum together to provide the absorption, scattering and attenuation of the medium:

$$I(r) = I(r=0)\exp\left(-r\sum_j \gamma_j\right)$$