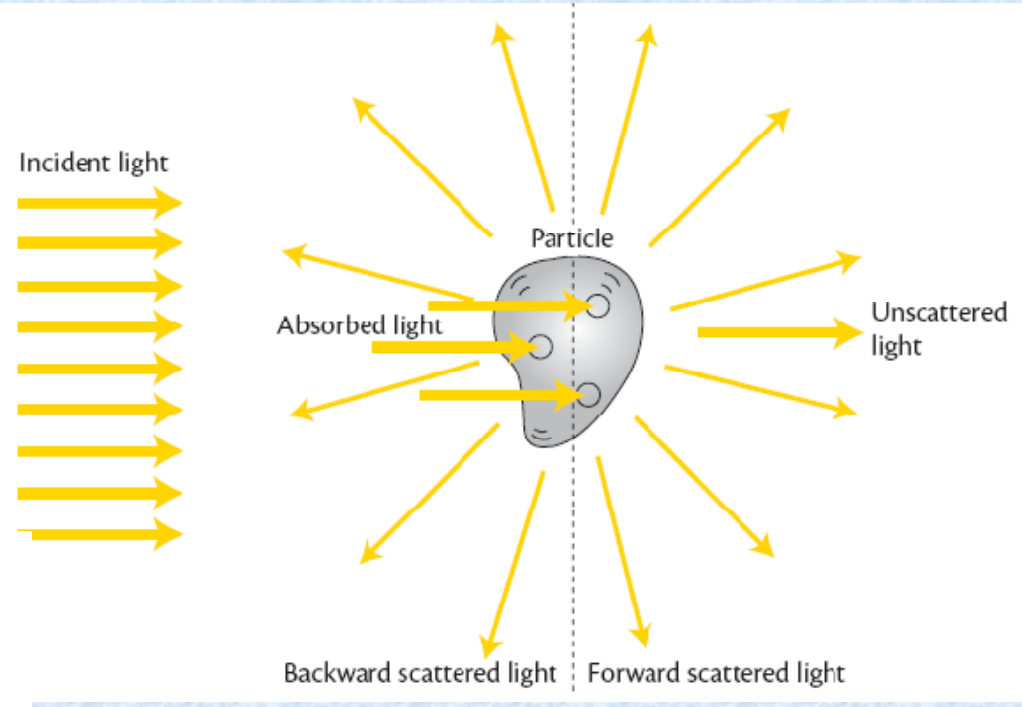
Lecture/Lab 5: Interaction of light with particles. Mie's solution.

Scattering of light by spherical particles (Mie scattering).

The problem (Bohren and Huffman, 1983):

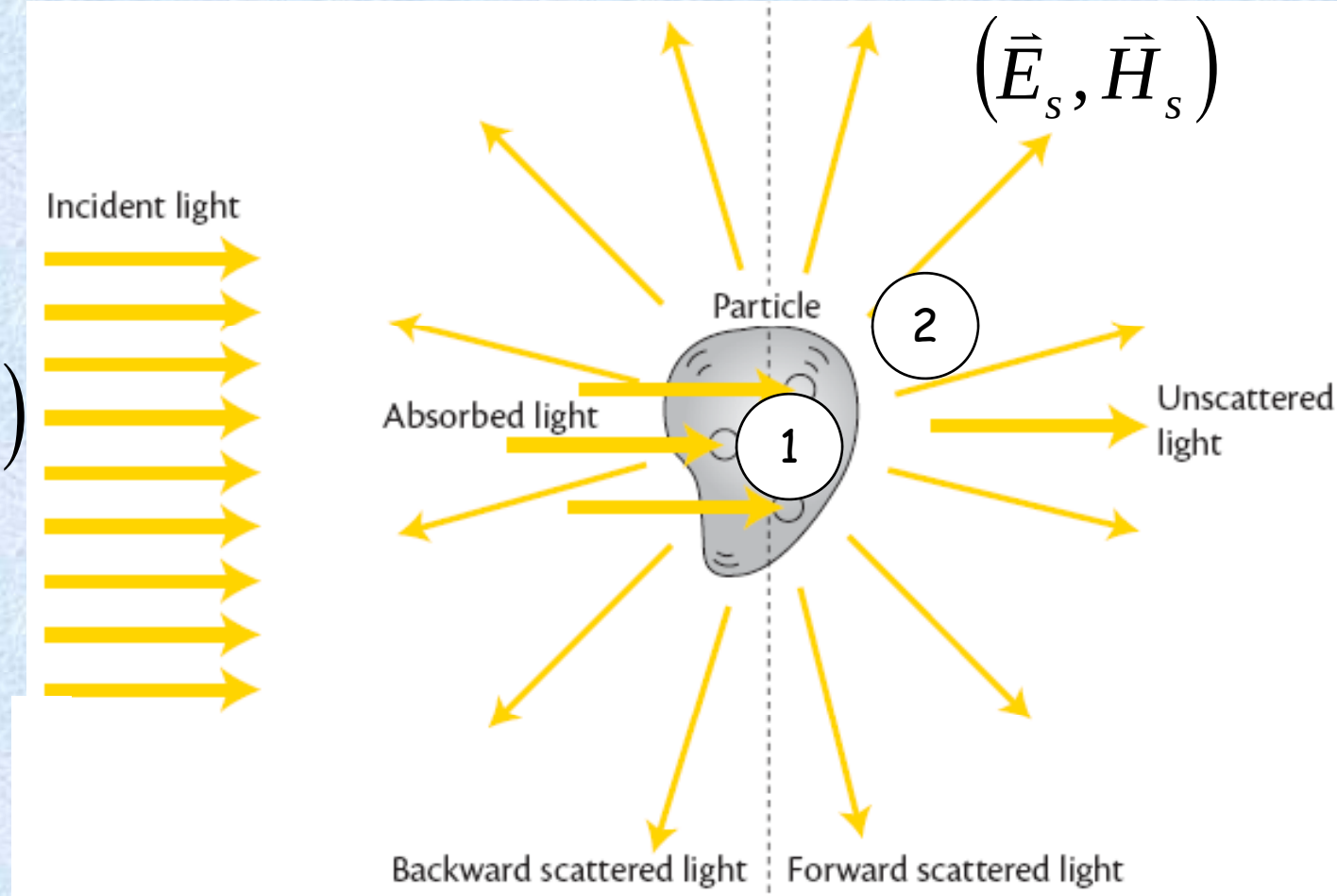
Given a particle of a specified size, shape and optical properties that is illuminated by an arbitrarily polarized monochromatic wave, determine the electromagnetic field at all points in the particles and at all points of the homogeneous medium in which it is embedded.



We will assume that the wave is plane harmonic wave and a spherical particles.

Define four fields: $\left(\vec{E}_1, \vec{H}_1\right), \left(\vec{E}_2, \vec{H}_2\right), \left(\vec{E}_i, \vec{H}_i\right), \left(\vec{E}_s, \vec{H}_s\right)$

$\left(\vec{E}_i, \vec{H}_i\right)$



Outside the particle: $\vec{E}_2 = \vec{E}_i + \vec{E}_s, \vec{H}_2 = \vec{H}_i + \vec{H}_s$

Plane parallel harmonic wave:

$$\vec{E}_i = \vec{E}_0 \exp\left(i\left(\vec{k} \cdot x - \omega t\right)\right)$$
$$\vec{H}_i = \vec{H}_0 \exp\left(i\left(\vec{k} \cdot x - \omega t\right)\right)$$

Must satisfy Maxwell's equation where material properties are constant:

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{H} = 0$$
$$\nabla \times \vec{E} = i\omega\mu\vec{H}$$
$$\nabla \times \vec{H} = -i\omega\varepsilon\vec{E}$$

ε is the *permittivity* μ is the *premeability*.

define:

$$k^2 = \varepsilon\mu\omega^2$$

The vector equation reduce to:

$$\nabla^2\vec{E} + k^2\vec{E} = 0$$

$$\nabla^2\vec{H} + k^2\vec{H} = 0$$

Boundary conditions: the tangential components of the electric and magnetic fields must me continuous across the boundary of the particle (analogous to energy conservation):

$$\left[\vec{E}_2 - \vec{E}_1\right]\times\hat{n} = 0$$

$$\left[\vec{H}_2 - \vec{H}_1\right]\times\hat{n} = 0$$

The equations and BCs are linear → superposition of solutions is a solution.

An arbitrarily polarized light can be expressed as a supperposition of two orthogonal polarization states:

Amplitude scattering matrix

$$\begin{pmatrix} E_{l,s} \\ E_{r,s} \end{pmatrix} = \frac{\exp(ik(r-z))}{-ikr} \begin{pmatrix} S_2(\theta,\varphi) & S_3(\theta,\varphi) \\ S_4(\theta,\varphi) & S_1(\theta,\varphi) \end{pmatrix} \begin{pmatrix} E_{l,i} \\ E_{r,i} \end{pmatrix}$$

For spheres:

$$\begin{pmatrix} E_{l,s} \\ E_{r,s} \end{pmatrix} = \frac{\exp(ik(r-z))}{-ikr} \begin{pmatrix} S_2(\theta) & 0 \\ 0 & S_1(\theta) \end{pmatrix} \begin{pmatrix} E_{l,i} \\ E_{r,i} \end{pmatrix}$$

$$\Rightarrow \begin{aligned} E_{l,s} &= \frac{\exp(ik(r-z))}{-ikr} S_1(\theta) E_{l,i} \\ E_{r,s} &= \frac{\exp(ik(r-z))}{-ikr} S_2(\theta) E_{r,i} \end{aligned}$$

Taking the real part of the squares of the electric fields we get the radiant intensity [W Sr^{-1}]:

$$\Rightarrow \begin{aligned} I_{r,s} &= \frac{\left|S_1(\theta)\right|^2}{k^2 r^2} I_{r,i} \\ I_{l,s} &= \frac{\left|S_2(\theta)\right|^2}{k^2 r^2} I_{l,i} \end{aligned}$$

For unpolarized light:

$$I_s = \frac{\frac{1}{2}\left\{\left|S_1(\theta)\right|^2 + \left|S_2(\theta)\right|^2\right\}}{k^2 r^2} I_i$$

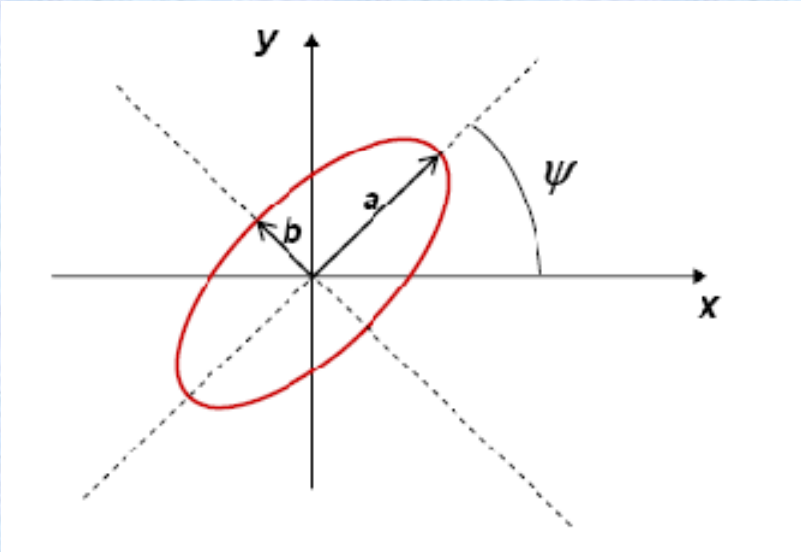
Polarization:

Describe the plane of propagation and phase of the EM radiation.

Assume a wave propagating in the z-direction, and the observer is at z=0.

General Case (A, ϕ):

$$\vec{E}(\vec{r},t)\Big|_{z=0} = E_o\left[\hat{x}\cos(\omega t) + \hat{y}\,A\cos(\omega t+\phi)\right]$$



Shape and Orientation of the Ellipse:

$$\tan(2\psi) = \frac{2A}{1-A^2}\cos(\phi)$$

$$a^2 + b^2 = 1 + A^2$$

$$ab = A\sin(\phi)$$

If $\chi = \tan^{-1}\left(\frac{b}{a}\right)$ then,

$$\sin(2\chi) = \frac{2A}{1+A^2}\sin(\phi)$$

http://instruct1.cit.cornell.edu/Courses/ece303/lecture15.pdf

Stokes notation and the scattering matrix:

$$I_s = \frac{k}{2\omega\mu}\left\langle \left|E_{l,s}\right|^2 + \left|E_{r,s}\right|^2 \right\rangle$$

No polarizers

$$Q_s = \frac{k}{2\omega\mu}\left\langle \left|E_{l,s}\right|^2 - \left|E_{r,s}\right|^2 \right\rangle$$

Horizontal - vertical polorizers

$$U_s = \frac{k}{2\omega\mu}\left\langle E_{l,s}E^*_{r,s} + E_{r,s}E^*_{l,s} \right\rangle$$

45° - (-45°) polorizers

$$V_s = i\frac{k}{2\omega\mu}\left\langle E_{l,s}E^*_{r,s} - E_{r,s}E^*_{l,s} \right\rangle$$

Right handed – left handed circular polorizers

$$\begin{pmatrix} I_s \\ Q_s \\ U_s \\ V_s \end{pmatrix} = \frac{1}{k^2r^2}\begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}\begin{pmatrix} I_i \\ Q_i \\ U_i \\ V_i \end{pmatrix}$$

For a sphere:

$$\begin{pmatrix} I_s \\ Q_s \\ U_s \\ V_s \end{pmatrix} = \frac{1}{k^2 r^2} \begin{pmatrix} S_{11} & S_{12} & 0 & 0 \\ S_{12} & S_{11} & 0 & 0 \\ 0 & 0 & S_{33} & S_{34} \\ 0 & 0 & -S_{34} & S_{33} \end{pmatrix} \begin{pmatrix} I_i \\ Q_i \\ U_i \\ V_i \end{pmatrix}$$

Link to amplitude scattering matrix:

$$\begin{pmatrix} E_{l,s} \\ E_{r,s} \end{pmatrix} = \frac{\exp(ik(r-z))}{-ikr} \begin{pmatrix} S_2(\theta) & 0 \\ 0 & S_1(\theta) \end{pmatrix} \begin{pmatrix} E_{l,i} \\ E_{r,i} \end{pmatrix}$$

$$S_{11} = \left\{ |S_1|^2 + |S_2|^2 \right\}, \qquad S_{12} = \left\{ |S_1|^2 - |S_2|^2 \right\}$$

$$S_{33} = \left\{ S_2^* S_1 + S_1^* S_2 \right\}, \qquad S_{34} = \left\{ S_2^* S_1 - S_1^* S_2 \right\}$$

$$S_{11}^2 = S_{12}^2 + S_{33}^2 + S_{34}^2, \qquad P \equiv -\frac{S_{12}}{S_{11}}$$

Solution method:

Expand incident and scattered fields in spherical harmonic functions for each polarization. Match solutions on boundary of particle and require them to be finite at large distances.

<u>Input to Mie code:</u>

Wavelength in medium (λ).

Size (diameter, D) in the same units as wavelength.

Index of refraction relative to medium ($n + in'$).

<u>Solution depends on:</u>

Size parameter: $\pi D/\lambda$

Index of refraction relative to medium

Output to Mie code:

Efficiency factors:

Q_a, Q_c (also called Q_{ext})

Scattering matrix elements:

S_1 and S_2

From which we can calculate:

$Q_b = Q_c - Q_a$

$\beta \propto S_{11} = |S_1|^2 + |S_2|^2$

Other polarization scattering matrix elements:

$S_{12} \propto |S_1|^2 - |S_2|^2$

$P = -S_{12}/S_{11}$

$S_{33} = \mathrm{Real}(S_2 \times S_1^*)/S_{11}$

$S_{34} = \mathrm{Imag}(S_2 \times S_1^*)/S_{11}$

Populations of particles:

Monodispersion- example, obtaining the scattering coefficient:

$b=NQ_bG$, $G=\pi D^2/4$. Watch for units!

Polydispersion: discrete bins:

$b=\Sigma N_i Q_{b,i} G_i$

When using continuous size distribtion:

$$N(D,\Delta D)=\int_{D-\Delta D/2}^{D+\Delta D/2} f(D)dD$$

$$b=\frac{\pi}{4}\int_{D_{\min}}^{D_{\max}} Q_b(D)D^2 f(D)dD$$

Similar manipulations are done to obtain the absorption and attenuation coefficients, as well as the population's volume scattering function.

Resources (among many others…):

Barber and Hill, 1990.

Bohren and Huffman, 1983.

Kerker, 1969.

Van de Hulst, 1981 (original edition, 1957).

Codes (among many others):
http://www.iwt-bremen.de/vt/laser/wriedt/index_ns.html