Radiative transfer in the environment. Weitzmann, fall 2008.

Problem set III:

0. Provide the answer to the Mie-lab homework.
1. **b. Class example (and homework): let's assume we are dealing with non-absorbing rain drops (m=1.33) and a wavelength in air of 550nm (0.55μm). How does the mass-specific attenuation (*c*=Q_{ext}*cross-section/volume/density*) vary as function of size for drops varying in size D=0.01, 0.1, 1, 10, 100, 1000. Use the Matlab routine callbh.m and the Mie solver (bhmie.m) to get Q_{ext}=Q_c.**

2. **Given a water fraction of 1g/m^3 calculate the transmission ($L/L_0=e^{-cR}$) through a cloud R=1km thick for water distributed uniformly with the above diameters.**

Diameter (μm)	Qc	G [m^2]	volume [m^3]	C_ext [m^2]	c* [m^2/kg]	mass of a single drop [kg]	# of drops/m^3
0.01	0.00000118	7.85398E-17	5.23599E-25	9.2677E-23	0.177	5.23599E-22	1.90986E+18
0.1	0.0114	7.85398E-15	5.23599E-22	8.95354E-17	171	5.23599E-19	1.90986E+15
1	3.93	7.85398E-13	5.23599E-19	3.08661E-12	5895	5.23599E-16	1.90986E+12
10	2.15	7.85398E-11	5.23599E-16	1.68861E-10	322.5	5.23599E-13	1909859317
100	2.03	7.85398E-09	5.23599E-13	1.59436E-08	30.45	5.23599E-10	1909859.317
1000	2.01	7.85398E-07	5.23599E-10	1.57865E-06	3.015	5.23599E-07	1909.859317

beam-c (m^{-1})	transmission 1km
0.000177	0.837779785
0.171	5.44056E-75
5.895	0
0.3225	8.7102E-141
0.03045	5.96668E-14
0.003015	0.049045835

The above results summarize the Mie calculations results for droplets with different diameters. Qc denotes the attenuation (= extinction) efficiency factor, G the drop's cross-section, C_ext its extinction cross section (Qc x G), c the mass specific extinction coefficient (C_ext/volume/density), the number of drops are calculated (N=total mass/mass of single drop) to obtain the attenuation (c = N x C_ext) and finally the transmission τ=exp(-c x 1000m).*

3. How does it explain visibility differences between fog and rain?
   We observe very different attenuations for different size droplets and the same liquid content in the air. Fog droplet size varies from 2-60μm encompassing a large variability in attenuation (smallest droplets have the highest attenuation per mass). Rain drop size are O(1mm), which have 2 order of magnitude less attenuation per mass than 10μm drops.

2. Build a 2-stream radiative transfer model as follows:

   Input irradiance from sky: E_d

   A layer between sky and ground with transmission T_1 and reflectance R_1 and absorbance A_1 (their sum is 1, as they represent probabilities of being transmitted, reflected and absorbed).

   Light reaching the Earth: T_1E_d
   Light reflecting back to sky: $E_u=R_1E_d$
   Light absorbed: A_1E_d

   Assume the ground absorbs all the radiation hitting it. Add one more layer and calculate the properties of this new atmosphere (R_2, T_2, & A_2) as function of the properties of the one layer atmosphere. Note that light can bounce between layers, so it may be useful to first calculate the light field (E_d & E_u) in the middle between the layers.

   Keep doubling the number of layers and find expressions for R_N, T_N, & A_N as function of (R_1, T_1, & A_1) where N is even.

Plot R_N and T_N for N=1,2,4…1024 (log the x-axis) for A_1=0, 0.01, 0.05 and R1=0.001, 0.1 and 0.9 (total on 9 curves for each graph).

This problem was first published by Stokes in the Proceedings of the Royal Society and appears in volume IV of his Mathematical and Physical Papers. If you are having difficulties feel free to consult others for the solution!

First let's solve it for none absorbing layers. Starting with 2 layers we get:
$E_d(middle)=T_1E_d(0)+R_1E_u(middle)$
$E_u(middle)=R_1E_d(middle)$
$\rightarrow E_d(middle)=T_1/(1-R_1^2)\, E_d(0)$, *and* $E_u(middle)=R_1T_1/(1-R_1^2)$

Light reflected from atmosphere $E_u(0) = R_2E_d(0)=R_1E_d(0)+T_1E_u(middle)= (R_1+R_1T_1^2/(1-R_1^2))E_d(0) \rightarrow R_2=(R_1+ R_1T_1^2/(1-R_1^2))$
Light transmitted through atmosphere $T_2E_d(0)= T_1E_d(middle)=T_1^2/(1-R_1^2)\, E_d(0)$
$\rightarrow T_2= T_1^2/(1-R_1^2)$

Without absorption, $T_1+R_1=1 \rightarrow R_2=2R_1/(1+R_1)$, $T_2=(1-R_1)/(1+R_1)$

Since the same argument works if we have two layers each with R_2 *and* T_2 *(hence 4 layers),* $R_4=2R_2/(1+R_2)=4R_1/(1+3R_1)$ *and* $T_4=(1-R_1)/(1+3R_1)$. *Check:* $T_4=1- R_4$.

By recursion, for an even N, $R_N=NR_1/(1+(N-1)R_1)$, $T_N=(1-R_1)/(1+(N-1)R_1)$.
As $N\rightarrow\infty$ $R_N\sim NR_1/(1+NR_1) \rightarrow 1$, $T_N\sim(1-R_1)/(1+NR_1) \rightarrow 0$.

Now that we have some experience we can add absorption. The only difference with the above analysis is that we cannot assume that $T_1+R_1=1$, *but rather* $T_1+R_1+A_1=1$. *Hence:*
$R_2=(R_1+ R_1T_1^2/(1-R_1^2))$, $T_2= T_1^2/(1-R_1^2)$ *and* $A_2=1-R_2-T_2$. *Using excel or Matlab we keep doubling the layers and obtain* A_N, R_N, *and* T_N *as function of* A_1, R_1, *and* T_1 *(a closed form solution as we found above was not found (see Matlab program appended).*

Note how a little bit of absorption affects the asymptotic values for reflectivity for N layers and how, as N increases, the transmission through N layers always asymptote zero. Increase absorptivity also increases the rate by which the asymptotic state is achieved.

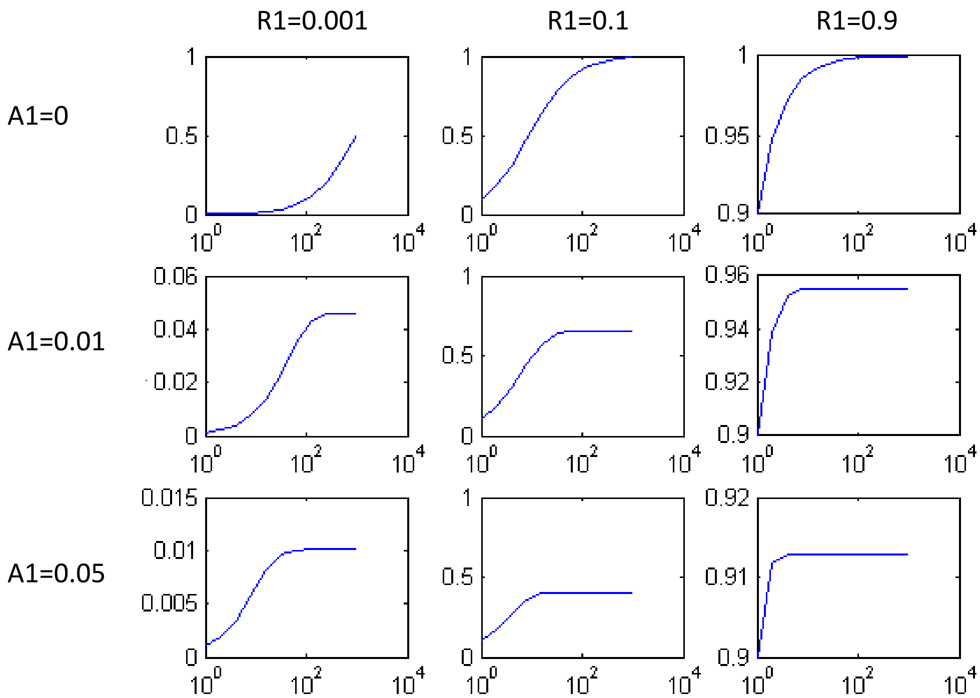

Figure 1. Reflectivity as function of number of layers for different values of absorptivity and reflectivity of a single layer.

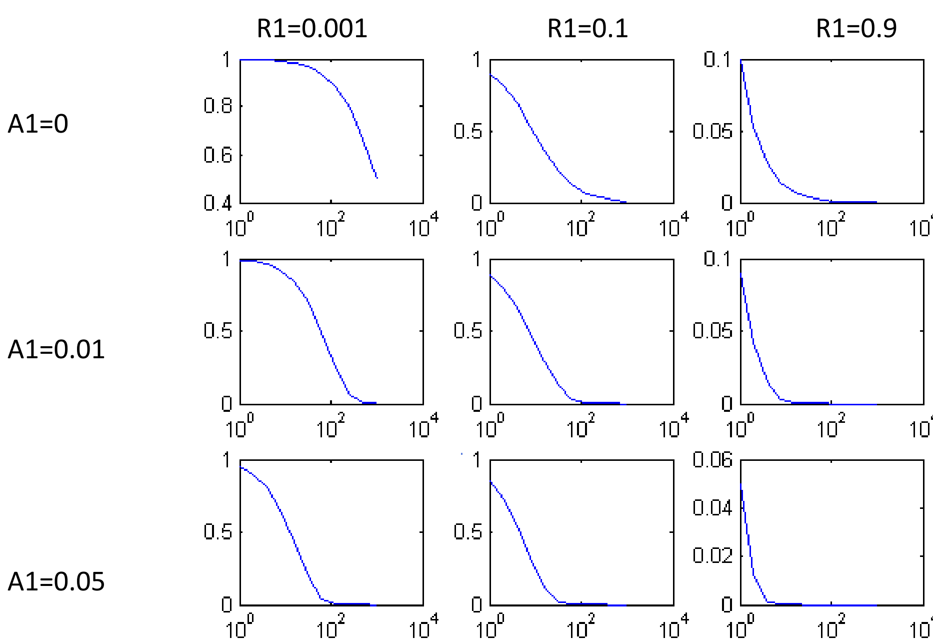

Figure 2. Transmisivity as function of number of layers for different values of absorptivity and reflectivity of a single layer.

```matlab
%program to solve question 3 in homework 3

A1=[0   0.01    0.05];
R1=[0.001 0.1   0.9];

for i=1:3
    for j=1:3
        Rn(1)=R1(i);
        An(1)=A1(j);
        Tn(1)=1-An(1)-Rn(1);
        n=1;
        for k=1:10
            n(k+1)=2^k;
            Rn(k+1)=Rn(k)+Rn(k)*Tn(k)^2/(1-Rn(k)^2);
            Tn(k+1)=Tn(k)^2/(1-Rn(k)^2);
            An(k+1)=1-Tn(k)-Rn(k);
        end
        figure(1)
        subplot(3,3,(j-1)*3+i)
        semilogx(n,Rn);
        figure(2)
        subplot(3,3,(j-1)*3+i)
        semilogx(n,Tn);
        figure(3)
        subplot(3,3,(j-1)*3+i)
        semilogx(n,An);
    end
end
```