SMS-618, Particle Dynamics, Fall 2003 (E. Boss, last updated: 10/22/2003)
Single Particle Dynamics

Solute transport to/from a sinking cell

The conservation equation for a solute is given by (as in lecture 4):

$$\frac{\partial C}{\partial t} + \vec{u}\nabla C = \nabla(D\nabla C), \nabla \equiv \partial/\partial x + \partial/\partial y + \partial/\partial z \qquad (1)$$

where $\vec{u}$ is the velocity field ([L T^{-1}])., and D the diffusion coefficient ([L^2 T^{-1}]). We wish to solve this equation for a particle. Let's assume a spherical particles (radius, r[L]) sinking steadily in the fluid ($\partial C/\partial t = 0, \vec{u}_{particle} = w_s$) and a constant diffusion coefficient D=const. around the cell:

$$\vec{u}\cdot\nabla C = D\nabla^2 C, \quad \nabla C = \hat{r}\frac{\partial C}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial C}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial C}{\partial \phi}$$
$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi^2} \qquad (2)$$

By invoking azimuthal symmetry of both velocity and concentration fields, $\partial/\partial\phi = 0$.
Equation (2) is non-dimensionalized as follows: $C^*=C/|C_\infty-C_0|$, $u^*=u/w_s$, $r^*=r/r_0$, 0-denoting the particles perimeter. The equation for the non-dimensional variables is:

$$Pe(\vec{u}^*\cdot\nabla C^*) = \nabla^2 C^*, \quad Pe \equiv \frac{w_s r_0}{D}. \qquad (3)$$

The non-dimensional Peclet number, $Pe=Re*D/\nu$, is a measure of the relative domination of advection over diffusion in terms of solute flux to/away of the sinking sphere.

The ratio of the flux to/from the particle in the presence of motion to the flux when there is no motion is another non-dimensional parameter, the Sherwood number, Sh:

$$Sh = \frac{-D\int_A \hat{n}\cdot\nabla C dA}{4\pi r_0 D(C_\infty - C_0)} \qquad (4)$$

(the diffusive solution was derived in lecture 4).
A relationship between Sh and Pe for $0.001<Pe<5000$ that is consistent with analytical solutions for small and large Pe is (Clift et al., 1978, Karp-Boss et al., 1996):

$$Sh = \frac{1}{2}\left(1 + (1+2Pe)^{\frac{1}{3}}\right) \qquad (5)$$

Solute transport to/from a cell in a turbulent fluid:

In the case of a turbulent fluid, the flow around small particles can be characterized based on the energy dissipation rate, ε, and the kinematic viscosity, ν. Assuming that particle to be smaller than the Kolmogorov length scale, $\eta=(\nu^3/\varepsilon)^{1/4}$, the velocity scale around the cell is given by $U= r_0(\varepsilon/\nu)^{1/2}$. The dependence of Sh on $Pe\equiv r_0^2(\varepsilon/\nu)^{1/2}/D$ is given by:

$$Sh = \begin{cases} 1+0.29Pe^{1/2} & Pe << 1 \\ 1.014+0.15Pe^{1/2} < Sh < 0.955+0.344Pe^{1/3} & 0.01 < Pe < 100 \\ 0.55Pe^{1/3} & Pe >> 1 \end{cases} \qquad (6)$$

Appendix: Flow around a sinking cell:
The Navier-Stokes equations around a sinking particle have been solved exactly for Re<<1. Assuming no flow into the cell, no flow relative to the cell on its perimeter, and -U velocity away from the cell (the coordinate system moves with the cell):

$$\psi_{Stokes} = \frac{1}{4}\left(2+\left(\frac{r_0}{r}\right)^3 - 3\frac{r_0}{r}\right)Ur^2\sin^2\theta,$$

$$u_\phi = 0,\ u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{1}{2}\left(2+\left(\frac{r_0}{r}\right)^3 - 3\frac{r_0}{r}\right)U\cos\theta, \qquad (5)$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{4}\left(4-\left(\frac{r_0}{r}\right)^3 - 3\frac{r_0}{r}\right)U\sin\theta$$

This solution is correct for $Re=Ur_0/\nu=0$. It turns our that as one goes away from the sphere, the neglect of the inertia term is unjustified, and O(Re) terms have to be incorporated.
The components of the stress on the sphere are:

$$\begin{aligned} \tau_\phi &= 0,\ \tau_r = -p+2\mu\frac{\partial u_r}{\partial r} = -p_\infty + \frac{3}{2}\frac{\mu U}{r_0}\cos\theta, \\ \tau_\theta &= \mu r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{\mu}{r}\frac{\partial u_r}{\partial\theta} = -\frac{3}{2}\frac{\mu U}{r_0}\sin\theta \end{aligned} \qquad (6)$$

where p_∞ is the pressure in the absence of motion (or very far from the sphere). The stress vector component on the sphere in the direction of the motion is:

$$\tau = \tau_r\cos\theta + \tau_\theta\sin\theta = -p_\infty\cos\theta + \frac{3}{2}\frac{\mu U}{r_0}, \qquad (7)$$

from which the net drag on the sphere is found to be:

$$D = \int_0^{2\pi}\int_0^{\pi}\tau r_0^2\sin\theta d\theta d\phi = 6\pi\mu U r_0, \qquad (8)$$

which was used to derive the Stokes settling speed. The next order correction for finite Re number gives a modified drag coefficient: $D=6\pi\mu Ur_0(1+3Re/8)$.

References:

Acheson, D. J., 1990. Elementery fluid dynamics. Oxford Press.
Batchelor, G. K., 1967. An introduction to fluid dynamics. Cambridge University Press.
Karp-Boss, L., E. Boss, and P. A. Jumars, 1996. Nutrient fluxes to planktonic osmotrophs in the presence of fluid motion. *Oceanography and Marine Biology: An annual Review*, **34**, 71-107.