Radiative transfer in the environment. Weitzmann, fall 2008.

Problem set I:

Answers (in italics):

1. Calculate the solar constant assuming the temperature of the sun is 5800K.

I will assume the sun to be a black-body radiator. The emmitted radiance from such a radiator integrated over all wavelengths is given by the Stephan-Boltzmann law

$$B(T) = \sigma T^4, \sigma = 5.7x10^{-8} Wm^{-2}K^{-4}$$

This radiance is emitted by each 1m^2 *of the sun, and I will assume the sun to be a Lambertian emmitor, hence:*

$$L(T) = \frac{B(T)}{\pi} \quad (1)$$

This radiance arrives to the top of the atmosphere unattenuated. In order to calculate the irradiance at the top of the atmosphere we have to integrate the radiance times the spatial angle from which this radiance arrives:

$$E_{earth} = \int_{\Omega} L(T) d\Omega \sim L(T) \Delta\Omega = \frac{B(T)}{\pi} \frac{\pi R_S^2}{R_{Earth-sun}^2} \quad (2)$$

Substituting for $R_s \sim 7x10^8$*m,* $R_{Earth-Sun} \sim 1.5x10^{11}$*m, we find:*

Solar constant ~1405Wm^{-2}*, consistent (given our rounding) with 1370Wm*$^{-2}$ *reported in the literature.*

Another way to arrive at the same result (equ. (2)) is to calculate the radiant power that is emitted by the sun (area of the sun times B(T)) and then divide it by a unit area at the position of the Earth.

2. You reached the top of the Everest with a broadband radiometer (UV to far IR). You point it at the center of the sun (your acceptance angle is small enough it is fully within the sun). Given the value of the solar constant, what do you expect the radiance reading to be?

We know that L(T) is conserved along the path. It value is related to the solar constant based on equ. 2 above, e.g.:

$$L = \frac{E_{earth}}{\Delta\Omega} = \frac{E_{earth} R^2_{Earth-sun}}{\pi R_s^2}$$

Asumming and albedo of 1388Wm^{-2} and the values reported above L~2x10^{7} Wm^{-2}sr^{-1} which is consistent with (1).

How would it compare to the radiance you will measure half way between the Earth and sun?

It should be the same since radiance does not change along a ray (unless there is interaction with matter).

3. What is the yearly and spatially *averaged* broadband irradiance reaching the Earth assuming no atmoshphere?

To obtain this value we need to multiply the solar constant by the cross-sectional area of the Earth and divide it by the Earth's surface area. Since for any convex shape, averaged over all orientation (not really the case here, but it does not matter) the average cross-sectional area is a ¼ of the surface area (a classic theorem by Cauchy) which is identically true of a sphere (not too bad of an assumption for the Earth) the value is:

Solar constant/4~1388/4=347 Wm^{2}

4. Assuming no atmosphere and an Earth with an average albedo of 0.3 (that is 30% of the incident radiation is reflected, the other 70% absorbed), what should be the yearly and spatially averaged temperature of the Earth?

Given the answer to 3, the input of energy to the earth per unit area is 0.7x347 Wm^{2}=243W/m^{2}. In thermal equilibrium this energy needs to be radiated away. Assuming the Earth to behave as a blackbody, in equilibrium:

$$\sigma T^4 = 243wm^{-2} \leftrightarrow T{\sim}255K{\sim}-17°\mathrm{C}.$$

Thus, without the atmosphere this planet would not be very hospitable…

5. (From Light and Water, Ch. 1) As a crude approximation, the sky radiance distribution on a clear day can be represnted as a collimated direct solar beam plus an istoropic diffuse sky radiance. Let $L(\theta,\phi)$ be th edirection of the direct beam which contributes a fraction f ($0 \le f \le 1$) to the total sun-plus-sky irradiance E_d. The diffuse sky

radiance contributes *1-f* of the total irradiance. Such a radiance distribution can be written as:

$$L(\theta,\phi) = C\left[f\,\delta(\cos\theta - \cos\theta_s)\,\delta(\phi - \phi_s) + \frac{1-f}{\pi}\right],$$

where *C* is a constant that sets the overall magnitude of *L*.

(a) What are the dimensions of *C*?

C has to have dimensions irradiance or of power per area, hence the division by π *in the second term.*

(b) Compute E_d and E_{od} in terms of *C*.

$$E_d = C\int_0^{2\pi}\int_0^{\pi/2}\left\{f\delta(cos\theta - cos\theta_s)\delta(\phi-\phi_s) + \frac{1-f}{\pi}\right\}cos\theta sin\theta d\theta d\phi =$$

$$= Cf\int_0^{2\pi}\delta(\phi-\phi_s)d\phi\int_0^{\pi/2}\delta(cos\theta - cos\theta_s)cos\theta sin\theta d\theta$$

$$+\frac{2\pi C(1-f)}{\pi}\int_0^{\pi/2}cos\theta sin\theta d\theta = C[fcos\theta_s + (1-f)]$$

$$E_{0d} = C\int_0^{2\pi}\int_0^{\pi/2}\left\{f\delta(cos\theta - cos\theta_s)\delta(\phi-\phi_s) + \frac{1-f}{\pi}\right\}sin\theta d\theta d\phi$$

$$= Cf\int_0^{2\pi}\delta(\phi-\phi_s)d\phi\int_0^{\pi/2}\delta(cos\theta - cos\theta_s)sin\theta d\theta$$

$$+\frac{2\pi C(1-f)}{\pi}\int_0^{\pi/2}sin\theta d\theta = C[f + 2(1-f)] = C[2-f]$$

Note: we made use of the fact that:

$\int_0^{\frac{\pi}{2}}\delta(cos\theta - cos\theta_s)cos\theta sin\theta d\theta = \int_0^1\delta(\mu-\mu_s)\mu d\mu = \mu_s$, while:

$$\int_0^{\pi/2}\delta(cos\theta - cos\theta_s)sin\theta d\theta = \int_0^1\delta(\mu-\mu_s)d\mu = 1$$

(c) Modify the above equation so that $L(\theta,\phi)$ represents a direct beam plus a cardioidal background sky.

A cardioidal-sky radiance obeys:

$$L(\theta,\phi) = L_o(1 + 2\cos\theta), \qquad 0 \le \theta \le \frac{\pi}{2}.$$

Such a radiance distribution might be a better approximation for a day with a uniform overcast, but with the sun's position still discernable through the cloud layer.

(c) Compute E_d and E_{od} for the sun-plus-cardioidal-sky radiance distribution of part (c).

The sky radiance is given by:

$$E_d = L_0 \int_0^{2\pi} \int_0^{\pi/2} \{1 + 2cos\theta\} cos\theta sin\theta d\theta d\phi = \frac{7\pi}{3} L_0$$

$$E_{0d} = L_0 \int_0^{2\pi} \int_0^{\pi/2} \{1 + 2cos\theta\} sin\theta d\theta d\phi = 4\pi L_0$$

Adding the direct sun component (given in b) gives:

$$E_d = E_s cos\theta_s + \frac{7\pi}{3} L_0 \quad E_{0d} = E_s + 4\pi L_0$$

Note E_{0d} is always larger than E_d, sometimes very substantially. Their ratio is termed the 'mean cosine' as exhibited in (b).