Particle dynamics class, SMS 618, (Emmanuel Boss 11/19/2003)
Hand out: Basic gravity wave theory (2-D, x-z):

Assume a homogeneous inviscid fluid at rest. The hydrostatic balance implies that the pressure due to water at depth z is:

$$p_0(z)=\rho g z. \tag{1}$$

A wave introduces a perturbation of the free surface, $z=\eta(x,t)$. Define,

$$p'(x,z,t)=p-\rho g z. \tag{2}$$

Non-divergence implies that the velocity field (u,w) obeys:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{3}$$

Linearizing the equation of motion in the horizontal and vertical we find:

$$\begin{aligned}\rho\frac{\partial u}{\partial t}&=-\frac{\partial p'}{\partial x}\\ \rho\frac{\partial w}{\partial t}&=-\frac{\partial p'}{\partial z}\end{aligned}. \tag{4a,b}$$

Differentiating the (4a) with repect to x and (4b) wrt z and using (3) we find p’ to obey Laplace’s equation:

$$\frac{\partial^2 p'}{\partial x^2}+\frac{\partial^2 p'}{\partial z^2}=0. \tag{5}$$

The boundary conditions are as follows:

1. w=0 @ z=-H (no flow into the bottom).
2. At the surface, $w=\partial\eta/\partial t$ @ $z=\eta$.

In addition the pressure must vanish at the free surface, i.e.

3. $p=p_0+p'=0$ ➔ $p'=-\rho g\eta$ @ $z=\eta$.

Since we are assuming linear (and thus small) solutions, we can replace $z=\eta$ by z=0 for the BCs.
Assuming simple sinusoidal wave solutions, $\eta=\eta_0\cos(\kappa x-\omega t)$, with phase speed $c=\omega/\kappa$. $\omega=2\pi/T$ is the frequency, and $\omega=2\pi/\lambda$ is the wavenumber. Assuming that p’ is proportional to η (basically that it is also periodic in x and t), we substitute into (5):

$$\frac{\partial^2 p'}{\partial z^2} - \kappa^2 p' = 0 \tag{6}$$

The solution of this equation that satisfies the BCs is:

$$p' = \frac{\rho g \eta_0}{\cosh(\kappa H)} \cosh(\kappa(z+H))\cos(\kappa x - \omega t)$$

$$w = \frac{\kappa g \eta_0}{\omega \cosh(\kappa H)} \sinh(\kappa(z+H))\sin(\kappa x - \omega t) \tag{7}$$

$$u = \frac{\kappa g \eta_0}{\omega \cosh(\kappa H)} \cosh(\kappa(z+H))\cos(\kappa x - \omega t)$$

where: $\sinh(y)=(e^y-e^{-y})/2$, $\cosh(y)=(e^y+e^{-y})/2$ (see appendix).
Note that u(z=-H) is not equal to zero. That is because our equations neglected friction. In order to satisfy the 3rd condition, the wavenumber and frequency have to satisfy the dispersion relation:

$$\omega^2 = \kappa g \tanh(\kappa H). \tag{8}$$

Near the bottom friction dominates and we have to match the above solution with one where u→0 as z→-H. That region where friction is important is the wave boundary layer. A solution due to Stokes is:

$$u = \frac{\kappa g \eta_0}{\omega \cosh(\kappa H)} \cosh(\kappa(z+H))\cos(\kappa x - \omega t)\left\{1 - \exp\left(-z\sqrt{\omega/2\nu}\right)\cos\left(\sqrt{\omega/2\nu}\,z\right)\right\}. \tag{9}$$

This solution not only decays to the bottom but is out of phase with the fluid above (as observed in wave tanks).
The phase of the wave, $\phi=\kappa x-\omega t$, represents a wave traveling towards increasing x in time with speed c. Similarly, $\phi=\kappa x+\omega t$ represents a wave traveling towards increasing x in time with speed c. Superposition (e.g. addition) of a left and right traveling waves result in a *standing* wave, $\eta = \eta_0 \cos(\kappa x - \omega t) + \eta_0 \cos(\kappa x + \omega t) = 2\eta_0 \cos(\kappa x)\cos(\omega t)$.

Now, let's fit a gravity wave in a tank. The lateral BC's of the tank are that u=0 at the sides (or u=u(paddle) at the paddle position if there is a wave maker, as we had in the previous lab). Assume the length of the tank is L. A solution for a standing wave in a tank with u=0 at the sides, must have $\lambda=2\pi/\kappa=L/\{0.5,1 \ldots. N-0.5,N\}$. When u=u(paddle) the solution is more complex but could fit if $\lambda=2\pi/\kappa=L/\{0.25,0.75 \ldots. N-0.25,N+0.25\}$. The frequency that will match it will be given as the solution of (8).

References:

Lighthill, J., 1978, Waves in Fluids, Cambridge University Press.
Gill, A. E., 1982, Atmosphere-Ocean Dynamics, Academic Press.

Appendix: graph of hyperbolic functions (from http://www.ping.be/~ping1339/hypf.gif)