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Theory of Polarized Light Fields in Discrete Spaces

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INTRODUCTION

The theory of radiative transfer on discrete spaces introduced in an earlier study ¹, is here extended to include within its formalism the phenomenon of polarized light. The present work is divided into two parts: First, a discussion of the phenomenological (or operational) definition of polarized light; Second, the casting of the principle of local interaction ¹ into polarized form.

The main purpose of the present report is to cast into polarized form the principle of local interaction introduced in reference 1. In this way we round out the development of the discrete space version of radiative transfer theory to include the phenomenon of polarization. In addition we also add to the evidence that the discrete-space formulation is capable of supplying relatively tractable discrete counterparts to the important continuous-space concepts and problems. The polarized form of local interaction principle leads to a set of vector equations which are solvable by the methods discussed in reference 1. Thus, while the polarized form of the local interaction

principle leads to a more complete phenomenological description of radiative transfer processes on discrete spaces, it adds no new mathematical problems to the discrete-space transfer theory.

Before we can realize the main purpose of this report, it is necessary to develop a phenomenological definition of the concept of polarized radiant flux. The phenomenological definition of polarized light given below leads to the Stokes vector representation of polarized light. (For an exposition of the idea of a Stokes vector, see reference 2.) The approach used here is inspired principally by the pioneering but still unpublished work of Mueller (M.I.T. circa 1946) in the form of his Phenomenological Algebra. (For an exposition of this, see reference 3.)

We take the time and space to develop these ideas in some detail because, aside from the work of Parke and Mueller, there does not appear to be available any well-defined procedure to show how to connect the experimental and theoretical aspects of polarized light fields in general scattering-absorbing media. (All existing procedures develop the Stokes vector from electromagnetic theory.) Since the discrete space formulations had been committed to a rigorous phenomenological approach¹ and since no rigorous phenomenological definition of polarized radiance and associated scattering matrix in general media appears to exist, we are now irreversibly committed to direct some attention to this matter.

PHENOMENOLOGICAL DEFINITION OF RADIANCE

A phenomenological, (or operational) definition of a physical concept is one which uses a given apparatus and prescribed set of operations with the apparatus to define the concept. The apparatus and the operations together define the concept. This type of definition is to be distinguished from such definitions that employ only constitutive concepts and the analytical or general symbolic connections between them. A constitutive definition of radiance would be: "Let U represent radiant energy (presumed known and of a given wavelength). U , as a mathematical concept, is an absolutely continuous totally finite additive measure with respect to the Lebesgue measure ν on $X \times \Xi$, where X is some subset of E_3 and Ξ is the unit sphere in E_3 . It follows from a general theorem in measure theory that the Radon-Nikodym derivative $N \equiv dU/d\nu$ of U with respect to ν on $X \times \Xi$ exists. We shall call N the radiance function on $X \times \Xi$, and we call the values $N(x, \xi)$ of N , the radiance at $(x, \xi) \in X \times \Xi$." A constitutive definition, therefore, makes use of both the established concepts within some mathematical or general symbolic framework representing a certain segment of physical reality, and the mathematical connections between these concepts. The connections usually are established in the non-trivial cases by means of theorems or corollaries of theorems in the representing framework; hence there is no immediate contact with physical reality in

such definitions. A phenomenological definition of radiance, on the other hand, appeals directly to physical phenomena for its formulation. We now consider the main details that should be covered in developing such a definition of radiance.

Unpolarized Radiance

Consider the schematic representation of the assembly shown in Figure 1. It consists of a radiant flux meter Φ and a cylindrical tube \top limiting the angular field of view of Φ to some small fixed solid angle Ω . In addition to Φ and \top , there is a filter F which allows only radiant flux of a single wavelength λ (plus some small fixed band of wavelengths about λ) to pass through it and be measured by Φ . The assembly ΦFT is called a radiance meter R , and is used as follows: R is taken to some point \mathcal{X} in an optical medium X , and is oriented so that the axis of the tube is directed along some direction ξ (which points down the tube toward Φ) at time t . The reading of R is denoted by $N(\mathcal{X}, \xi, t, \lambda)$ and is called the specific radiance at \mathcal{X} in the direction ξ at time t for the wavelength λ . It is found that, by holding \mathcal{X} , ξ , t fixed (the latter may be accomplished by simultaneous readings with two or more radiance meters), N varies with λ . That is, N is wavelength-dependent. There is another wavelength-dependent phenomenon available for study in the radiative transfer field, and that is the phenomenon of polarization.

The Standard Euclidean Reference Frame

The experimental activities described below will take place within an Euclidean reference frame. It will be assumed that the x , y , and z coordinate axes in E_3 have been established. Then the z axis defines the vertical direction. Planes parallel to the z axis are defined as vertical planes. Directions are defined in terms of the unit vectors ξ in Ξ , i.e., the elements of the unit sphere Ξ in E_3 centered at point $(0,0,0)$ in E_3 .

An auxiliary method of directional measurement in addition to the unit vector method will be needed in the experimental and theoretical procedures. This method is depicted in Figure 3. Consider the unit vector ξ at point x in E_3 . Both x and ξ are ordered triples of real numbers; x is the location vector, ξ the direction vector which designates (as in Figure 1) the direction of flow of the radiant energy.) It will be necessary to consider rotations about the direction ξ (the purpose of these rotations will be made clear later). To describe these rotations, we may use the vertical plane through ξ as a base for measurement of the rotation angle ψ , which will be defined to increase with clockwise rotation about ξ when viewed in the direction defined by ξ . If ξ is parallel to the unit vector k along the z axis, then angles ψ will be measured from the xz plane as shown in Figure 3.

Polarized Radiance; Standard Stokes Vector

Suppose we add to R the following two devices:^{*} (a) a polarizer P (b) a variable wave-plate W . The polarizer can be manufactured from some natural polarizing material such as tourmaline (a dichroic crystal) or perhaps some man-made polarizing material such as polaroid sheets. It is mounted so that its optic axis may be rotated about the axis of T . The wave-plate is made of some negatively doubly refracting material, such as calcite, and is assembled in a form which will allow a wide range of optical path lengths through it; e.g., a Babinet compensator type of device may be used. The new assembly with the relative positions of P and W on R is indicated schematically in Figure 2.

* We are assuming that steps leading to the successful phenomenological definition of these devices has been covered. Thus we may pass on with minimum digression to the operations of particular interest here.

As is well known, the polarizer P and variable wave-plate W , each have a well-defined optic axis. They are then assembled in R so that their optic axes have the following relative orientations within R . First of all, R is mounted on some support so that at any point \mathcal{X} it may be directed in every direction ξ about \mathcal{X} . Furthermore, when R is first directed along ξ_1 , and then along ξ_2 , the net result of the motion is such that T does not rotate on its axis of symmetry. (The mechanical details of such a mount are easy to realize). This means that if a vertical plane is passed through the axis of T when T is directed along ξ_1 , the plane intersects T in two of its generator-lines, say g and g' . Then if T is directed along ξ_2 , and a vertical plane is again passed through the axis of T , the plane will once again intersect T in g and g' . Thus if the polarizer P and wave-plate W are fixed within T so that their optic axes are in the vertical plane when T is directed along ξ_1 , then the optic axes of P and W are still in a vertical plane when T is directed along ξ_2 . When the optic axes of P and W have this common orientation, we will say that they are in the preferred orientation. With these formalities in mind we may now consider the following set of experiments.

EXPERIMENT ONE: Orient R at \mathcal{X} along the direction ξ . Fix λ and assume the light field is in steady state. Set the wave retardation ϵ on W equal to zero. Now rotate the optic axis of P clockwise (when viewed in the direction ξ as in Figure 3) from

$\psi = 0$ (the preferred orientation) to $\psi = 2\pi$. The resultant plot of N vs. ψ is given in Figure 4.

The pattern is found to repeat with a period π radians (180° of turn of the optic axis of P) and is evidently of sinusoidal form (references to x , ξ , λ , and t are momentarily dropped for simplicity):

$$N(\psi, 0) = \bar{N} + \Delta N \cos 2(\psi - \psi_0), \quad (1)$$

where

$$\bar{N} = \frac{N_{\max} + N_{\min}}{2},$$

$$\Delta N = \frac{N_{\max} - N_{\min}}{2},$$

and where ψ_0 is the value of ψ at which $N(\psi, 0) = N_{\max}$, the maximum observed radiance reading. The symbol 0 in $N(\psi, 0)$ indicates that we have set $\epsilon = 0$.

In order to make contact with the usual terminology for Stokes vectors, we let the symbols I , Q and U represent the following observed magnitudes:

$$\bar{N} = \frac{1}{2} I, \quad (2)$$

$$\Delta N \cos 2\psi_0 = \frac{1}{2} Q, \quad (3)$$

$$\Delta N \sin 2\psi_0 = \frac{1}{2} U. \quad (4)$$

With this notation the simple empirical relation (1) resulting from EXPERIMENT ONE becomes:

$$N(\psi, 0) = \frac{1}{2} [I + Q \cos 2\psi + U \sin 2\psi] . \quad (5)$$

EXPERIMENT TWO: With α , ξ , λ , and t as in EXPERIMENT ONE, we now hold ψ fixed and vary ϵ from 0 to 2π , and observe that the plot of the following quantity:

$$\frac{2N(\psi, \epsilon) - [I + Q \cos 2\psi]}{\sin 2\psi} ,$$

for some fixed $\psi > 0$, is of the form shown in Figure 5. A trigonometric analysis of this empirical curve shows that it is a linear combination of $\sin \epsilon$ and $\cos \epsilon$. For the special value $\epsilon = 0$, we see from (5) that the linear combination is equal to U . Hence the coefficient of $\cos \epsilon$ is U . The remaining coefficient, namely that of $\sin \epsilon$, is then determinable. Denote it by $-V$. Hence the empirical curve of this experiment is:

$$U \cos \epsilon - V \sin \epsilon . \quad (6)$$

Thus for a general orientation ψ of P and wave retardation setting ϵ of W , we have:

$$N(\psi, \epsilon) = \frac{1}{2} [I + Q \cos 2\psi + (U \cos \epsilon - V \sin \epsilon) \sin 2\psi] , \quad (7)$$

where, of course, I, Q, U, V generally depend on x, ξ, t and λ .

Definition: The ordered quadruple $\underline{S} = [I, Q, U, V]$ of radiance numbers defined above is called the Standard Stokes Vector at x, ξ, t and λ in X .^{*}

The Standard Observable Vector

The ordered quadruple $[I, Q, U, V]$ is, by definition, the Standard Stokes Vector. The representation of $N(x, \xi, t, \lambda, \psi, \epsilon)$ in terms of the vector's components is given in (7). There are, however, many ways of representing the radiance $N(x, \xi, t, \lambda, \psi, \epsilon)$ other than by means of the Standard Stokes Vector. For example, suppose the radiance meter is fixed at x, ξ, t and λ and the following four readings are taken:

$$\left. \begin{array}{lll} N(0,0) & \equiv & {}_1 N \\ N(\frac{\pi}{2},0) & \equiv & {}_2 N \\ N(\frac{\pi}{4},0) & \equiv & {}_3 N \\ N(\frac{\pi}{4},\frac{\pi}{2}) & \equiv & {}_4 N \end{array} \right\} \quad (8)$$

*Recall that the optic axes of the P and W elements of R are initially in the preferred orientations.

The ordered quadruple $\underline{N} = [{}_1N, {}_2N, {}_3N, {}_4N]$ of radiance magnitudes is called the Standard Observable Vector. This vector may be formed directly from readings of a radiance meter fitted with a three-setting polarizer P and a quarter-wave plate W which is snapped into place for the determination of the component ${}_4N$. In all that follows, the word "observable" will refer to this particular vector. The word "standard" refers to the fact that the initial orientations of the optic axes of P and W are in a vertical plane. The experimental advantages of the Standard Observable Vector are obvious. Further, the physical meaning of each component is immediately clear.

The remainder of the report will be devoted to the development of the scattering matrices in terms of the Standard Observable Vector. The principle of local interaction will be couched in terms of these concepts. However, in order to maintain contact with the theoretical descriptions of the light field as given by the Stokes and Chandrasekhar Vectors, we will establish ties between the various Stokes, Chandrasekhar, and Observable concepts as they arise.

General Connections Between the Standard Stokes
Vectors and Standard Observable Vectors

The general representation of the polarized radiance given in (7) allows an explicit evaluation of the components of the Observable vector \underline{N} as defined in (8). Suppressing references to α , δ , t , and λ as usual, and setting ψ and ϵ in (7) equal to the values indicated in (8), we have:

$$\left. \begin{aligned} {}_1N &= N(0,0) = \frac{1}{2} [I + Q] \\ {}_2N &= N(\frac{\pi}{2},0) = \frac{1}{2} [I - Q] \\ {}_3N &= N(\frac{\pi}{4},0) = \frac{1}{2} [I + U] \\ {}_4N &= N(\frac{\pi}{4},\frac{\pi}{2}) = \frac{1}{2} [I - V] \end{aligned} \right\} \quad (9)$$

From this set of equations we may immediately write down the matrix \mathcal{Q} which maps \underline{S} into \underline{N} :

$$\underline{N} = \underline{S} \mathcal{Q} \quad (10)$$

i.e.,

$$[{}_1N, {}_2N, {}_3N, {}_4N] = [I, Q, U, V] \left(\frac{1}{2} \right) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where

$$Q = \left(\frac{1}{2} \right) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (11)$$

The matrix Q^{-1} which maps \underline{N} into \underline{S} , i.e.,

$$\underline{S} = \underline{N} Q^{-1} \quad (12)$$

is of the form:

$$Q^{-1} = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} . \quad (13)$$

It should be observed that there are many possibilities for observable vectors: any set of four readings of $N(\psi, \epsilon)$ which yields a generally uniquely solvable set of equations in \mathbb{I} , Q , U , V , by means of (7) suffices to define a vector which may be used to represent the polarized light field. We have chosen the \underline{N} defined above on the basis of its mathematical and conceptual simplicity.

Special Connections Between the Standard Stokes
and Standard Observable Vectors.

It is well known² that the component I of the Standard Stokes Vector represents the radiance of the field as it appears to a radiance meter from which the P and W elements have been removed. (It is assumed here that P and W have ideal transmittances: $1/2$ for P for unpolarized light, and 1 for W for all fluxes of interest.) This radiance may be expressed in terms of the ${}_i N$ components by using (12) and equating the first components of \underline{S} and $\underline{N} \mathcal{Q}^{-1}$. It follows that ${}_1 N + {}_2 N$ is the observable radiance recorded by the radiance meter after the P and W elements are removed (all other settings are kept unchanged and assuming ideal transmittances for P and W). We shall denote this radiance, as usual, by the scalar N . Thus, formally, we set:

$$N \equiv {}_1 N + {}_2 N, \quad (14)$$

It is a particularly simple matter to define for example the concepts of linearly polarized, circularly polarized, and unpolarized radiance in terms of the Standard Observable Vector. Thus, we define,

(Standard Observable Vectors)*

Verbal Description	Vector Representation
vertically linearly polarized radiance:	$\frac{1}{2} [2N, 0, N, N]$
horizontally linearly polarized radiance:	$\frac{1}{2} [0, 2N, N, N]$
linearly polarized radiance at $+45^\circ$:	$\frac{1}{2} [N, N, 2N, 0]$
right circularly polarized radiance:	$\frac{1}{2} [N, N, N, 0]$
left circularly polarized radiance:	$\frac{1}{2} [N, N, N, 2N]$
unpolarized radiance:	$\frac{1}{2} [N, N, N, N]$

* To tie in these definitions with the electromagnetic definitions, recall first that the optic axes of P and W are initially in a vertical plane (the standard preferred orientation). Then in electromagnetic terminology, the E-vector in vertically polarized light is vertical as it crosses a plane perpendicular to its direction of travel. In $+45^\circ$ linearly polarized light, the vector is inclined along a $+45^\circ$ line from the vertical as seen from the incident side of the plane (+angles measured clockwise as in Figure 3). Finally, the projection of the E-vector of a right circularly polarized wave train describes a clockwise motion as seen on the incident side of the plane as the wave train passes through the plane. (Hence the space curve traced out by the tip of the E-vector on a right circularly polarized wave train is a left-handed helix!)

By means of the transformation (12) we may obtain the usual Stokes representations of these special polarized radiances:

(Standard Stokes Vectors)

Verbal Description	Vector Representation
vertically linearly polarized radiance	$[N, N, 0, 0]$
horizontally linearly polarized radiance	$[N, -N, 0, 0]$
linearly polarized radiance at $+45^\circ$	$[N, 0, N, 0]$
right circularly polarized radiance	$[N, 0, 0, N]$
left circularly polarized radiance	$[N, 0, 0, -N]$
unpolarized radiance	$[N, 0, 0, 0]$

SCATTERING MATRICES

The present phenomenological--or operational--development of the concepts of the polarized radiance field will now be extended to include the description of the polarized version of the volume scattering function. The basic experimental procedure for determining the volume scattering function in the unpolarized case is outlined elsewhere⁵. Except for elevating this procedure from the scalar to the vector level it will remain essentially unchanged. But first some necessary additional concepts concerning the geometric description of polarized radiance fields must be developed.

The Local Euclidean Reference Frame

The Standard Observable and Standard Stokes Vectors were defined by fixing the initial orientations of the optic axes of P and W in a vertical plane. In carrying out scattering measurements, however, the initial orientations of the optic axes are most naturally set perpendicular to the plane of scattering defined by the directions of the incident and scattered beams of flux at a point. (See Figure 6.) Now this plane is generally not horizontal, so that the optic axes of P and W would then not initially lie in a vertical plane as in the standard case. Therefore, to develop the concept of the Observable Scattering Matrix, it would be helpful to adopt the following conventions: (a) The incident (ξ^o) and scattered (ξ) directions define the plane of scattering; (b) When the radiance meter analyzes the observable incident radiance vector $\underline{N}^o(\xi^o)$, the optic axis of P is initially in a plane normal to the plane of scattering; (c) When the radiance meter analyzes the observable scattered radiance vector $\underline{N}(\xi)$ the optic axis of P is initially in a plane normal to the plane of scattering. The initial orientations of the optic axis of P defined in (b) and (c) are called the local preferred orientations. The resultant Observable and Stokes vectors are called Local Observable Vectors and Local Stokes Vectors, respectively. The connections between these vectors are still given by means of \mathcal{Q} and \mathcal{Q}^{-1} as defined in (10) and (12).

The Rotation Matrices

After defining the concepts of Local Stokes and Local Observable Vectors, the question arises: What is the connection between the Local and Standard radiance Vectors at a given point? This question can be answered on strictly operational grounds by making use of the now well-defined concepts of Standard and Local Observable Vectors, and the operationally determinable relations between the components of the Local Observable Vector (call it $\underline{N}\phi$) and the Standard Observable Vector \underline{N} at $(\mathcal{X}, \mathcal{f})$ after a rotation of the frame clockwise about \mathcal{f} at \mathcal{X} through an angle of magnitude ϕ . However, in the interests of expediting the discussion we choose an analytical route to the desired connection and make use of a rotation matrix developed by Chandrasekhar.² To establish the required rotation transformation between $\underline{N}\phi$ and \underline{N} we must then establish the connection between the Chandrasekhar Vectors and the present Observable Vectors, which we shall now do.

The Standard Chandrasekhar (specific radiance) Vector² is defined as

$$\underline{I} = [I_\ell, I_\tau, U, V], \quad (15)$$

where $I_{\ell} = {}_1N$, $I_r = {}_2N$ in the present notation, and where U and V are the third and fourth components of the Standard Stokes Vector $\underline{S} = [I, Q, U, V]$. Furthermore, $I = I_{\ell} + I_r = N$. It follows that the linear transformation C which maps \underline{S} into \underline{I} i.e.,

$$\underline{I} = \underline{S} C \quad (16)$$

has the matricial representation:

$$C = \left(\frac{1}{2}\right) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad (17)$$

with inverse:

$$C^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

The desired map of \underline{I} into \underline{N} is then found by means of equations (10) and (11):

$$\underline{N} = \underline{Q} \mathcal{O} = (\underline{I} \mathcal{C}^{-1}) \mathcal{O}$$

that is,

$$\underline{N} = \underline{I} (\mathcal{C}^{-1} \mathcal{O}), \quad (19)$$

which of course holds in all reference frames--Standard or Local.

Now Chandrasekhar showed² that if the frame of reference for $\underline{I} = [I_x, I_y, U, V]$ is rotated clockwise about the direction ξ (see Figure 1) associated with \underline{I} , then the representation \underline{I}_ϕ of \underline{I} in this new frame is given by:*

$$\underline{I}_\phi = \underline{I} \underline{L}(\phi), \quad (20)$$

where

$$\underline{L}(\phi) = \begin{pmatrix} \cos^2 \phi & \sin^2 \phi & -\sin 2\phi & 0 \\ \sin^2 \phi & \cos^2 \phi & \sin 2\phi & 0 \\ \frac{1}{2} \sin 2\phi & -\frac{1}{2} \sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (21)$$

* We use the transpose of $\underline{L}(\phi)$ as given in reference 2.

Suppose that \underline{N} is the Standard Observable Vector associated with \underline{I} before rotation of the axes, and that \underline{N}_ϕ corresponds to \underline{I}_ϕ after the clockwise rotation of the axes through an angle ϕ . We may then easily determine the observable correspondent $\underline{L}(\phi)$ to $\underline{I}(\phi)$ which relates \underline{N}_ϕ to \underline{N} . Thus, from (19),

$$\begin{aligned}\underline{N}_\phi &= \underline{I}_\phi (C^{-1}\mathcal{O}) \\ &= \underline{I} \underline{L}(\phi) (C^{-1}\mathcal{O}).\end{aligned}$$

But

$$\underline{I} = \underline{N} (\mathcal{O}^{-1}C),$$

which again follows from (19). Hence,

$$\underline{N}_\phi = \underline{N} \mathcal{O}^{-1}C \underline{L}(\phi) C^{-1}\mathcal{O}. \quad (22)$$

On the basis of (22), we define

$$\underline{L}(\phi) = \mathcal{O}^{-1}C \underline{L}(\phi) C^{-1}\mathcal{O}, \quad (23)$$

which is the observable counterpart to $\underline{L}(\phi)$. The matrix $\mathcal{L}(\phi)$ is readily determinable since \mathcal{Q} , \mathcal{C} , and $\underline{L}(\phi)$ are known. It may be verified that:

$$\mathcal{L}(\phi) = \begin{pmatrix} \cos^2 \phi - \frac{1}{2} \sin 2\phi & \sin^2 \phi + \frac{1}{2} \sin 2\phi & \sin^2 \phi - \frac{1}{2} \sin 2\phi & 0 \\ \sin^2 \phi - \frac{1}{2} \sin 2\phi & \cos^2 \phi + \frac{1}{2} \sin 2\phi & \sin^2 \phi + \frac{1}{2} \sin 2\phi & 0 \\ \sin 2\phi & -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (24)$$

Some Properties of $\mathcal{L}(\phi)$

It can be easily verified that $\mathcal{L}(\phi)$ is an isomorphism on the additive group of reals mod (π) into the multiplicative group of all 4×4 real matrices; that is,

$$\mathcal{L}(\phi_1 + \phi_2) = \mathcal{L}(\phi_1) \mathcal{L}(\phi_2), \quad 0 \leq \phi < \pi.$$

Hence in particular,

$$\mathcal{L}(0) = \mathcal{I},$$

the 4×4 identity matrix; and

$$\mathcal{L}^{-1}(\phi) = \mathcal{L}(-\phi),$$

which shows how to compute the inverse of $\mathcal{L}(\phi)$..

The matrices $\mathcal{L}(\phi)$ in (24) also have some interesting physical properties. For example, observe that the sum of the elements in each column of $\mathcal{L}(\phi)$ is unity. This corresponds to the requirement that for unpolarized radiance, the Standard Observable Vector $\underline{N} = \frac{1}{2} [N, N, N, N]$ should be invariant under the present type of reference frame rotations. Furthermore, the physical requirement that the observable scalar radiance ${}_1N + {}_2N = N$ be invariant under rotations of the reference frame under all polarized conditions is also borne out by the properties of (24). Finally, the component ${}_4N$, (like the corresponding component V of \underline{S}) is invariant under rotations of the reference frame.

The Local Observable Scattering Matrix

Suppose an element of volume at a point \mathcal{X} of an optical medium X (generally anisotropic, non-reciprocal) is irradiated in the direction ξ^0 by a beam of polarized radiance:

$$\underline{N}^0(\xi^0) = [{}_1N^0(\xi^0), {}_2N^0(\xi^0), {}_3N^0(\xi^0), {}_4N^0(\xi^0)] ,$$

expressed in Local Observable Vector form (Figure 6). Let the scattered radiance, also in Local Observable Vector form, be designated by:

$$\underline{N}_{\ell}^*(\xi) = [{}_1N_{\ell}^*(\xi), {}_2N_{\ell}^*(\xi), {}_3N_{\ell}^*(\xi), {}_4N_{\ell}^*(\xi)] .$$

The ℓ denotes that the line of sight through the irradiated element of volume is of length ℓ .

Suppose further the source irradiating the element subtends a small solid angle Ω at \mathcal{X} . Then $\underline{N}^o(\xi^o)$ and $\underline{N}_l^*(\xi)$ define a 4×4 matrix $\underline{\Sigma}(\xi^o; \xi)$ at \mathcal{X} by means of the following formula:

$$\underline{N}_l^*(\xi) \equiv \frac{\underline{N}_l^*(\xi)}{\Omega} = \underline{N}^o(\xi^o) \underline{\Sigma}(\xi^o; \xi), \quad (25)$$

where

$$\underline{\Sigma}(\xi^o; \xi) = \left[\begin{matrix} \sum_{ij} \end{matrix} \underline{\Sigma}(\xi^o; \xi) \right] \quad (26)$$

$i, j = 1, 2, 3, 4,$

is the Local Observable Scattering Matrix.

The sixteen quantities $\sum_{ij} \underline{\Sigma}(\xi^o; \xi)$ for each pair $(\xi^o; \xi)$ of directions (which explicitly depend also on \mathcal{X} , λ , and t) are generally determined by suitably irradiating the element of volume at \mathcal{X} with four generally different sets of $\underline{N}^o(\xi^o)$ vectors but holding ξ^o fixed. However, the number of independent components of $\underline{\Sigma}(\xi^o; \xi)$ is considerably reduced if the space \mathcal{X} (as happens in most applications) exhibits at \mathcal{X} both isotropy and reciprocity properties. In this latter instance there will be at most six independent components.

The details of such regularity considerations have been worked out in detail, and need not be considered here⁶. The important point to note is that the Local Observable Scattering Matrix is definable in terms of observable radiances in a manner which is completely analogous to that used in the unpolarized case considered earlier⁵. Finally, we recall that in the discrete space context we define $\underline{\Sigma}(\xi^0; \xi)$ by setting $\mathcal{L}\Omega = 1$.

The Standard Observable Scattering Matrix

The Local Observable Scattering Matrix can be cast into Standard form by converting the Local Observable Vectors, occurring in the definition (25), into Standard form using the rotation matrix $\mathcal{L}(\phi)$ defined in (23). Suppose that a rotation of axes through an angle ϕ^0 is required to convert $\underline{N}^0(\xi^0)$ into standard form (Figure 7). The sign of ϕ^0 (in accordance with the clockwise convention) is determined completely by initially giving ξ^0 and ξ . Similarly, we determine the angle ϕ through which the local frame must be rotated about ξ to effect a transformation of $\underline{N}_*(\xi)$ to standard form. Then from (25),

$$\underline{N}_*(\xi)\mathcal{L}(\phi) = \underline{N}^0(\xi^0)\mathcal{L}(\phi^0) [\mathcal{L}^{-1}(\phi)\underline{\Sigma}(\xi^0; \xi)\mathcal{L}(\phi)].$$

Since $\underline{N}_*(\xi)\mathcal{L}(\phi)$ and $\underline{N}^0(\xi^0)\mathcal{L}(\phi^0)$ are the required standard observable vectors at \mathcal{X} , it follows that

$$\boxed{\underline{\Pi}(\xi^0; \xi) = \mathcal{L}(-\phi^0) \underline{\Sigma}(\xi^0; \xi) \mathcal{L}(\phi)} \quad (27)$$

is the required Standard Observable Scattering Matrix.

POLARIZED FORM OF THE PRINCIPLE OF LOCAL INTERACTION

We shall now use the results of the preceding discussions to formulate the main object of study of the present report: the polarized form of the principle of local interaction.

Consider three points x_k, x_i, x_j , in a discrete space X_n , $n \geq 3$, and suppose X_n is imbedded in E_3 ,

$$E_3 = \{(x, y, z) : -\infty \leq x, y, z \leq \infty\}.$$

(See Figure 7.) Let the z axis define the vertical direction in X_n . Let the (specific) radiances occurring in the discrete-space theory, expressed in Standard Observable Vector form, be specifically denoted as $\underline{N}_{k,i}$, and $\underline{N}_{i,j}$, using the notation conventions for a discrete space set forth in reference 1. The radiance $\underline{N}_{k,i}$ will be scattered at point x_i in the direction of x_j . Finally, let $\phi_{k,i}$ and $\phi_{i,j}$ be defined analogously to the ϕ^0 and ϕ occurring in the preceding derivation of the form of the Standard Observable Scattering Matrix $\underline{\Pi}$.

The polarized form of the local interaction principle (Eq. (13), reference 1) then takes the following form:

$$\begin{aligned}
 \underline{N}_{ij} = & \sum_{k=1}^n \underline{N}_{ki} \underline{\Pi}(x_i; \xi_{ki}; \xi_{ij}) \\
 & + \sum_{k=1}^{A_i} N_{ki}^0 \underline{\Pi}^0(x_i; \xi_{ki}^0; \xi_{ij}) \\
 i, j = & 1, \dots, n,
 \end{aligned}
 \tag{23}$$

where \underline{N}_{ki}^0 denotes the source vector at x_i and $\underline{\Pi}^0(x_i; \xi_{ki}^0; \xi_{ij})$ denotes the Standard Observable Scattering Matrix adapted to the source condition at x_i . This completes the required formulations.

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BASIC ELEMENTS OF A RADIANCE METER

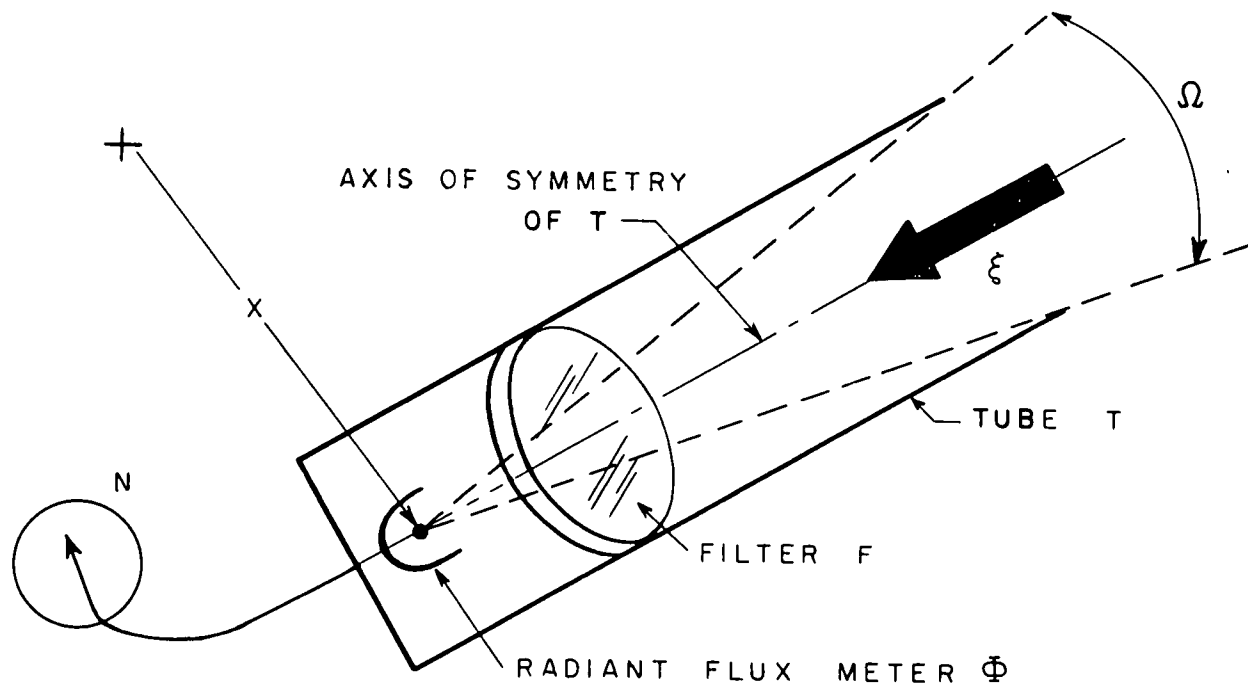


Figure 1

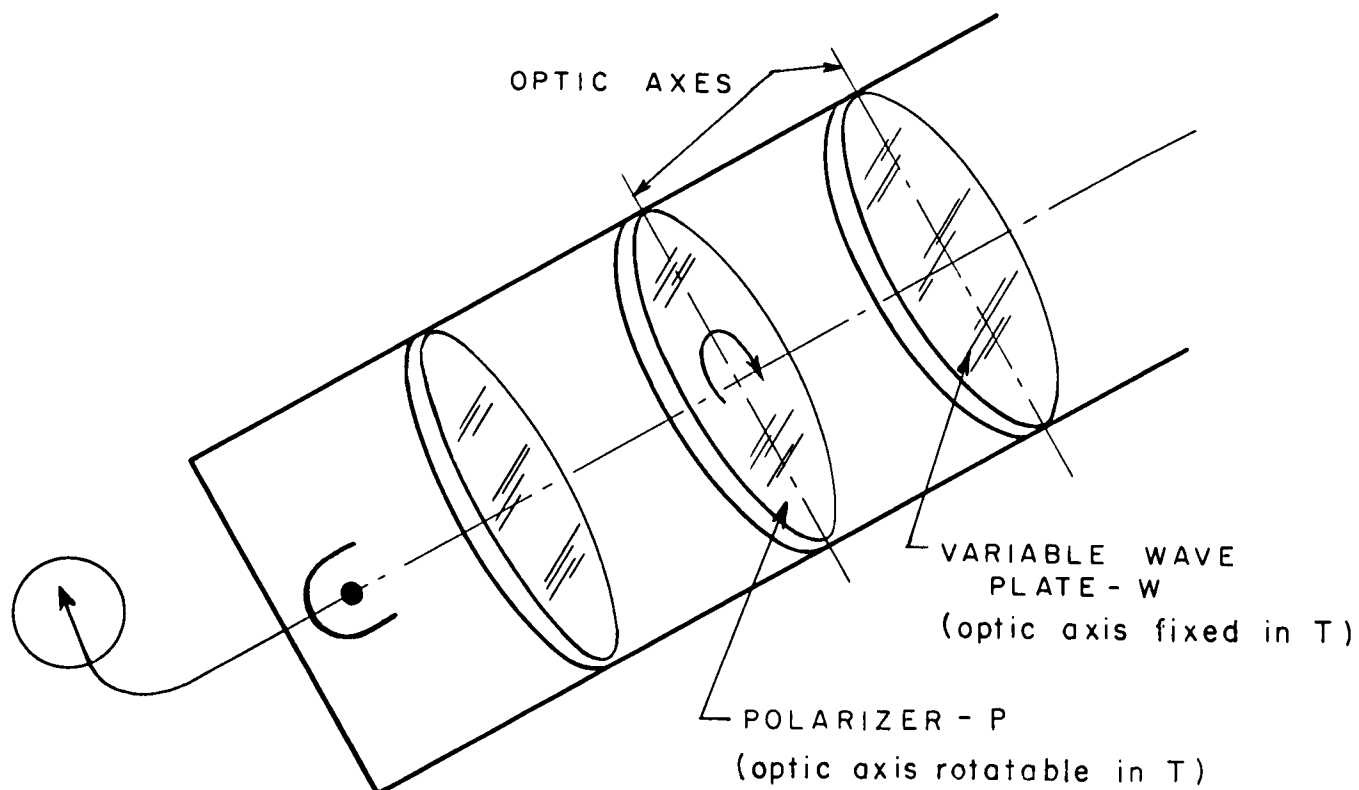


Figure 2

STANDARD EUCLIDEAN REFERENCE FRAME

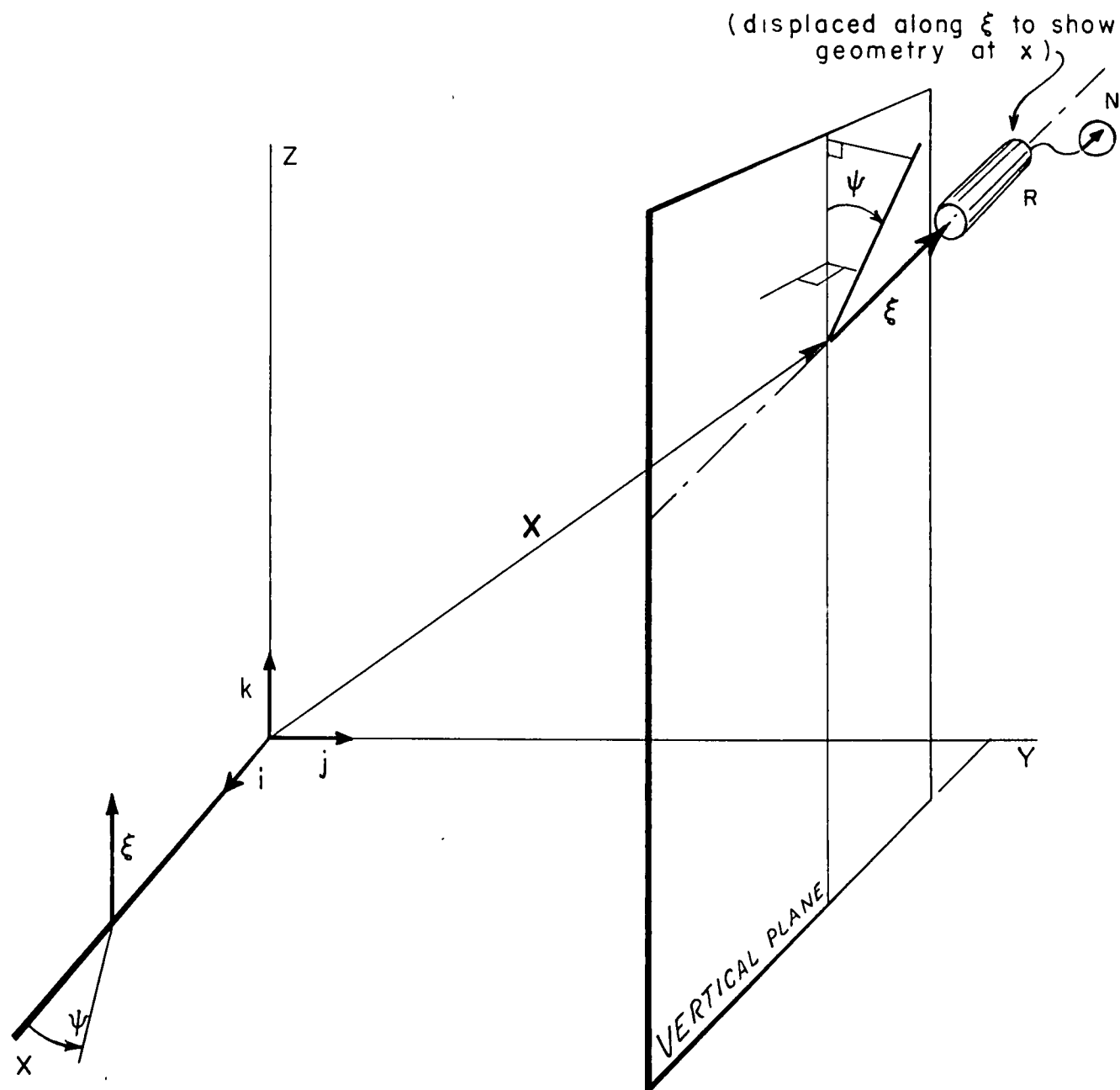


Figure 3

GRAPHIC RESULTS OF EXPERIMENTS ONE AND TWO

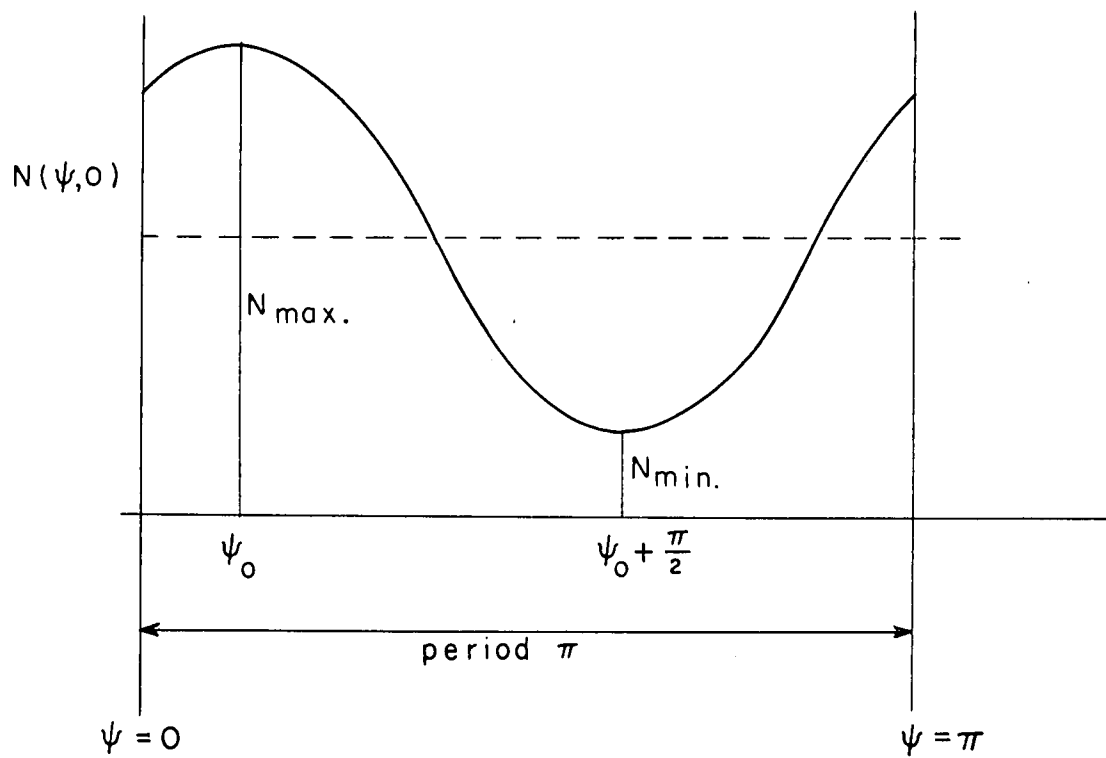


Figure 4

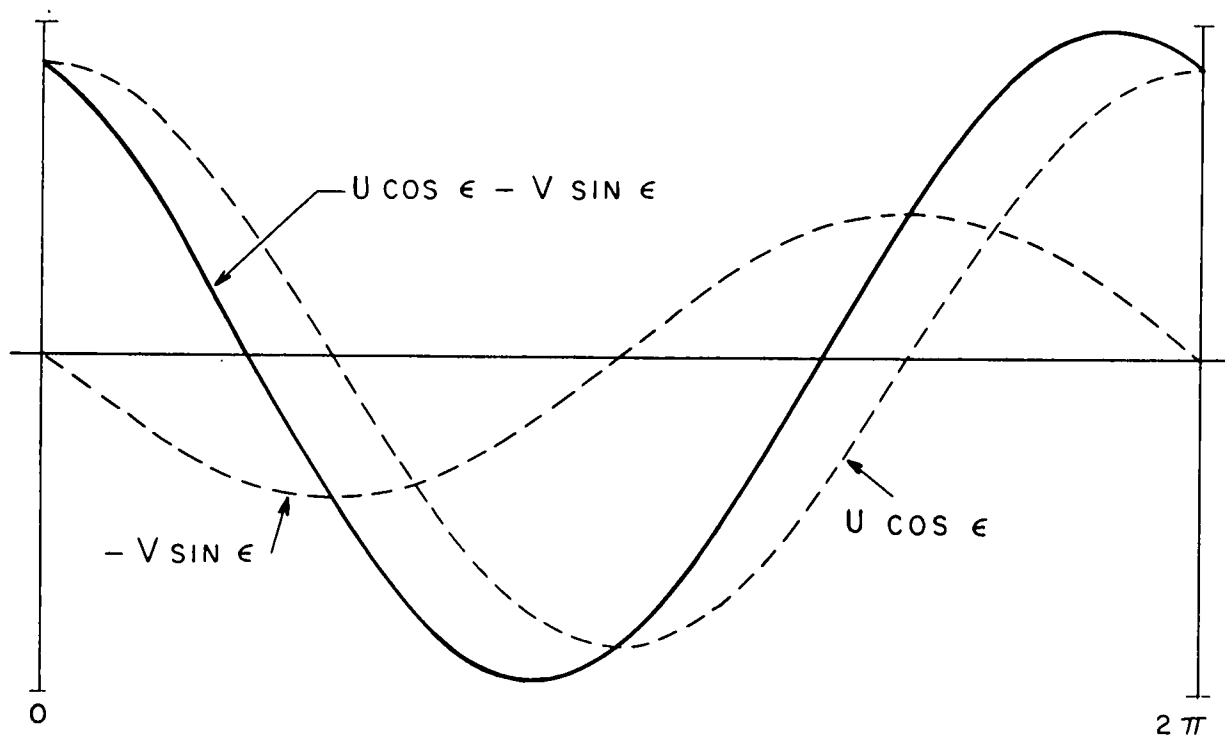


Figure 5

LOCAL EUCLIDEAN REFERENCE FRAME

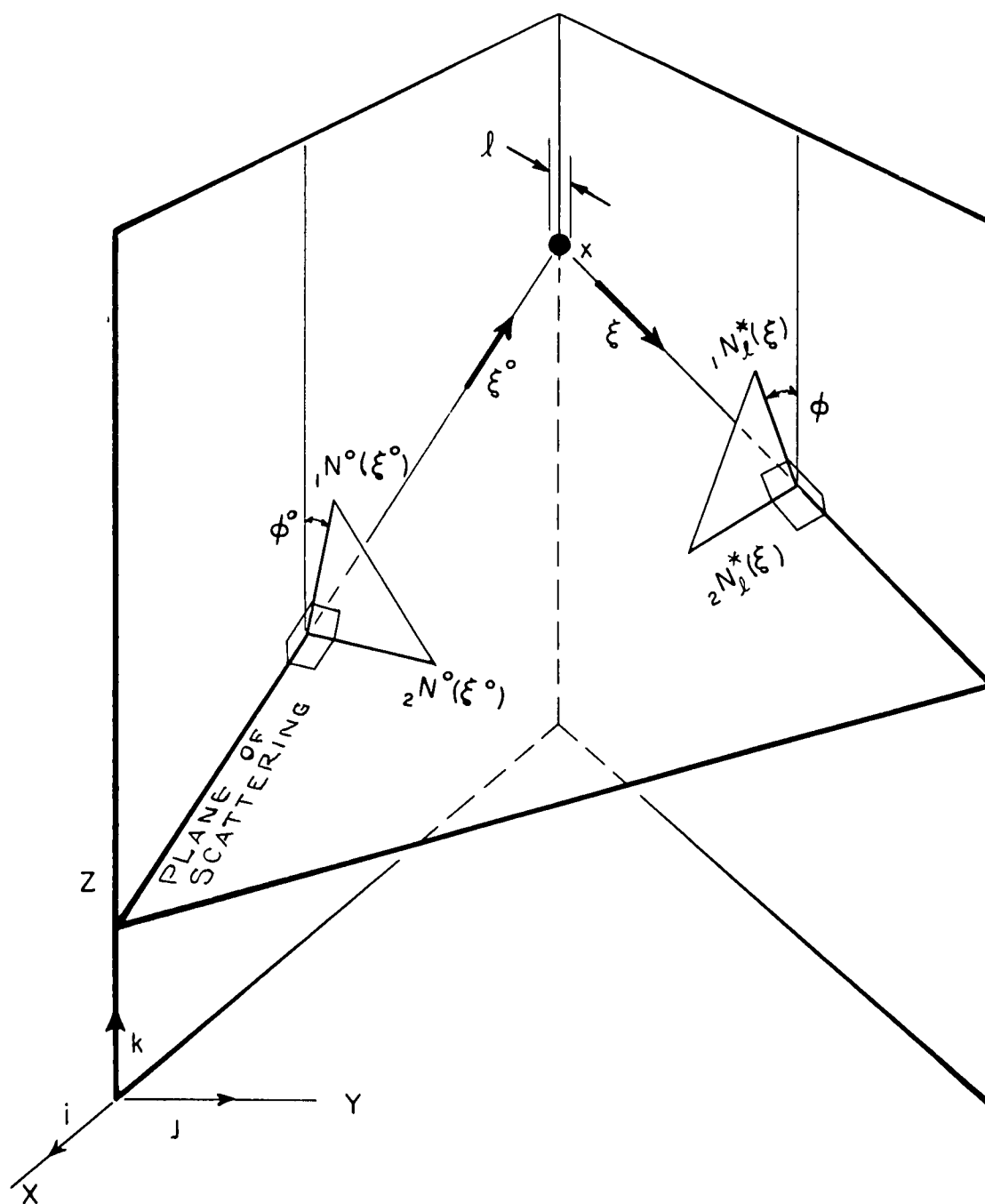


Figure 6

GEOMETRY FOR THE POLARIZED FORM OF
THE PRINCIPLE OF LOCAL INTERACTION

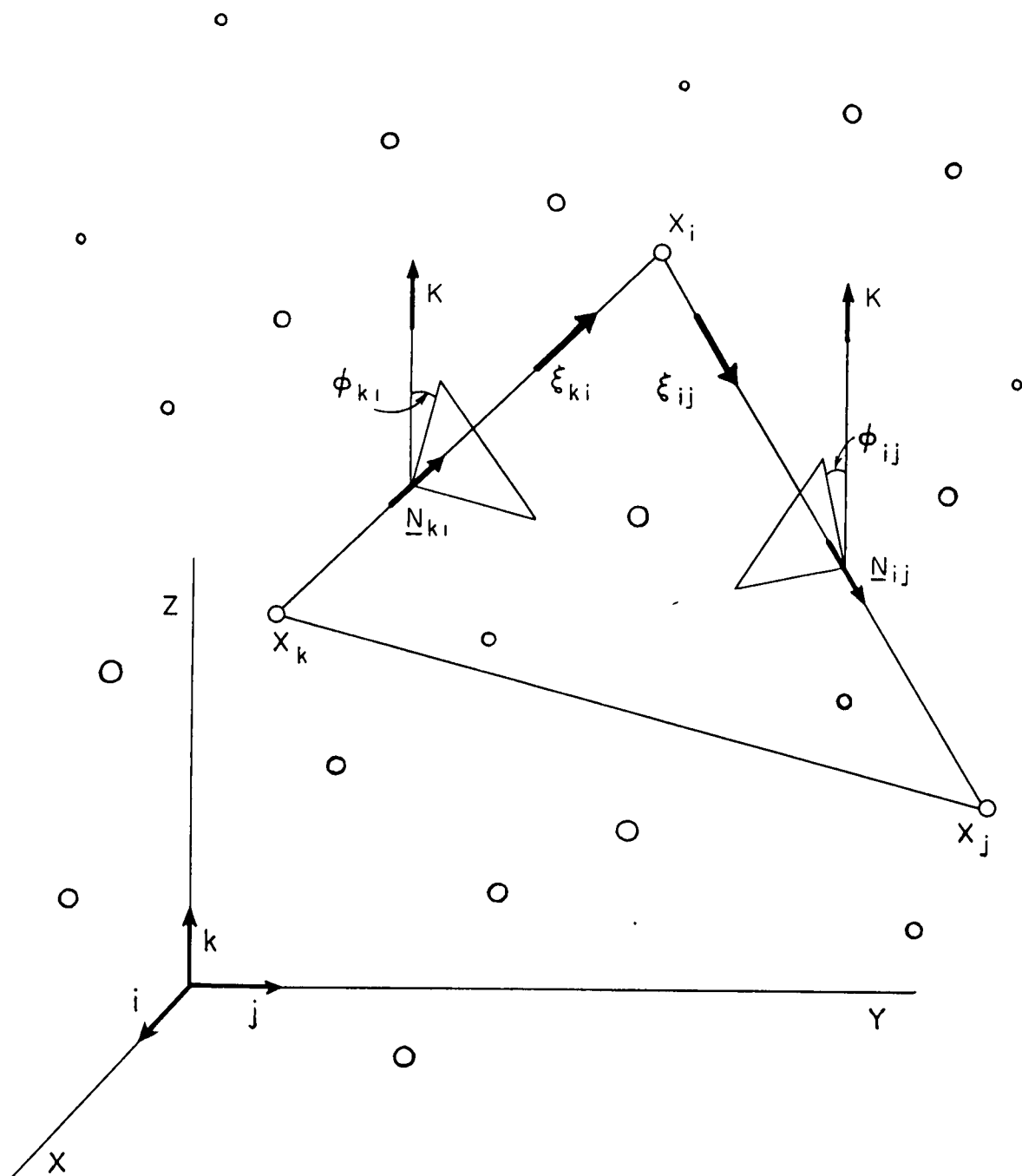


Figure 7