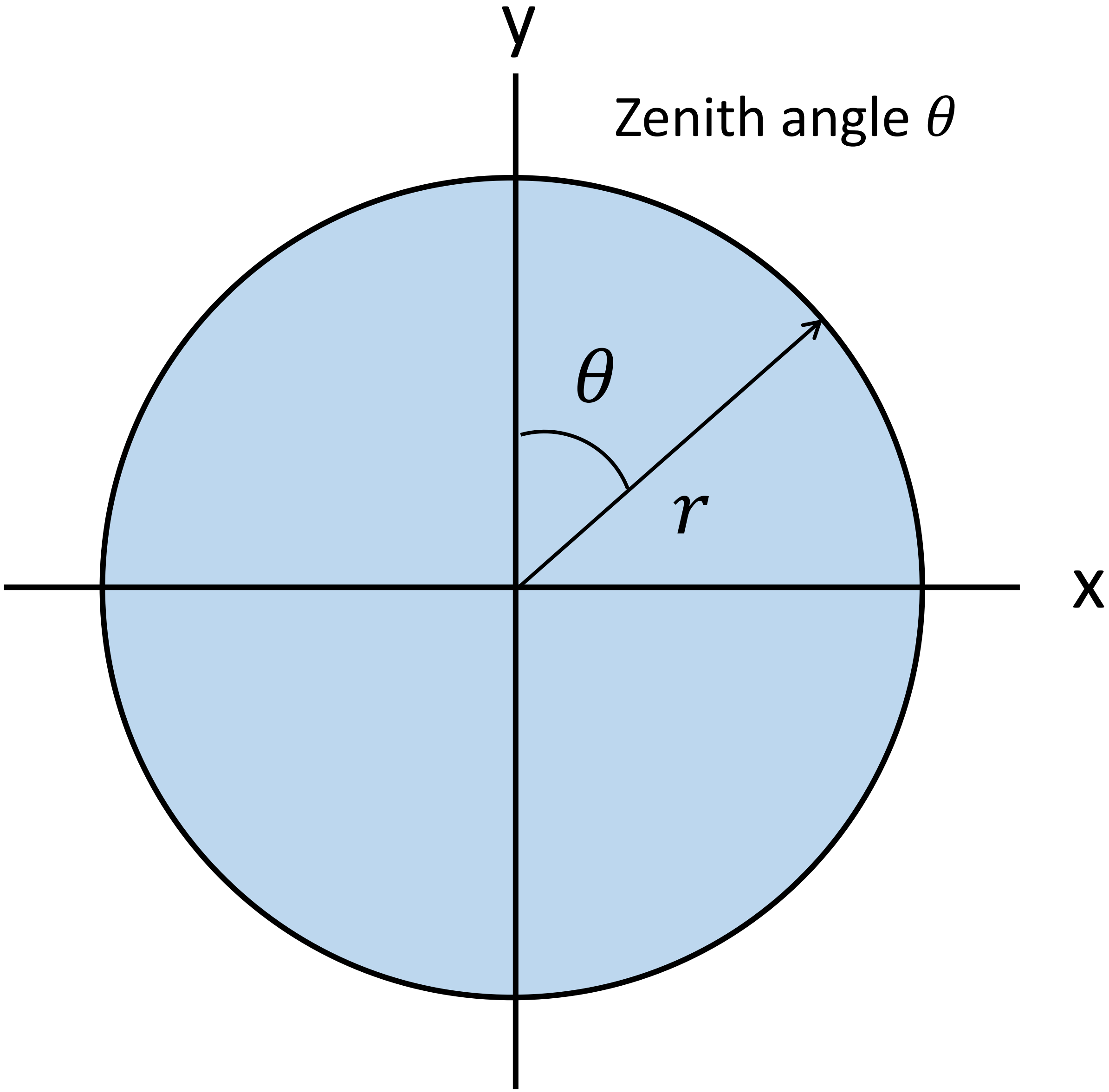
Lecture 17
Radiative Transfer Equation

Collin Roesler

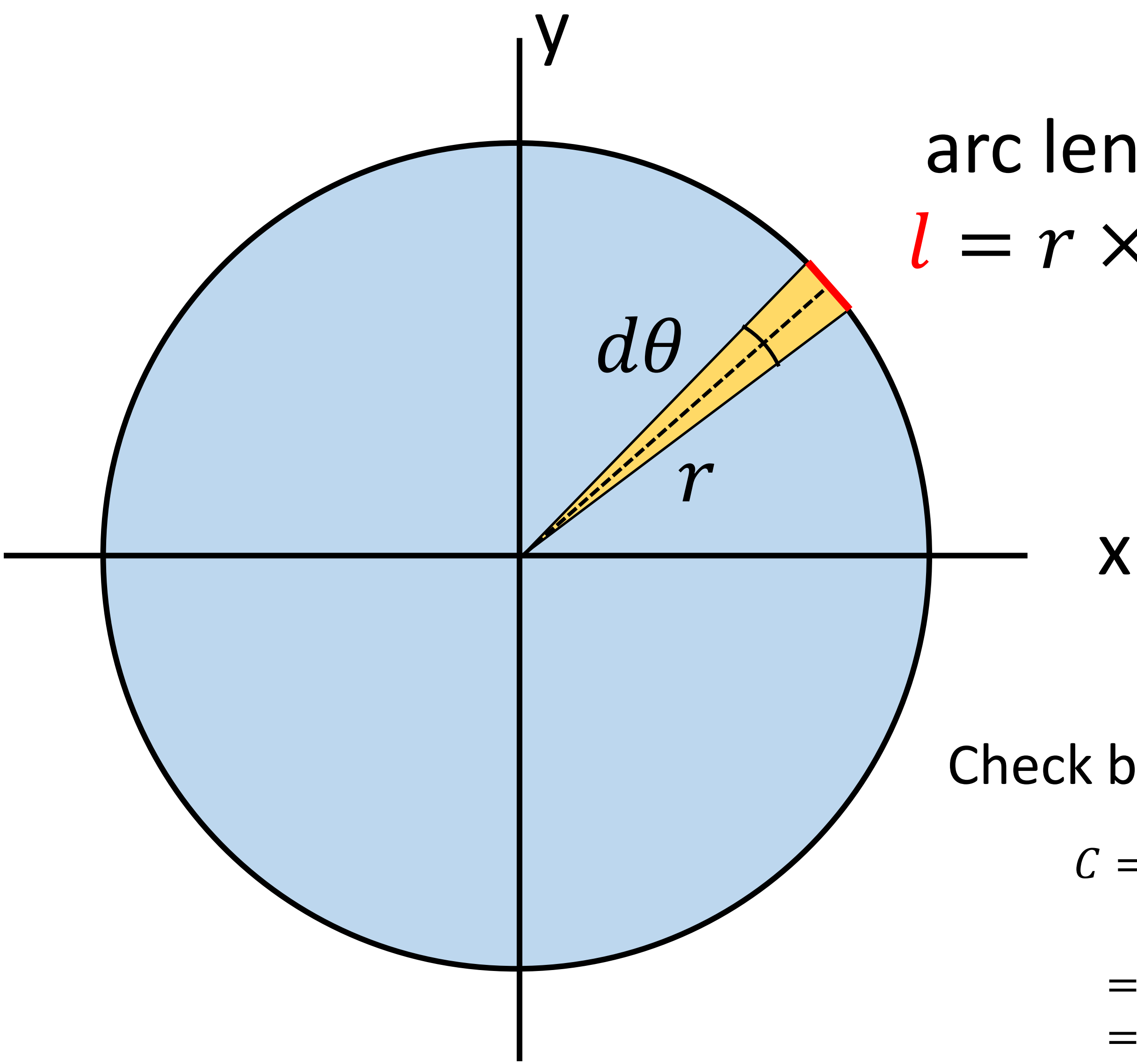
Outline

- Review of
  - spherical geometry to derive solid angle
  - radiometric measurements
- Derive the Radiative Transfer Equation
- Simplifications of the RTE
  - Gershun’s Equation
  - Remote Sensing Reflectance

2d Geometry Review



2d Geometry Review



arc length

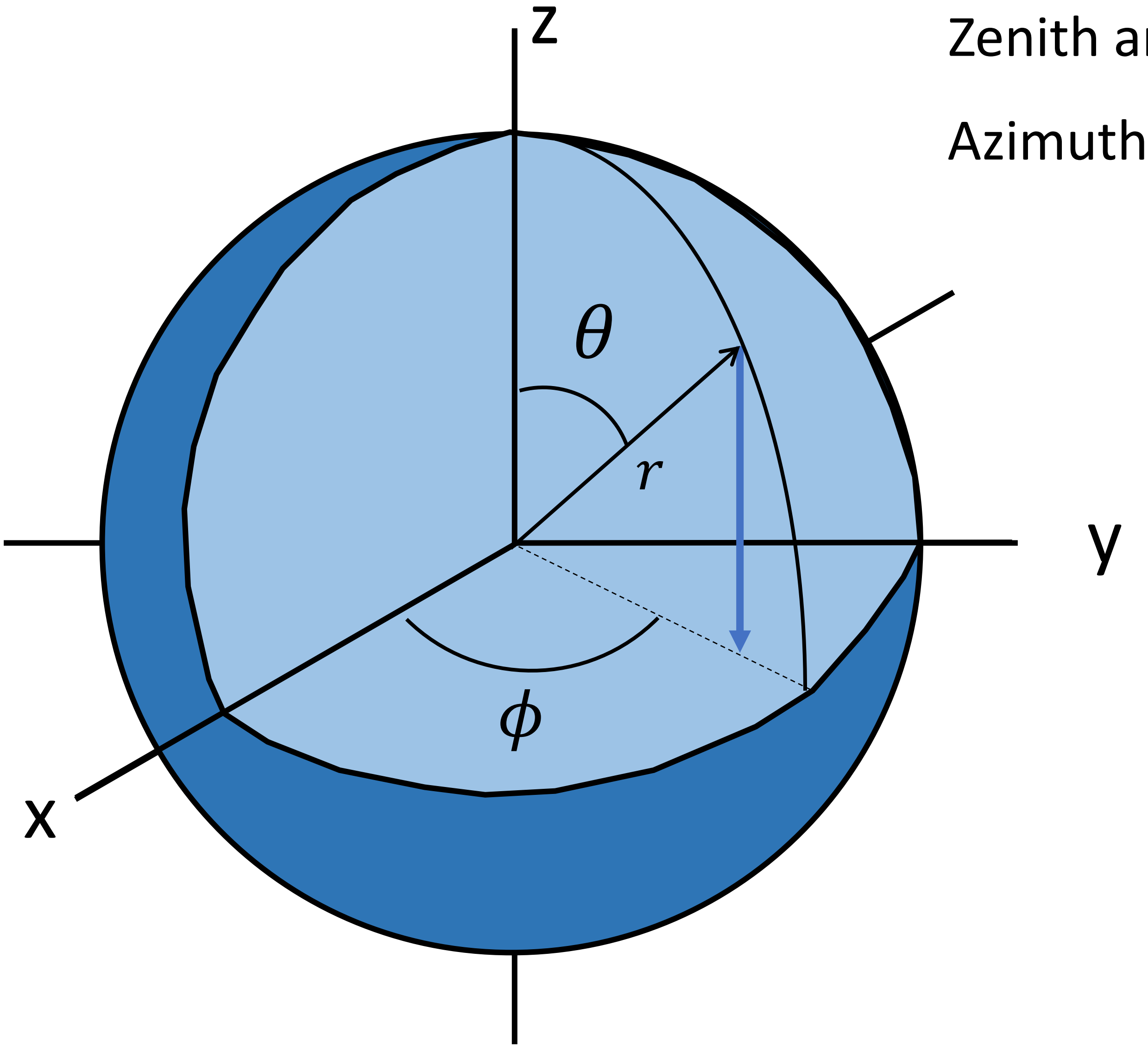
$$l = r \times d\theta$$

Check by integration

$$C = \int_0^{2\pi} r\, d\theta$$

$$= r \times (\theta_2 - \theta_1)$$
$$= r \times (2\pi - 0)$$
$$= 2\pi r$$

3d Geometry Review

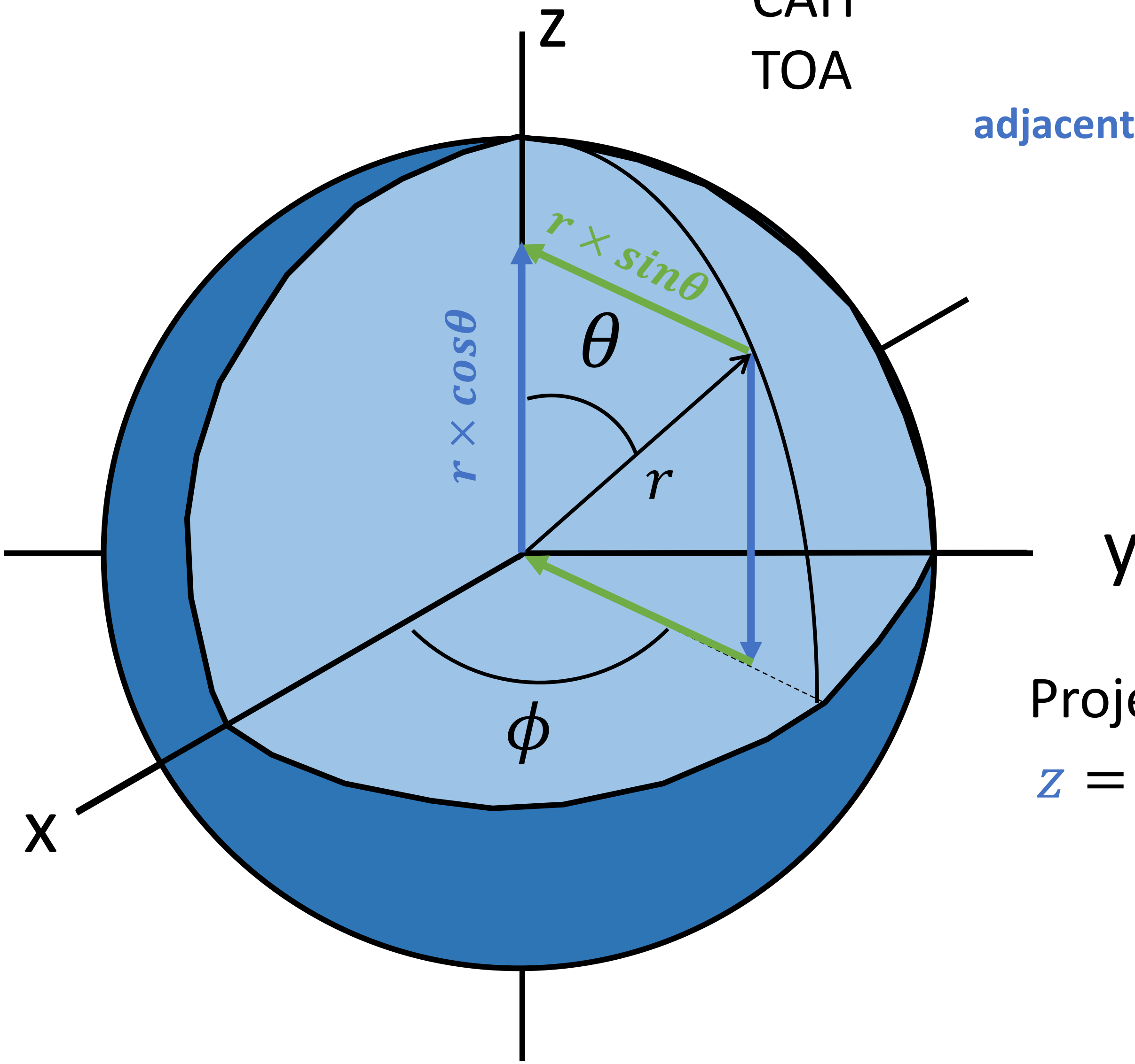


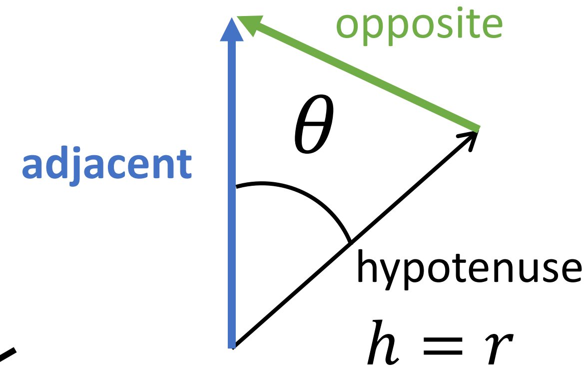
Zenith angle θ

Azimuth angleϕ

3d Geometry Review

SOH
CAH
TOA





$h = r$

$sin\theta = \frac{o}{r}$

$cos\theta = \frac{a}{r}$

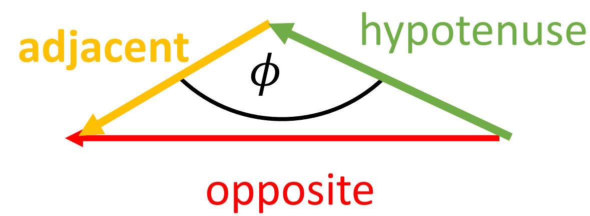
Project to axes

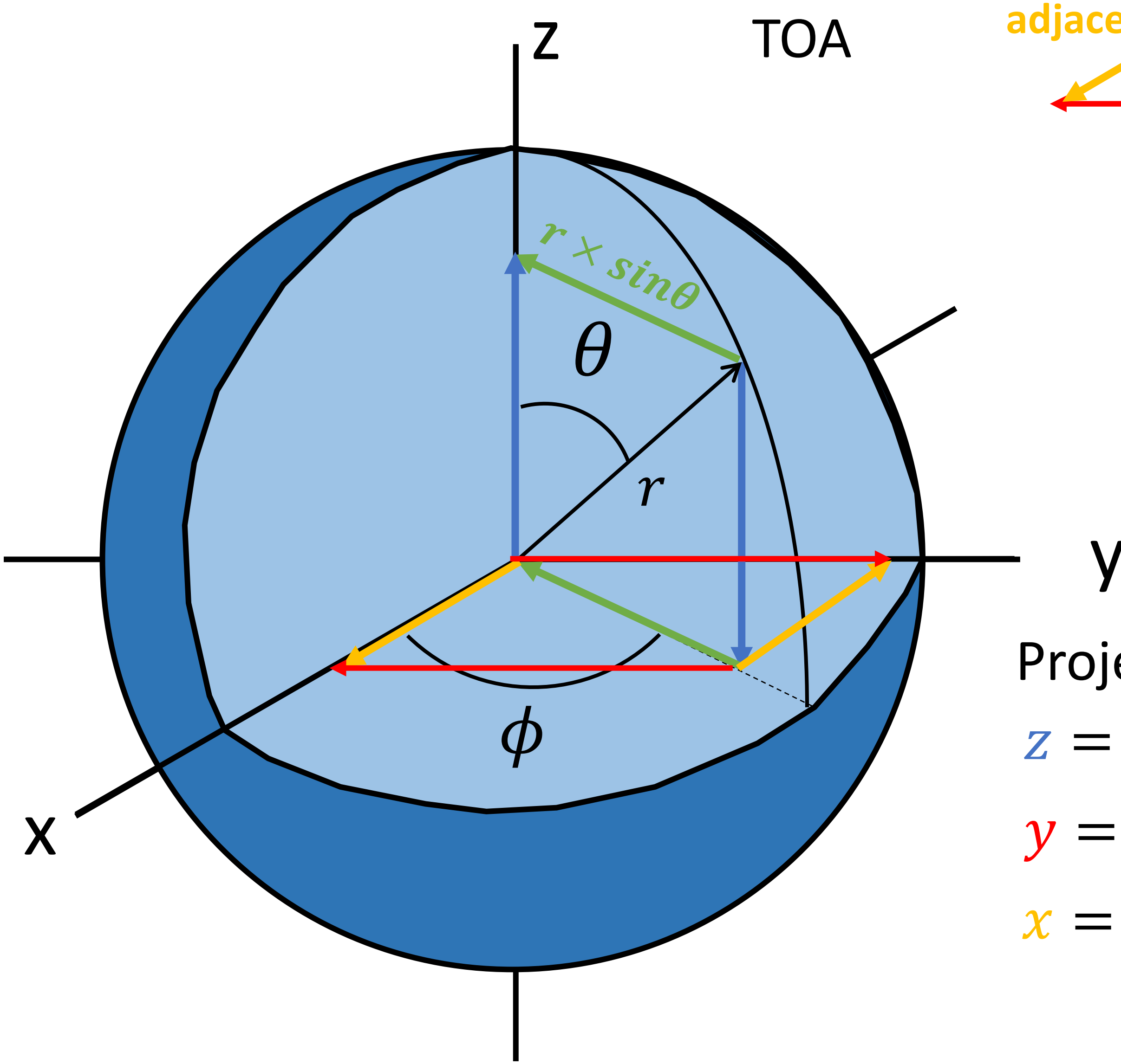
$z = r \times cos\theta$

3d Geometry Review

SOH
CAH
TOA

$$h = r \times sin\theta$$





$$sin\phi = \frac{\mathbf{0}}{r \times sin\theta}$$

$$cos\phi = \frac{\mathbf{a}}{r \times sin\theta}$$

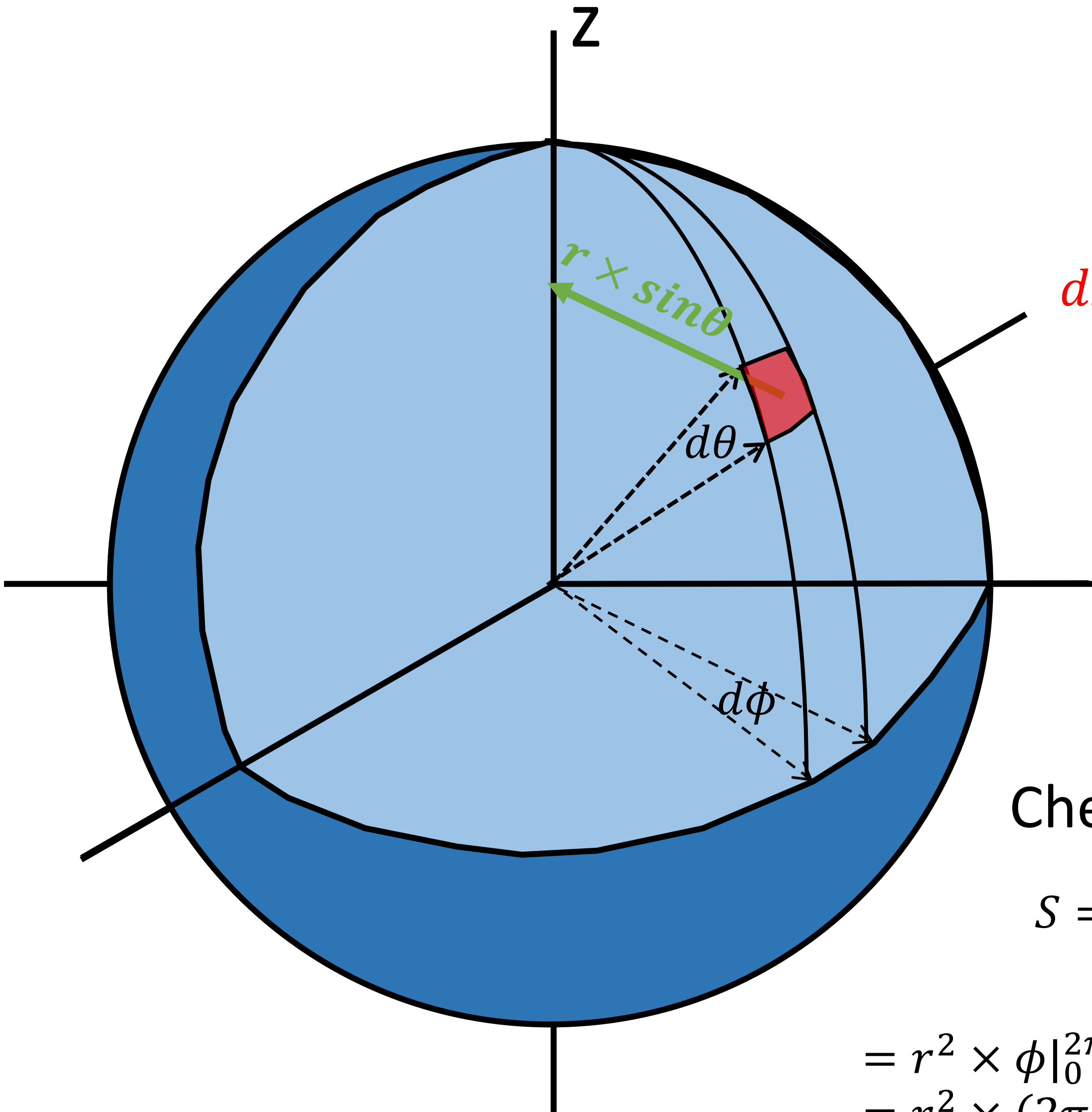
Project to axes

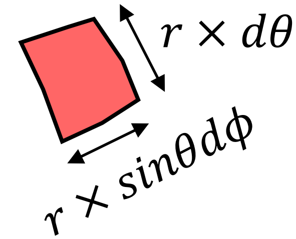
$$z = r \times cos\theta$$

$$y = r \times sin\theta sin\phi$$

$$x = r \times sin\theta cos\phi$$

3d Geometry Review





Surface area

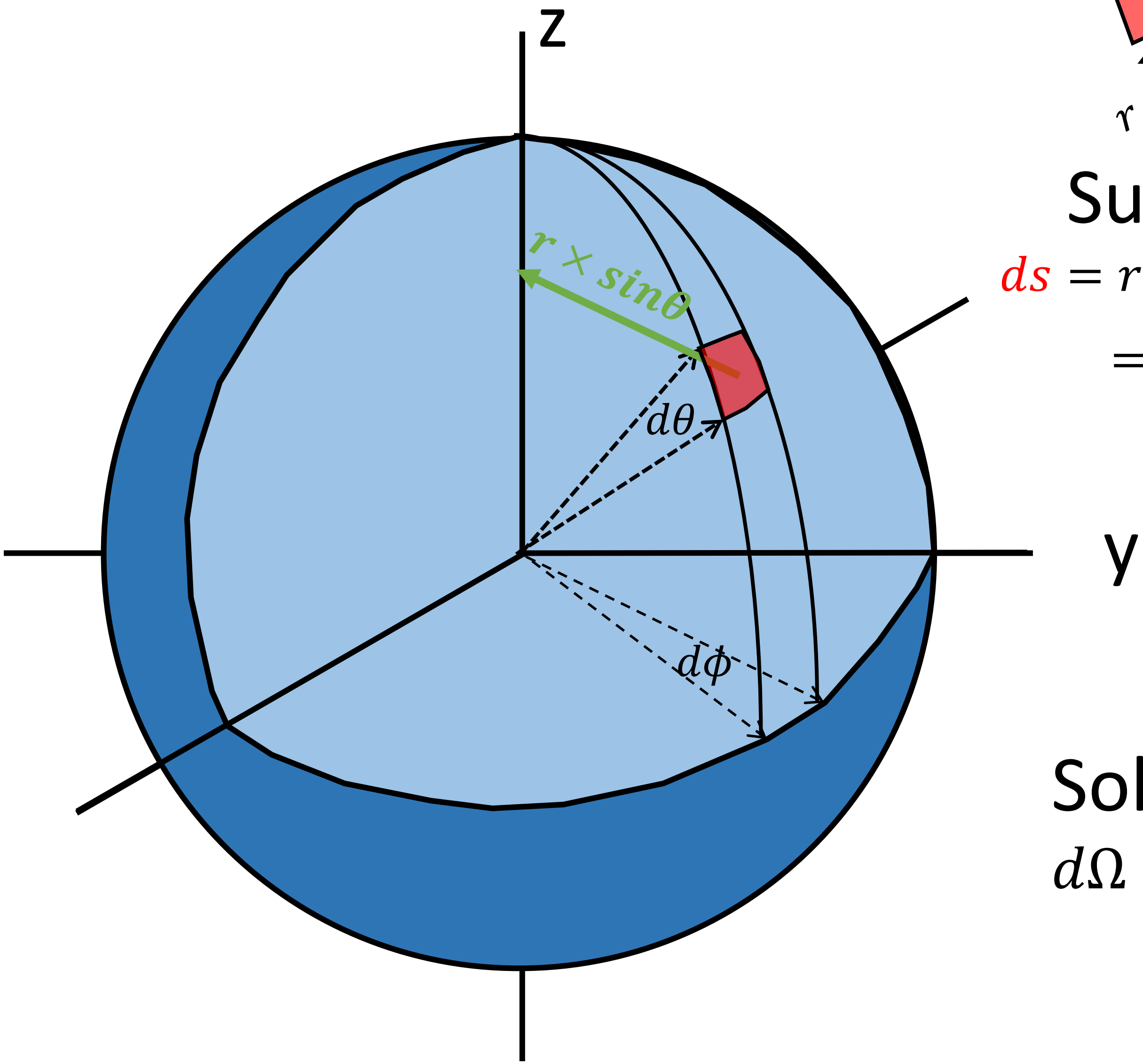
$$ds = r \times d\theta \times r \times sin\theta d\phi$$
$$= r^2 \times sin\theta d\theta d\phi$$

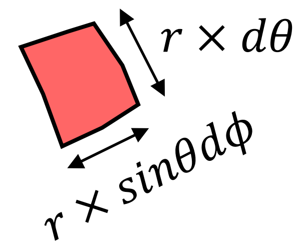
Check by integration

$$S = \int_0^{2\pi} \int_0^{\pi} r^2 \times sin\theta d\theta d\phi$$

$$= r^2 \times \phi|_0^{2\pi} \times -cos\theta|_0^{\pi}$$
$$= r^2 \times (2\pi - 0) \times (-cos\pi - -cos0)$$
$$= 4\pi r^2$$

3d Geometry Review





Surface area

$$ds = r \times d\theta \times r \times sin\theta d\phi$$
$$= r^2 \times sin\theta d\theta d\phi$$

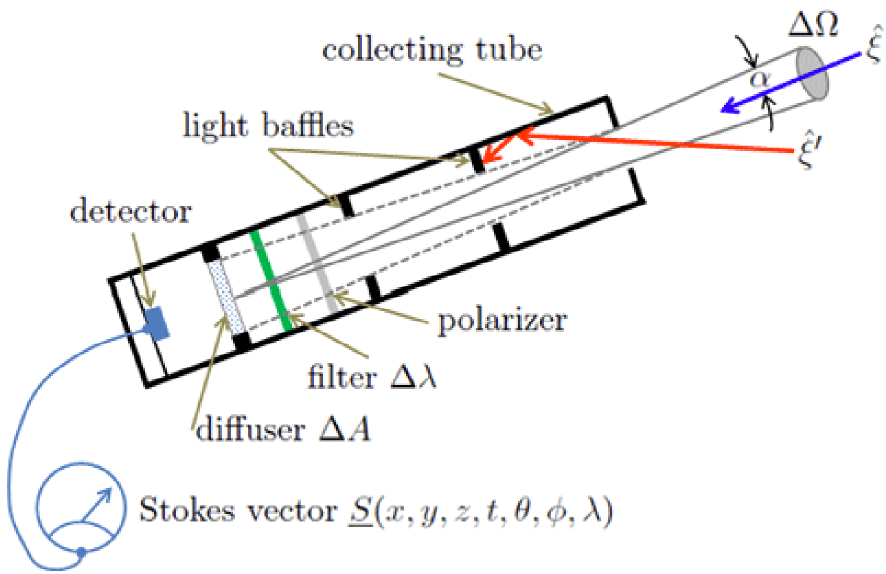
Solid Angle

$$d\Omega = ds/\, r^2$$
$$= sin\theta \; d\theta \; d\phi$$

Outline

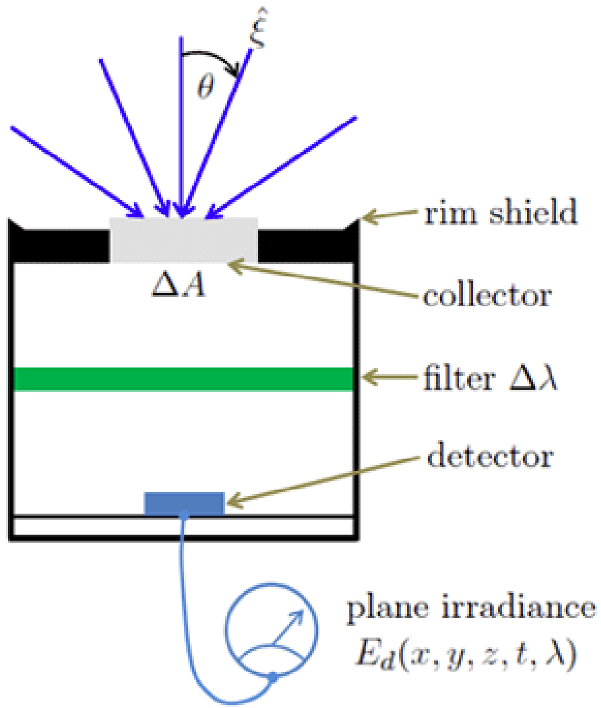
- Review of
  - spherical geometry to derive solid angle
  - **radiometric measurements**
- Derive the Radiative Transfer Equation
- Simplifications of the RTE
  - Gershun's Equation
  - Remote Sensing Reflectance

Radiometric measurements



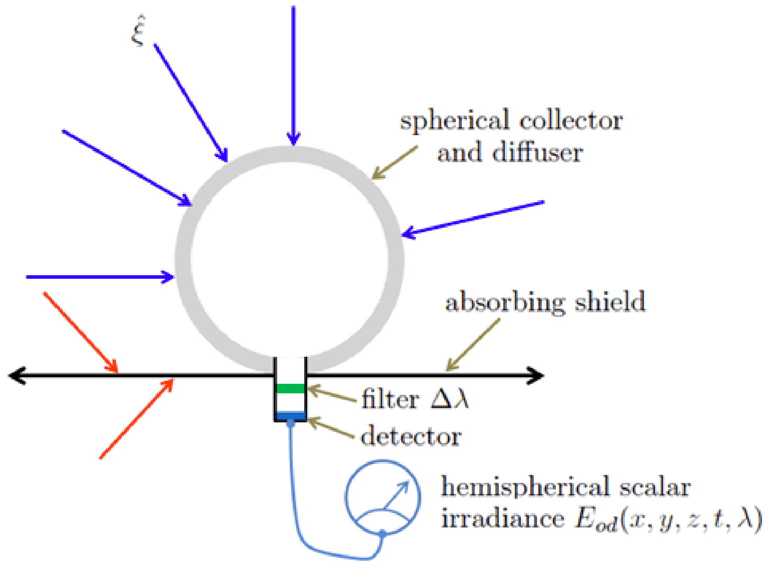
- Radiance
- $L(\theta, \phi)$ (µmol photons m^{-2} s^{-1} sr^{-1})

Radiometric measurements



- Irradiance
- $E_d = \int_0^{2\pi} \int_0^{\pi/2} L(\theta, \phi)\ cos\theta\ d\Omega$ (µmol quanta m^{-2} s^{-1})

Review of radiometric properties



- Scalar Irradiance
- $E_{od} = \int_0^{2\pi} \int_0^{\pi/2} L(\theta, \phi)\, d\Omega$ (μmol photons m^{-2} s^{-1})

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- **Derive the Radiative Transfer Equation**
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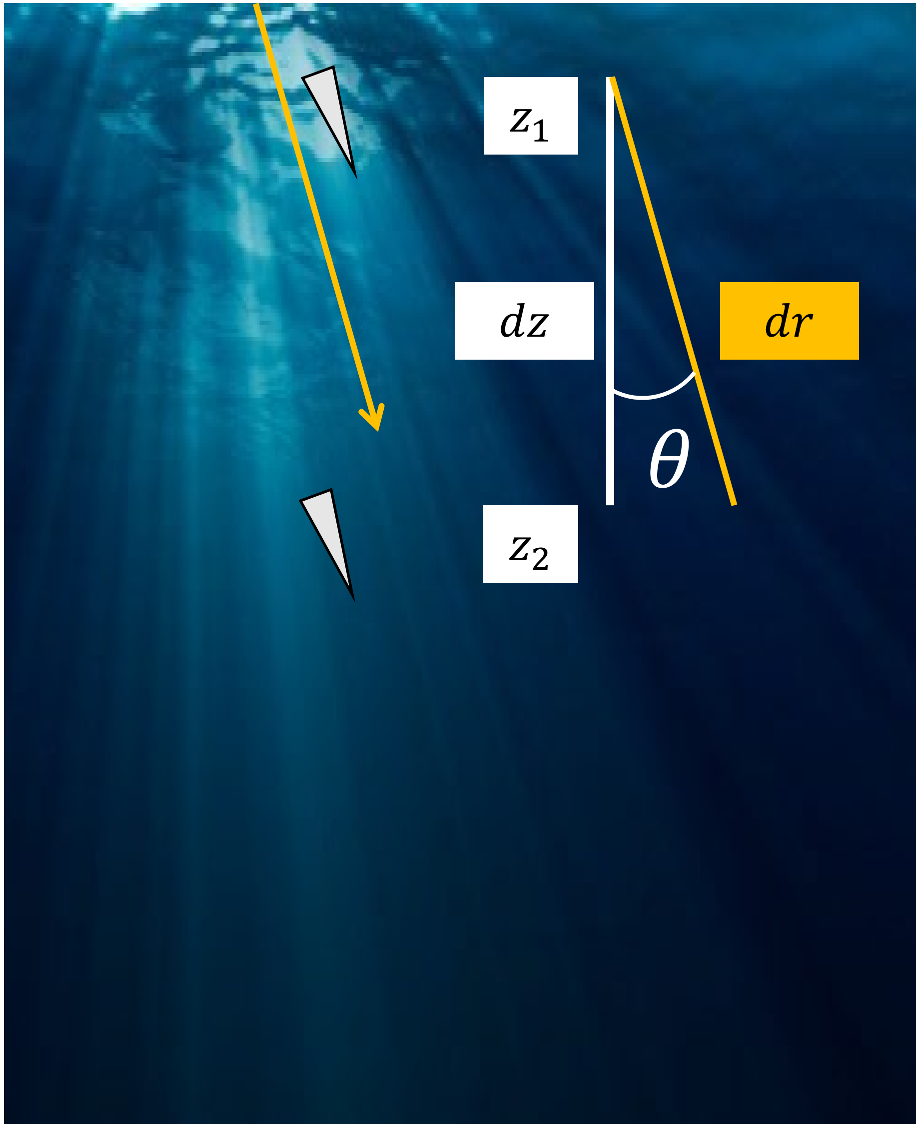
Radiative Transfer Equation

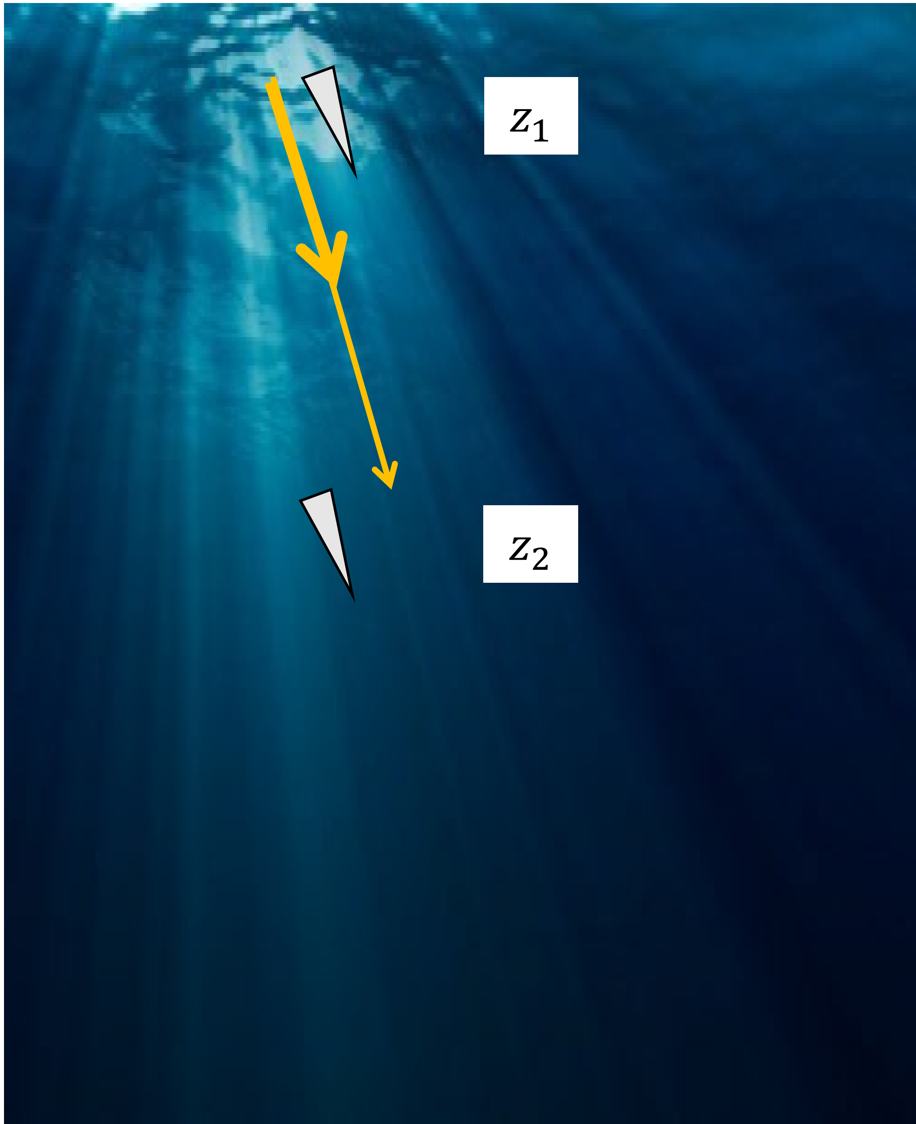
- Describes how radiance changes along the path of propagation, $\frac{dL(z,\lambda,\theta,\phi)}{dr}$
- Place it in the context of measuring a radiance profile
- Note that $cos\theta = dz/dr$

$$\frac{dL(z,\lambda,\theta,\phi)}{dr} = cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz}$$



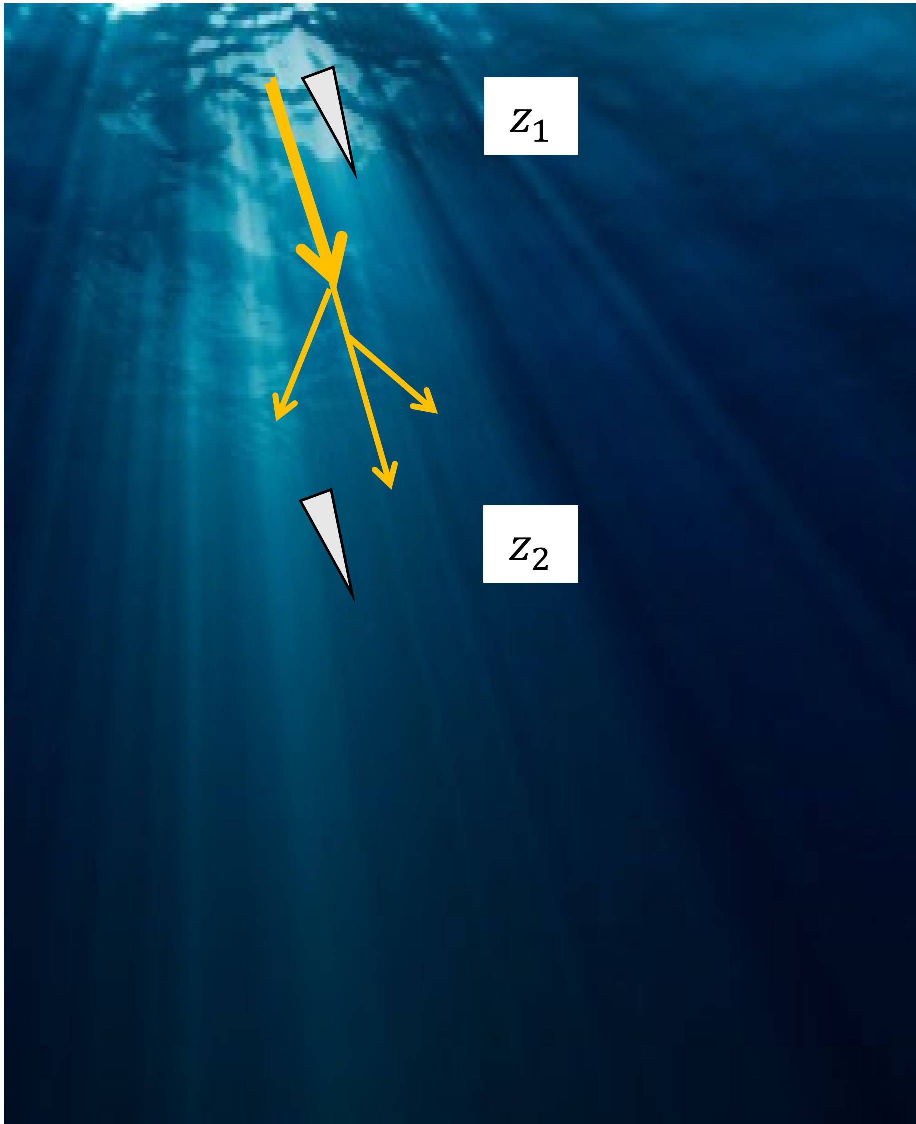
Radiative Transfer Equation

- What processes will impact the radiance?
- Absorption
- The amount of radiance absorbed along the path (loss) is equal to
- $-a(z, \lambda) \times L(z, \lambda, \theta, \phi)$



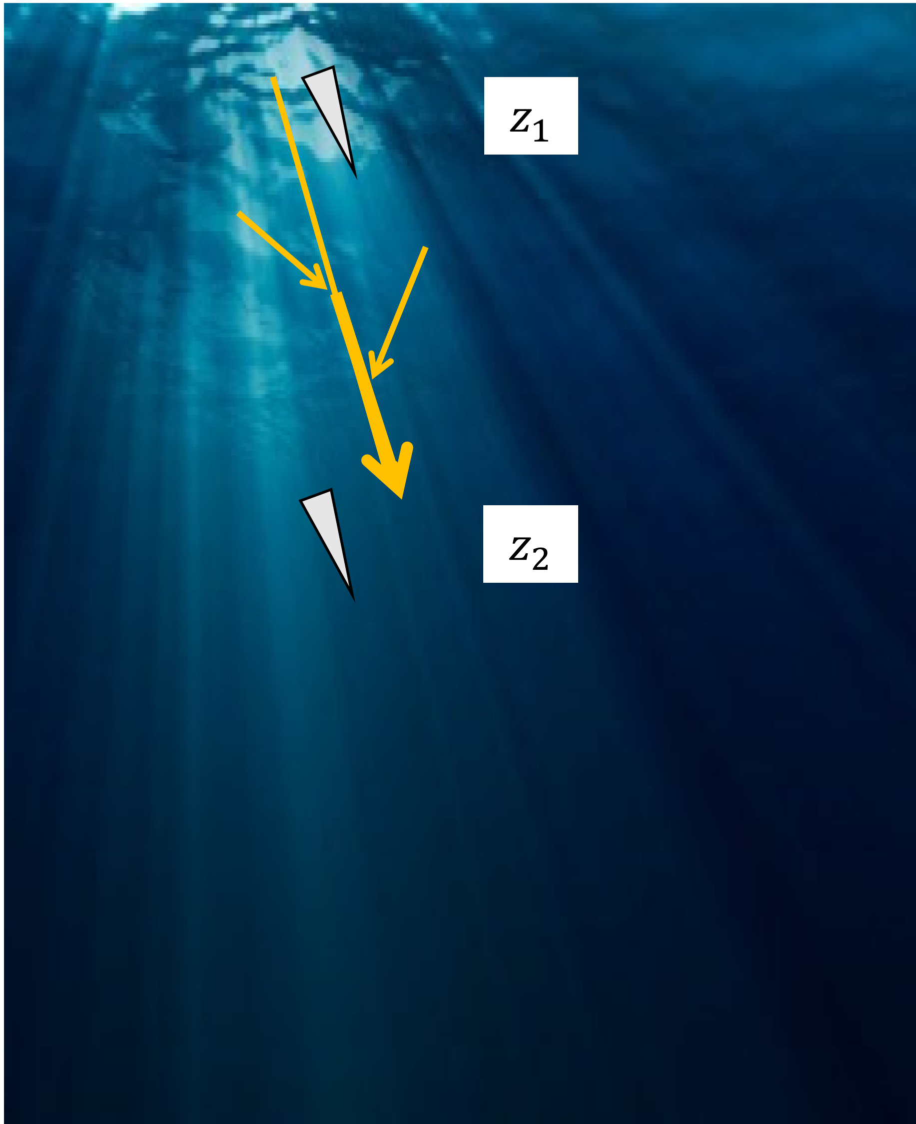
Radiative Transfer Equation

- What processes will impact the radiance?
- Scattering out of path
- The amount of radiance scattered out of the path (loss) is equal to
- $-b(z, \lambda) \times L(z, \lambda, \theta, \phi)$



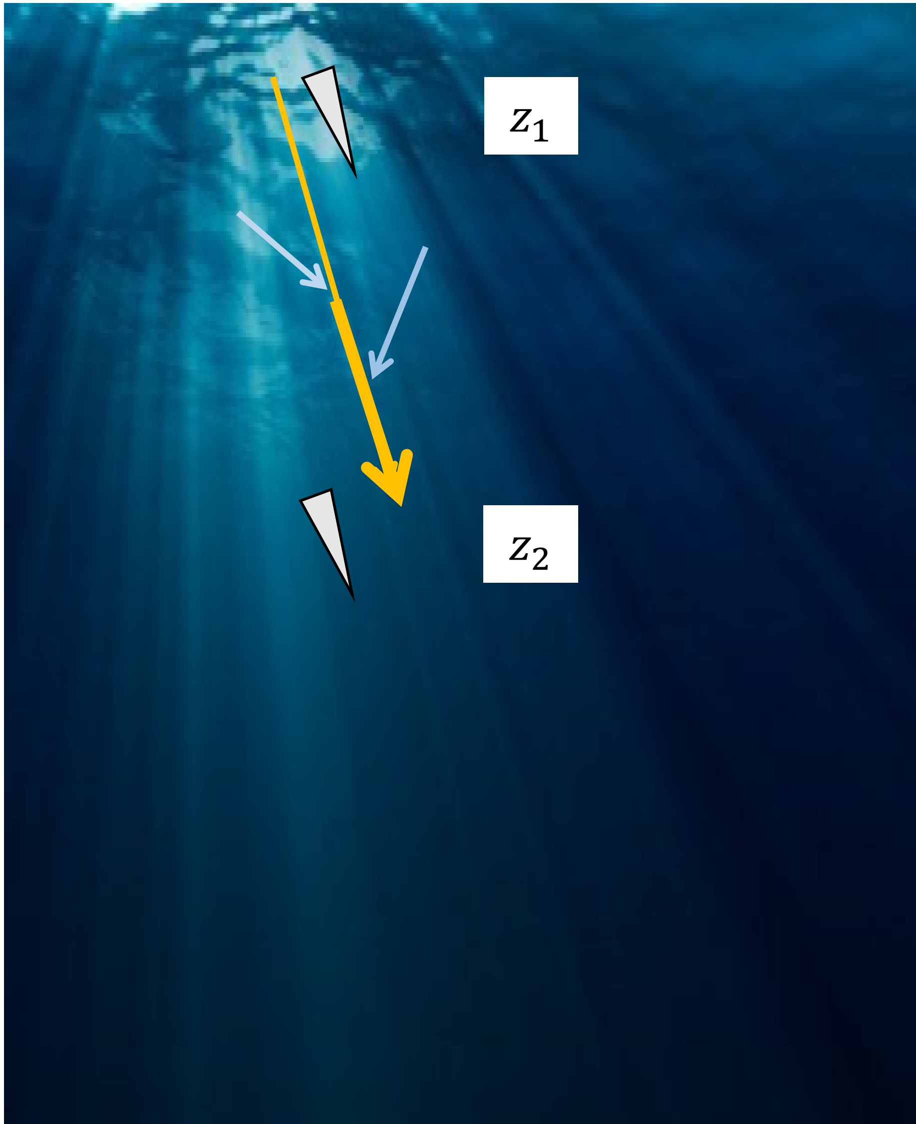
Radiative Transfer Equation

- What processes will impact the radiance?
- Scattering into the path
- The amount of radiance scattered into the path (gain) is equal to
- $\int_{4\pi} \beta(z, \lambda, \theta', \phi' \rightarrow \theta, \phi) \times L(z, \lambda, \theta', \phi') d\Omega'$



Radiative Transfer Equation

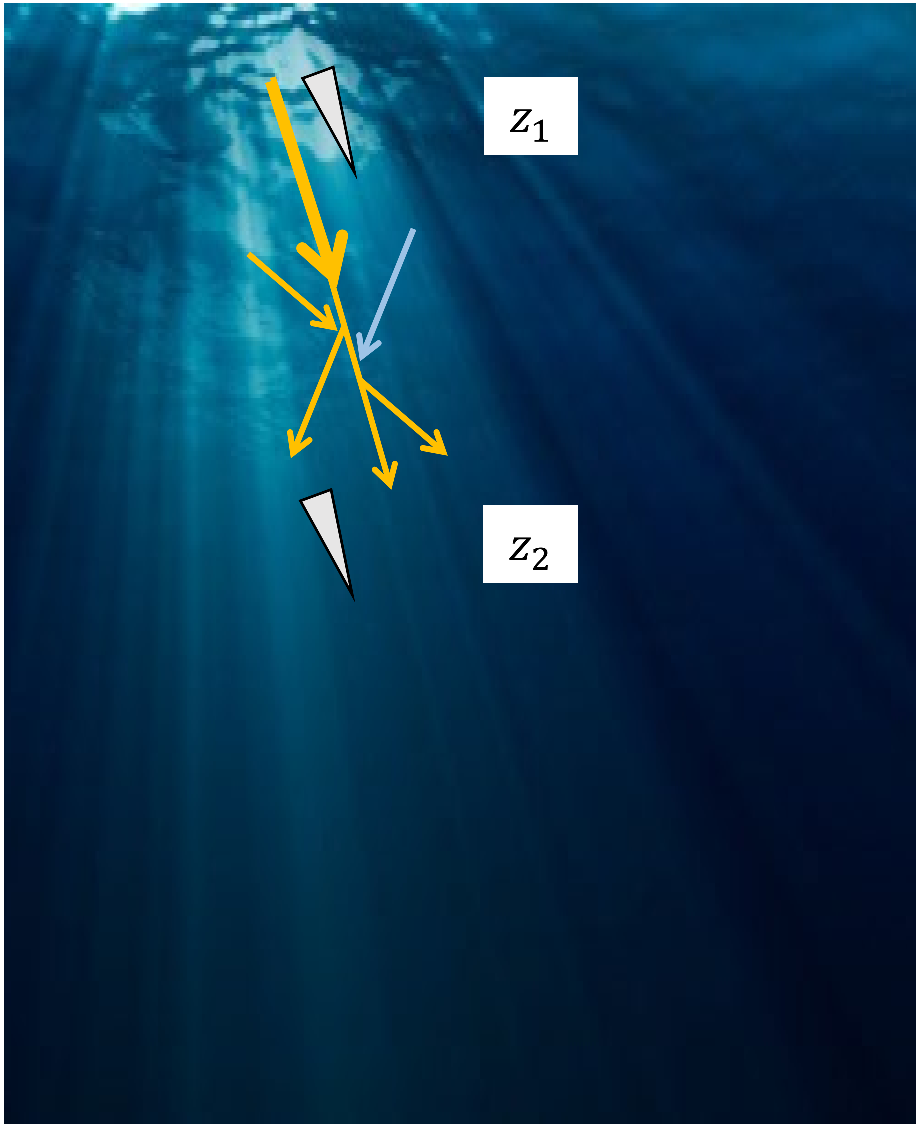
- What processes will impact the radiance?
- Radiance fluoresced into the path in the wavelength of interest
- Fluorescence into the path (gain) is a source term representing a transference of energy from one wavelength to the one of interest
- $S(z, \lambda' \rightarrow \lambda, \theta, \phi)$



Radiative Transfer Equation

- Putting it all together

$$cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz} =$$
$$-a(z,\lambda) \times L(z,\lambda,\theta,\phi)$$
$$-b(z,\lambda) \times L(z,\lambda,\theta,\phi)$$
$$+\int_{4\pi} \beta(z,\lambda,\theta',\phi' \rightarrow \theta,\phi) \times L(z,\lambda,\theta',\phi')d\Omega'$$
$$+S(z,\lambda' \rightarrow \lambda,\theta,\phi)$$



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Approximation to RTE
Gershun's Equation

$$cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz} =$$

$$-c(z,\lambda) \times L(z,\lambda,\theta,\phi) + \int_{4\pi} \beta(z,\lambda,\theta',\phi',\theta,\phi) \times L(z,\lambda,\theta',\phi') d\Omega'$$

- Integrate over all solid angles
- $\int_{4\pi} cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz} d\Omega =$
- $\frac{d\vec{E}(z,\lambda)}{dz}$ where $\vec{E}(z,\lambda) = E_d(z,\lambda) - E_u(z,\lambda)$
  net downward irradiance

Approximation to RTE
Gershun's Equation

$$cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz} =$$

$$-c(z,\lambda) \times L(z,\lambda,\theta,\phi) + \int_{4\pi} \beta(z,\lambda,\theta',\phi'\theta,\phi) \times L(z,\lambda,\theta',\phi')d\Omega'$$

- Integrate over all solid angles
- $\int_{4\pi} -c(z,\lambda) \times L(z,\lambda,\theta,\phi)\, d\Omega =$
- $-c(z,\lambda) \times E_o(z,\lambda)$
- Where $E_o(z,\lambda)$ is the scalar irradiance

Approximation to RTE Gershun's Equation

$$cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz} =$$

$$-c(z,\lambda) \times L(z,\lambda,\theta,\phi) + \int_{4\pi} \beta(z,\lambda,\theta',\phi',\theta,\phi) \times L(z,\lambda,\theta',\phi')d\Omega'$$

- Integrate over all solid angles
- $\int_{4\pi} \int_{4\pi} \beta(z,\lambda,\theta',\phi',\theta,\phi) \times L(z,\lambda,\theta',\phi')d\Omega' \, d\Omega =$
- $b(z,\lambda) \times E_o(z,\lambda)$
- Where $E_o(z,\lambda)$ is the scalar irradiance

Approximation to RTE Gershun's Equation

$$\frac{d\vec{E}(z,\lambda)}{dz} = -c(z,\lambda) \times E_o(z,\lambda) + b(z,\lambda) \times E_o(z,\lambda)$$

$$\frac{d\vec{E}(z,\lambda)}{dz} = -a(z,\lambda) \times E_o(z,\lambda)$$

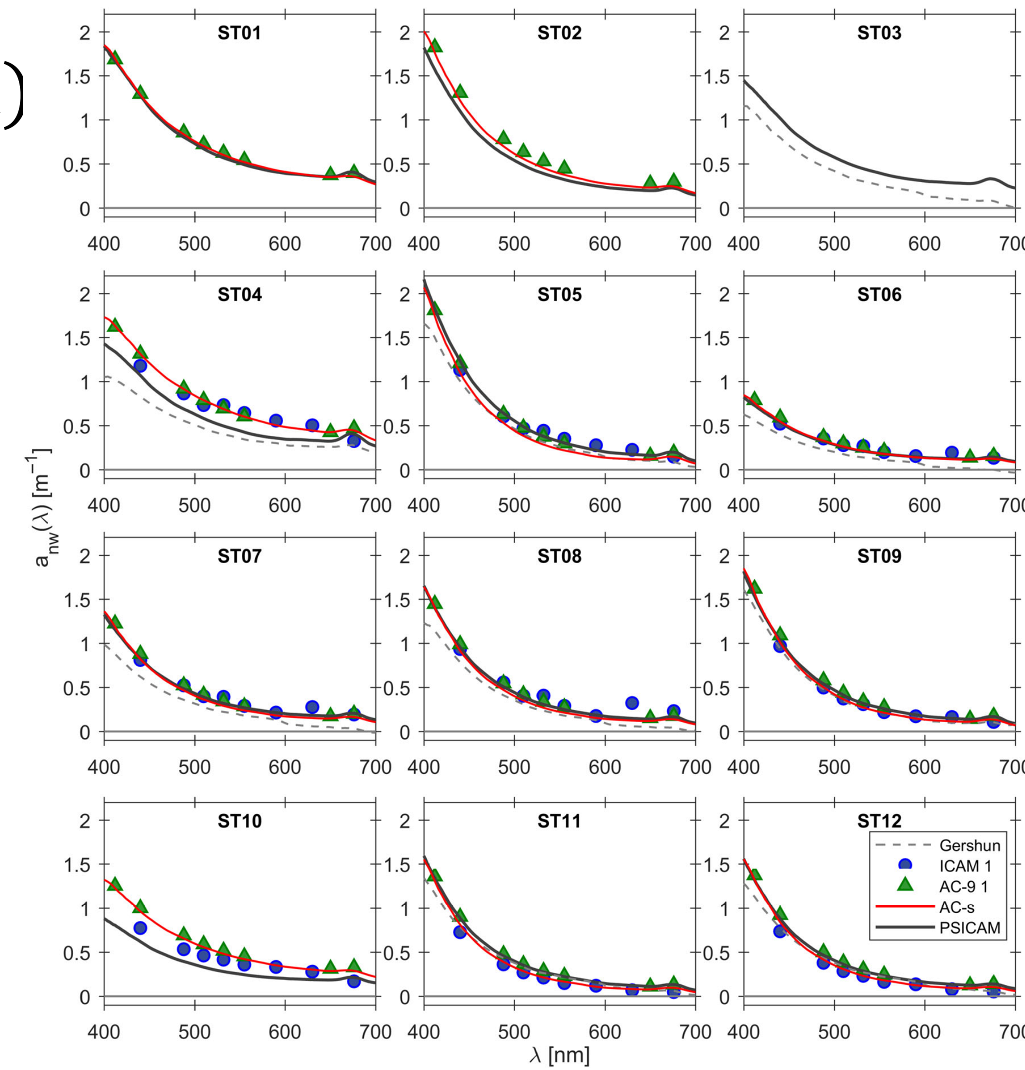
Divide both sides by $-\vec{E}(z,\lambda)$ and solve for $a(z,\lambda)$

$$a(z,\lambda) \times \frac{E_o(z,\lambda)}{\vec{E}(z,\lambda)} = \frac{1}{-\vec{E}(z,\lambda)} \times \frac{d\vec{E}(z,\lambda)}{dz}$$

$$a(z,\lambda) = \vec{K}(z,\lambda) \times \bar{\mu}(z,\lambda)$$

Approximation to RTE Gershun's Equation

$$a(z,\lambda) = \vec{K}(z,\lambda) \times \bar{\mu}(z,\lambda)$$



Kostakis et al. 2021

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Approximation of RTE

$$cos\theta \frac{dL(z,\lambda,\theta,\phi)}{dz} =$$
$$-c(z,\lambda) \times L(z,\lambda,\theta,\phi) + \int_{4\pi} \beta(z,\lambda,\theta',\phi',\theta,\phi) \times L(z,\lambda,\theta',\phi')d\Omega'$$

- successive order scattering
  - separate radiance into unscattered, single scattered, twice scattered… contributions
  - $L(z,\lambda,\theta,\phi) \Rightarrow L_0 + L_1 + L_2 + \cdots + L_3$
- single scattering approximation
  - consider only the unscattered and single scattered radiance terms
  - $L(z,\lambda,\theta,\phi) \Rightarrow L_0 + L_1 + \cancel{L_2 + \cdots + L_3}$

Approximation of RTE

- successive order scattering
- single scattering approximation
- quasi-single scattering approximation
  - noting that the volume scattering functions in the ocean are highly peaked in the forward direction
  - forward scattering *is like no scattering at all*
  - replace b with b_b and c with $a + b_b$
- The math is pretty tricky but solving the approximation means that
  - The upward light field results from single scattering of the downward irradiance, so backscattering, b_b
  - The loss of downward irradiance is due to attenuation, $a + b_b$
  - So upward/downward radiation $R \approx \frac{L_u}{E_d} \approx \frac{b_b}{a+b_b}$
  - What are the assumptions in using this relationship?