where

$$h^*(\mathbf{x}) = \int_X h_s(\mathbf{x}')\ dV(\mathbf{x}') \tag{37}$$

and

$$h^o(\mathbf{x}') = N^o\Omega(\mathbf{x}')e^{-\alpha r'}s \tag{38}$$

and where

$$r' = |\mathbf{x}'| \quad , \tag{39}$$

and $\Omega(\mathbf{x}')$ is the solid angular subtense of the point source as measured at $\mathbf{x}'$. The source is actually a small finite sphere of surface radiance N^o in the direction $\xi' = \mathbf{x}/|\mathbf{x}'|$. V is the volume measure in X. We shall not go into further details here. See (66) of Sec. 6.6 in particular, and Sec. 6.6 in general for complete details.

1.4 Some Deductions from the Light Field Models

The three models for natural and artificial light fields derived above allow us to explain and interrelate many of the observed features of light fields in natural hydrosols. We shall consider here and in subsequent sections a small representative sample of such activity, based on simple deductions from the three models.

The Decay of the General Light Field with Depth

We shall now show how (7) of Sec. 1.2 follows from the two-flow model for light fields. Toward this end, we let the scattering medium X be infinitely deep and be absorbing, i.e., a>0. Then we compute the net downward irradiance at a general depth, using (6), (7) of Sec. 1.3.

$$\begin{aligned}\bar{H}(z,-) &= H(z,-) - H(z,+)\\ &= (g_- - g_+)\left[\ m_+e^{kz} - m_-e^{-kz}\right]\end{aligned} \tag{1}$$

Now from (16) of Sec. 1.3 we find, by integrating between depths 0 and z, and noting that h(z) is a non negative quantity for all z:

$$\bar{H}(z,-)-\bar{H}(0,-) = \int_0^z \frac{d\bar{H}(z',-)}{dz'} = -a\int_0^z h(z')\ dz' \leq 0$$

Hence for all z:

$$\bar{H}(z,-) \leq \bar{H}(0,-) \tag{2}$$

This shows that the net downward irradiance is bounded. Indeed, from Tables 2, 3 of Sec. 1.2 we can estimate an upper bound of $\bar{H}(z,-)$ as 1396 watts/m², and infer that $\bar{H}(z,-) \geq 0$ in real optical media. It follows that (2) and (1), along with $a > 0$, force m_+ to be zero; otherwise we could find a depth z at which (2) would be violated. Some further general inequalities related to (2) are given in Sec. 9.2.

Having established that $m_+=0$ in infinitely deep absorbing media, (6), (7), of Sec. 1.3 yield the requisite forms of $H(z,\pm)$ for every z:

$$H(z,-) = m_- g_+ e^{-kz} \tag{3}$$

$$H(z,+) = m_- g_- e^{-kz} \quad . \tag{4}$$

From (3), (4) we have, on setting z=0:

$$H(0,-) = m_- g_+$$

$$H(0,+) = m_- g_-$$

Let us write

$$\text{"}R_\infty\text{"} \quad \text{for} \quad H(0,+)/H(0,-)$$

Clearly, we then have from (3), (4):

$$R_\infty = \frac{H(0,+)}{H(0,-)} = \frac{H(z,+)}{H(z,-)} = \frac{g_-}{g_+} = \frac{1-\frac{aD}{k}}{1+\frac{aD}{k}} = \frac{k-aD}{k+aD} \tag{5}$$

This shows that the reflectance R_∞ of the medium is independent of depth and determinable once a, k, and D are known. Hence for every z,

$$H(z,+) = H(z,-)R_\infty$$

where

$$m_- = \frac{H(0,+)}{g_+ R_\infty} = \frac{H(0,+)}{g_-} = \frac{H(0,-)}{g_+}$$

Thus we have shown, among other things that:

$$H(z,\pm) = H(0,\pm)e^{-kz} \tag{6}$$

for all z.

Furthermore, by definition of the distribution factor D (cf. (1) of Sec. 1.3) we have, with the help of (8) of Sec. 1.1:

$$h(z) = h(z,+) + h(z,-)$$
$$= D(\ H(z,+) + H(z,-)\ )$$
$$= D(\ H(0,+) + H(0,-)\ )\ e^{-kz}$$
$$= h(0)\ e^{-kz} \qquad (7)$$

which is the theoretical basis for (7) of Sec. 1.2.

Observe how the assumption that $a>0$, is needed in various parts of the arguments above. This assumption is quite reasonable in terrestrial settings; indeed, in such settings the condition $a=0$ for every wavelength is never observed. What would the light field look like in an infinitely deep medium in which $a=0$? Equation (1) shows us that if $a=0$ for all wavelengths, then: since $g_- = g_+ = 1$,

$$\overline{H}(z,-) = 0$$

so that

$$H(z,-) = H(z,+)$$

at all depths z and for all wavelengths. The sea would be of the same general brightness and color of the sky in this case --at every depth!

Reflectance and Transmittance of Finitely Deep Hydrosols

The simple two-flow model allows us to estimate the reflectances and transmittances of finitely deep layers of water. We return to (6), (7) of Sec. 1.3 and consider a finitely deep homogeneous layer whose upper surface is at 0 and whose lower surface is at z. The upper surface is irradiated with a given irradiance $H(0,-)$ and we set $H(z,+)=0$, which simulates zero irradiation at the lower boundary (Fig. 1.40 (a)). We then find the m_+, m_- corresponding to these two given irradiances, and solve for $H(0,+)$. Thus, if under these conditions we write

"$R_\gamma(\tau)$" for $H(0,+)/H(0,-)$

then $R_\gamma(\tau)$ is the *reflectance* of the slab of (diffuse) optical depth* $\tau = kz$, and $R_\gamma(\tau)$ is found to be of the form:

$$R_\gamma(\tau) = (1-\gamma^2)\ \frac{e^\tau - e^{-\tau}}{(1+\gamma)^2 e^\tau - (1-\gamma)^2 e^{-\tau}} \qquad (8)$$

*There are many 'optical depths' possible in radiative transfer theory; one for each scattering or absorbing concept. In the present case we use k as a base for optical depth.

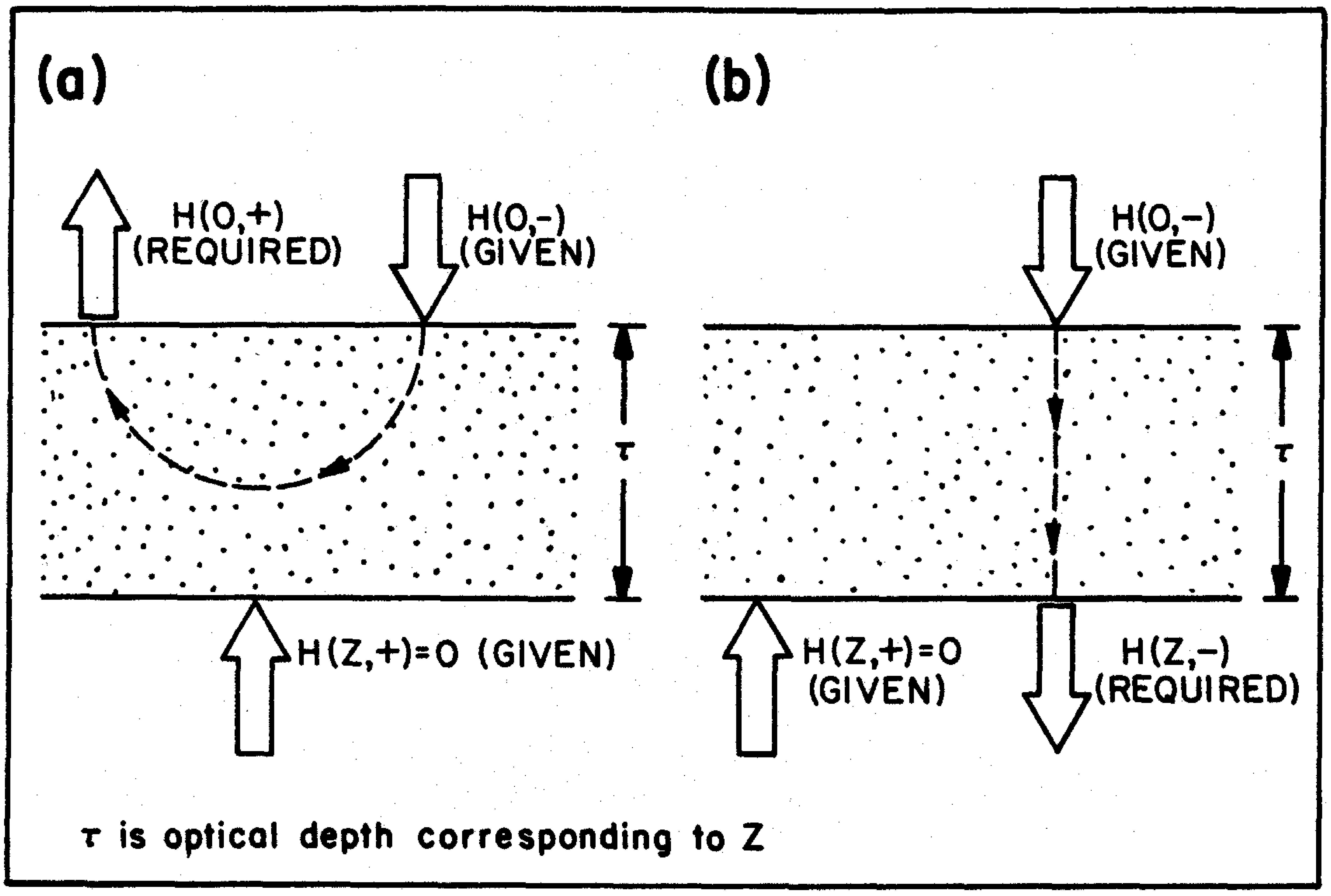


FIG. 1.40 Boundary conditions for the reflectance and transmittance of finitely deep layers in a hydrosol.

where we have written:

$$\text{"}\gamma\text{"} \quad \text{for} \quad \frac{aD}{k} \quad . \tag{9}$$

The transmittance $T_\gamma(\tau)$ of the slab of optical depth τ can be found in an analogous manner (Fig. 1.40 (b)) by now seeking $H(z,-)$ under the same conditions. Thus if we write:

$$\text{"}T_\gamma(\tau)\text{"} \quad \text{for} \quad H(z,-)/H(0,-) \quad ,$$

then it follows that:

$$\boxed{T_\gamma(\tau) = \frac{4\gamma}{(1+\gamma)^2 e^{\tau} - (1-\gamma)^2 e^{-\tau}}} \tag{10}$$

One should see that, because the medium is homogeneous, $R_\gamma(\tau)$ and $T_\gamma(\tau)$ depend spatially only on the optical depth τ, so that (8) and (10) pertain to any slab of thickness τ in the medium regardless of its vertical location within the medium.

It will also be interesting to look at some of the limiting values of $R_\gamma(\tau)$ and $T_\gamma(\tau)$ for various extreme values of τ and γ. For example, one may verify that:

$$\lim_{\tau \to 0} R_\gamma(\tau) = 0 \tag{11}$$

$$\lim_{\tau \to \infty} R_\gamma(\tau) = \frac{1-\gamma}{1+\gamma} = R_\infty \tag{12}$$

$$\lim_{\tau \to 0} T_\gamma(\tau) = 1 \tag{13}$$

$$\lim_{\tau \to \infty} T_\gamma(\tau) = 0 \tag{14}$$

$$\lim_{\tau \to 0} \frac{R_\gamma(\tau)}{\tau} = \frac{1-\gamma^2}{2\gamma} = \frac{b}{k} \tag{15}$$

$$\lim_{\tau \to 0} \frac{1-T_\gamma(\tau)}{\tau} = \frac{1+\gamma^2}{2\gamma} = \frac{aD+b}{k} \tag{16}$$

From (15) we see that the reflectance of very thin slabs is proportional to the backscattering coefficient b. Indeed,

$$\lim_{z \to 0} \frac{R_\gamma(\tau)}{z} = b$$

so that:

$$R_\gamma(\tau) \cong bz \tag{17}$$

for small τ. From (16) we see that the transmittance of very thin slabs is:

$$T_\gamma(\tau) = 1 - (aD+b)z \quad . \tag{18}$$

From (17), (18) we conclude that for thin slabs:

$$R_\gamma(\tau) + T_\gamma(\tau) = 1 - (aD)z$$

and if in general we write:

$$\text{"}A_\gamma(\tau)\text{"} \quad \text{for} \quad 1 - \left[R_\gamma(\tau) + T_\gamma(\tau)\right] \tag{19}$$

we see that in particular for thin slabs:

$$A_\gamma(\tau) \cong (aD)z \quad . \tag{20}$$

Clearly $A_\gamma(\tau)$ for general τ is the amount of irradiance absorbed by a slab of optical thickness τ and with optical properties a, b, and D. From (19) we have the general conservation law:

$$\boxed{A_\gamma(\tau) + R_\gamma(\tau) + T_\gamma(\tau) = 1} \tag{21}$$

Figs. 1.41, 1.42 represent $R_\gamma(\tau)$ and Fig. 1.43 represents $T_\gamma(\tau)$ for a selected set of γ and τ values. Values of k and γ can be obtained by direct computation from the definitions of k and γ, or by their graphs in Figs. 1.44, 1.45. The computations were done by Mrs. Judith Marshall.

Invariant Imbedding Relations for Irradiance

We now wish to investigate a particularly interesting property of the reflectance and transmittance functions $R_\gamma(\tau)$ and $T_\gamma(\tau)$ defined above. This property will allow us to write down Eqs. (6),(7) of Sec. 1.3 by sight for homogeneous media with transparent boundaries. We shall fix attention on an arbitrary medium X whose upper boundary is at optical depth 0 and whose lower boundary is at optical depth c (= zk), where z is the geometric depth of the medium. Since X is fixed throughout the present paragraph, we can drop the "γ" from the R and T notation. Furthermore, to emphasize the geometric limits of X we shall denote it by "X(0,c)".

Now suppose X(0,c) is irradiated at the upper boundary only. Then by definition of R(c) and T(c) we have:

$$H(0,+) = H(0,-)\ R(c) \tag{22}$$

$$H(c,-) = H(0,-)\ T(c) \tag{23}$$

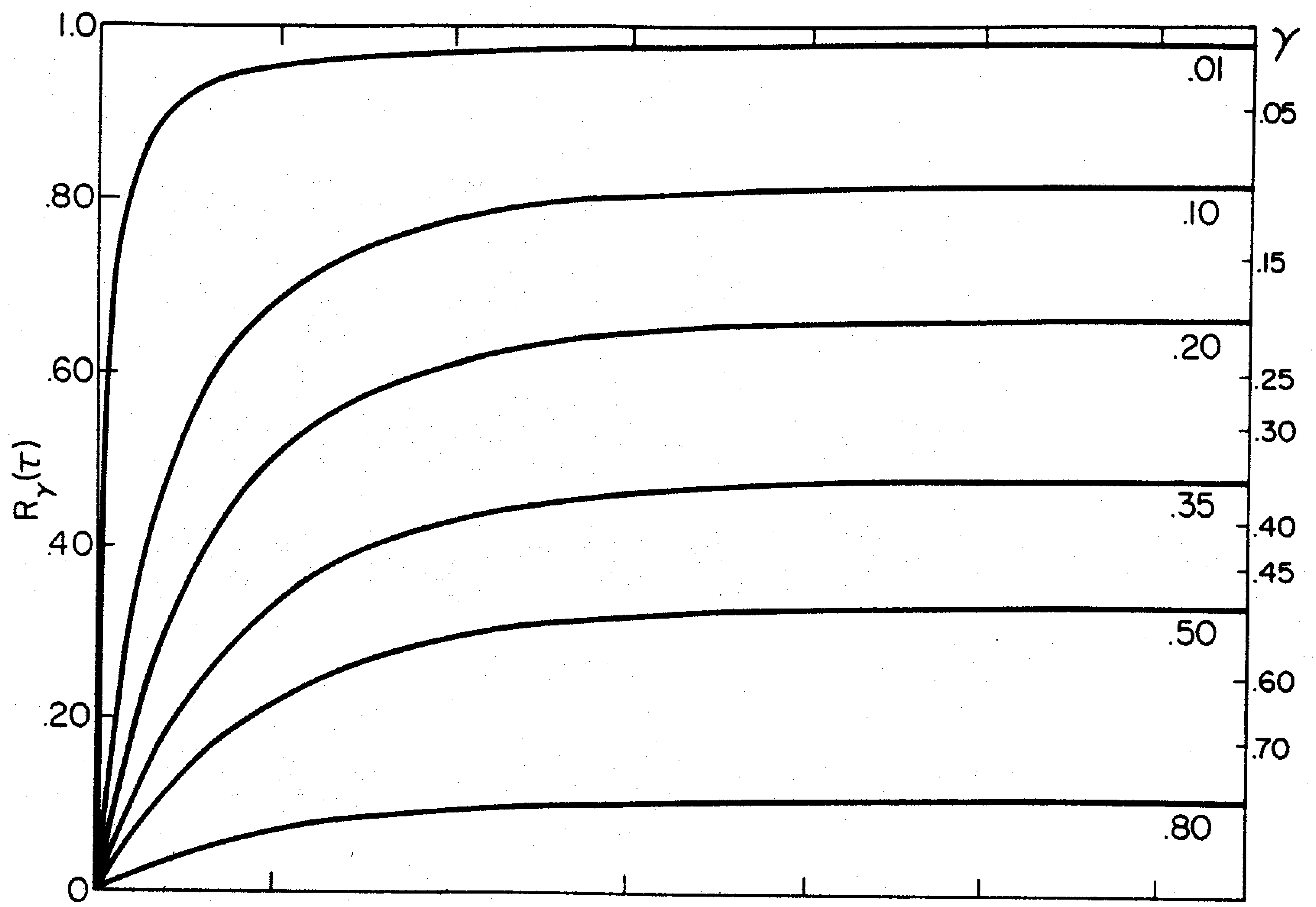
This is a simple application of (8) and (10) and the basic meanings of R(c) and T(c). Next, assume that X(0,c) is irradiated only on its lower boundary. Then, by the same token:

$$H(0,+) = H(c,+)\ T(c) \tag{24}$$

$$H(c,-) = H(c,+)\ R(c) \tag{25}$$

These formulas follow rigorously using the pattern of derivation leading to (8) and (10). However, they should be intuitively clear simply on the basis that T(c) and R(c) are transmittances and reflectances of homogeneous slabs of scattering absorbing material of optical thickness c in which complete symmetry of the light field has been assumed (in the form of (1) of Sec. 1.3).

Furthermore, and this is a crucial step, because the basic differential equations of the two-flow model are linear, we have at our beck and call the mathematical principle of linear superposition of solutions of these equations. Thus, if X(0,c) is irradiated simultaneously at levels 0 and c,



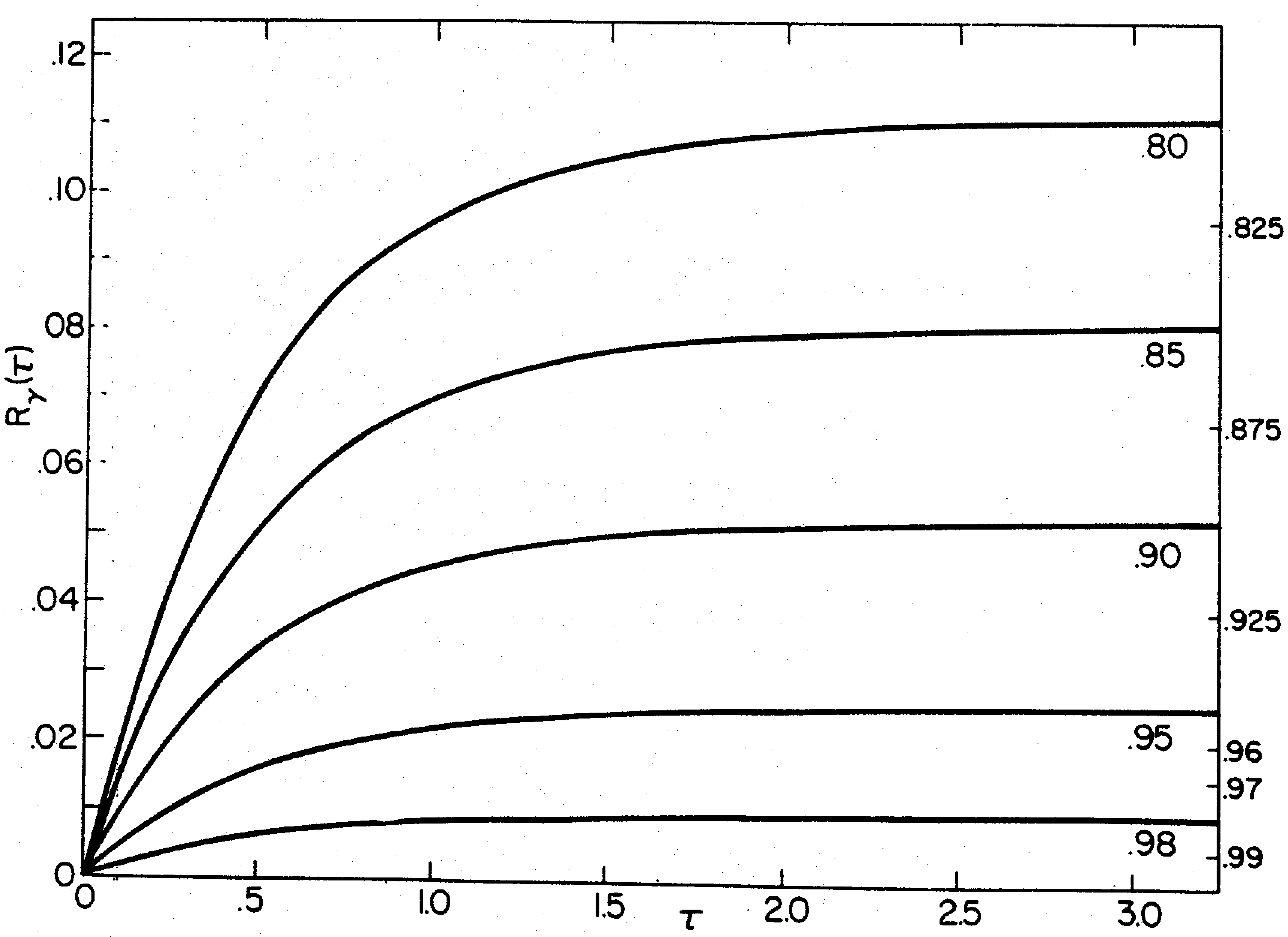


FIG. 1.41 Calculated reflectance $R_\gamma(\tau)$ versus τ, for $.01 \le \gamma \le .80$.

FIG. 1.42 Calculated reflectance $R_\gamma(\tau)$ versus τ, for $.80 \le \gamma \le .98$.

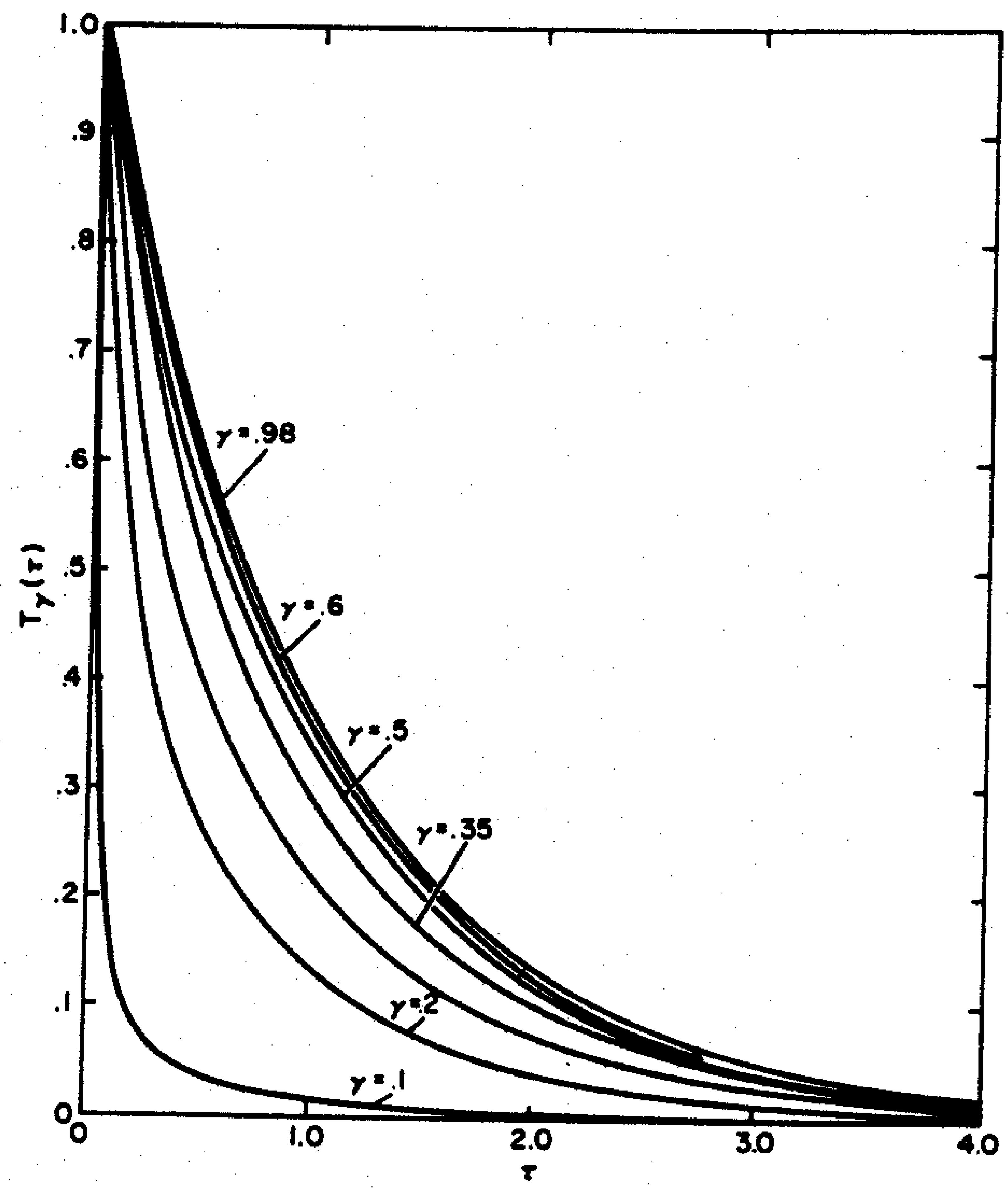


FIG. 1.43 Calculated transmittance $T_\gamma(\tau)$ versus τ, for $.01 \le \gamma \le .98$.

then we would be correct in writing the observed emergent irradiances at levels 0 and c as:

$$H(0,+) = H(0,-)R(c) + H(c,+)T(c) \qquad (26)$$

$$H(c,-) = H(0,-)T(c) + H(c,+)R(c) \ . \qquad (27)$$

These equations are readily forthcoming from (6), (7) of Sec. 1.3; however, we shall imagine for the moment that they form a relatively new basis for approaching radiative transfer problems, and that they are just as basic (as indeed they are) as the two-flow equations (4), (5) of Sec. 1.3 in setting up the foundations of the two-flow model. We shall spend much time on this point of view and its generalizations in Chapters 3, 7, and 8. For the moment we adopt it in the form of (26) and (27) and apply it in a simple and direct manner so as to explain the essential ideas behind it.

In order to illustrate in a relatively concrete manner the properties of (26) and (27), we shall consider an actual natural hydrosol in the framework of the one-D model. Thus let us suppose that

$D = 2$ (diffuse light distribution factor)

$a = .117/m$ (volume absorption coefficient)

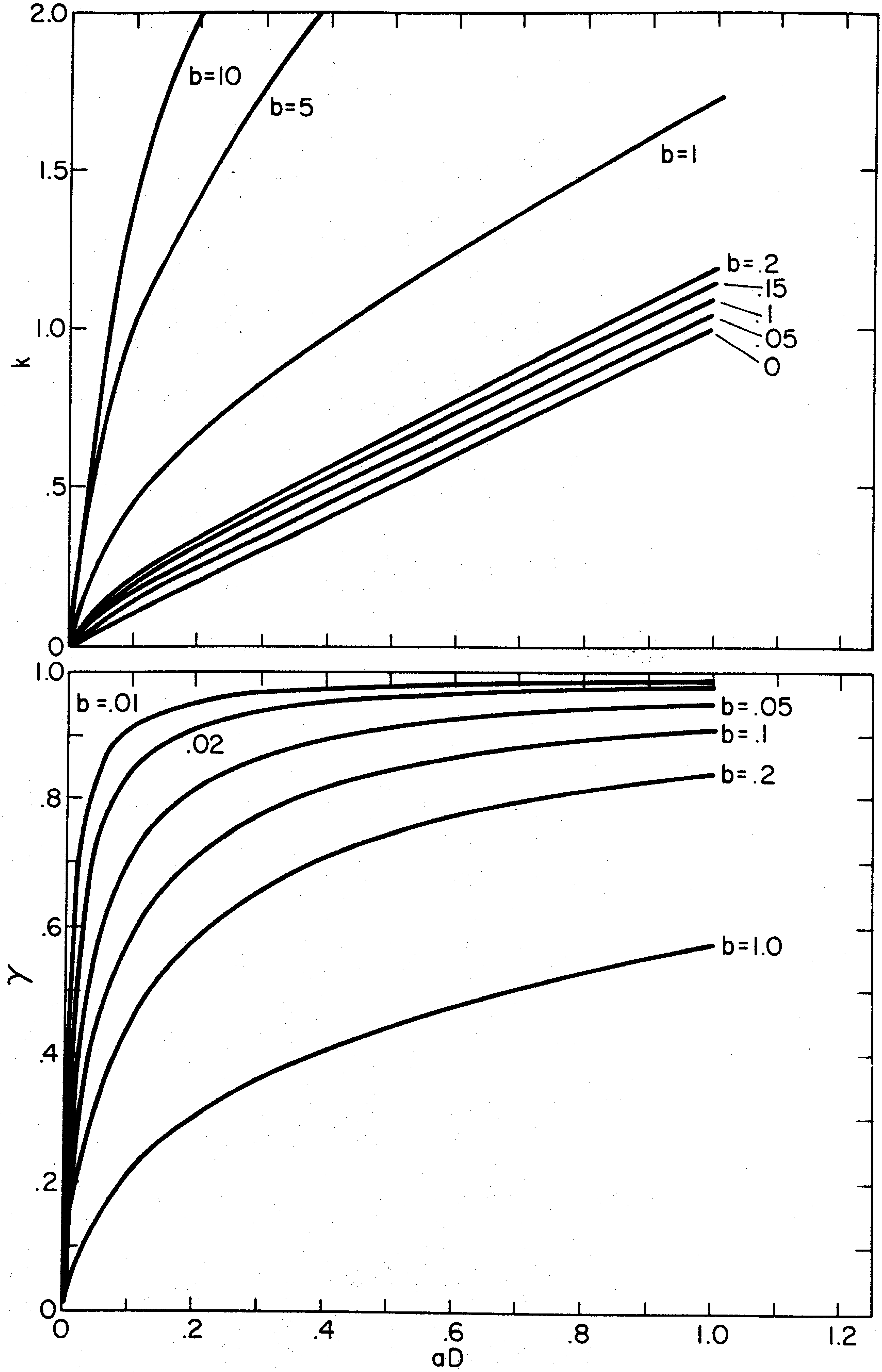


FIG. 1.44 Calculated k versus aD, for $0 \leq b \leq 10$.
FIG. 1.45 Calculated γ versus aD, for $.01 \leq b \leq 1.0$

Hence:

$aD = .234/m$ (volume absorption coefficient for diffuse light)

Further we suppose:

$s = .325/m$ (volume total scattering coefficient)

and

$b = .010/m$ (volume backward scattering coefficient for diffuse light)

From the graphs for γ, and k, we find that for this medium

$\gamma = .96$

$k = .250/m.$ (diffuse attenuation coefficient)

For later reference we note that:

$\alpha = a+s = .442/m$ (volume attenuation coefficient)

For a medium of depth:

$$z = 4 \text{ meters},$$

we have an optical depth of:

$$c = zk = 1.$$

According to the graphs for R and T, for such a medium:

$$R(1) = .018$$

$$T(1) = .360$$

and so:

$$A(1) = .622.$$

All these optical properties are to be considered for illustrative purposes only. In the present example, they pertain not to a single wavelength but to average values over the visible spectrum. Suppose that the medium X(0,1) has transparent upper and lower boundaries and that it is irradiated such that:

$$H(0,-) = 500 \text{ watt/m}^2$$

$$H(1,+) = 100 \text{ watt/m}^2 \quad .$$

The H(0,-) chosen here simulates a typical visible spectrum irradiance produced by a noonday sun at sea level on a horizontal plane, under a sky with clear dry air (cf. Table 2 of Sec. 1.2). Then the upward irradiance at the upper boundary is, according to (26):

$$H(0,+) = 500 \times .018 + 100 \times .360$$

$$= 45 \text{ watts/m}^2 \; .$$

The downward irradiance at the lower boundary is:

$$H(1,-) = 500 \times .360 + 100 \times .018$$

$$= 182 \text{ watts/m}^2 \quad .$$

Finally, the number of incident watts absorbed per square meter of boundary within X(0,1) are:

$$(H(0,-) + H(1,+)) \times .622 = 373 \text{ watts/m}^2$$

Suppose now that X(0,1) has a reflecting lower boundary. We wish to show next that the upward irradiance H(1,+) just above the lower boundary of X(0,1) can be computed directly, if the reflectance r of the lower boundary is known. Suppose that

$$r = .050$$

and suppose that only H(0,-) is given. Let the associated light field be set up in X(0,1). Then if we know the irradiance H(1,-) on the lower boundary, we have:

$$H(1,+) = H(1,-)r \qquad (28)$$

H(1,+) is the incident irradiance on the body of X(0,1) just within its lower boundary. Then, by (27),

$$H(1,-) = H(0,-)T(1) + H(1,+)R(1) \qquad (29)$$

Combining (28), (29) we have, on solving for H(1,-):

$$H(1,-) = \frac{H(0,-)T(1)}{1-rR(1)} \qquad (30)$$

Suppose that H(0,-) = 500 watts/m^2, then (30) yields:

$$H(1,-) = \frac{500 \times .360}{1-.05 \times .02}$$

$$= 180 \text{ watts/m}^2 \quad .$$

In other words, on comparing this H(1,-) with that worked out above, a bottom boundary reflecting by an amount r = .050 will contribute essentially nothing measurable to H(1,-). By (28) we have

$$H(1,+) = 180 \times .050 = 9 \text{ watts/m}^2$$

What should the reflectance r of the lower boundary be in order to yield the H(1,+) = 100 watts/m^2 we used in the first illustration above? Multiplying each side of (30) by r, and using (28) we have

$$H(1,+) = \frac{H(0,-)rT(1)}{1-rR(1)}$$

Solving for r:

$$r = \frac{H(1,+)}{H(1,+)R(1) + H(0,-)T(1)}$$

$$= \frac{100}{100 \times .018 + 500 \times .360}$$

$$= \frac{100}{182} = .550$$

which could for example simulate a light sandy bottom. Observe that the denominator in the preceding expression for r is simply H(1,-), under the present boundary conditions.

These examples begin to show the use of the one-D model in making elementary calculations concerning everyday matters in the study of hydrologic optics, including the effects of nontransparent boundaries.

We continue with another illustration which shows how to find the internal irradiances in X(0,1) knowing the incident irradiances on its transparent upper and lower boundaries. Suppose we have the incident irradiances:

$$H(0,-) = 500 \text{ watts/m}^2$$

$$H(1,+) = 100 \text{ watts/m}^2$$

We want to find H(1/2,±), i.e., the irradiances at the mid-level of the present medium. Now since (26), (27) hold for arbitrary media of optical depths c, let us apply them to the two subslabs X(0,1/2), and X(1/2,1) which comprise the upper and lower halves of X(0,1), respectively (see Fig. 1.46).

Applying (26), (27) to X(0,1/2):

$$H(0,+) = H(0,-)R(1/2) + H(1/2,+)T(1/2) \qquad (31)$$

$$H(1/2,-) = H(0,-)T(1/2) + H(1/2,+)R(1/2) \qquad (32)$$

Of the irradiances we know only H(0,-), and we want to find H(1/2,±). We also do not know H(0,+). We therefore need more relations. Returning to (26), (27), and applying them to X(1/2,1), we have:

$$H(1/2,+) = H(1/2,-)R(1/2) + H(1,+)T(1/2) \qquad (33)$$

$$H(1,-) = H(1/2,-)T(1/2) + H(1,+)R(1/2) \qquad (34)$$

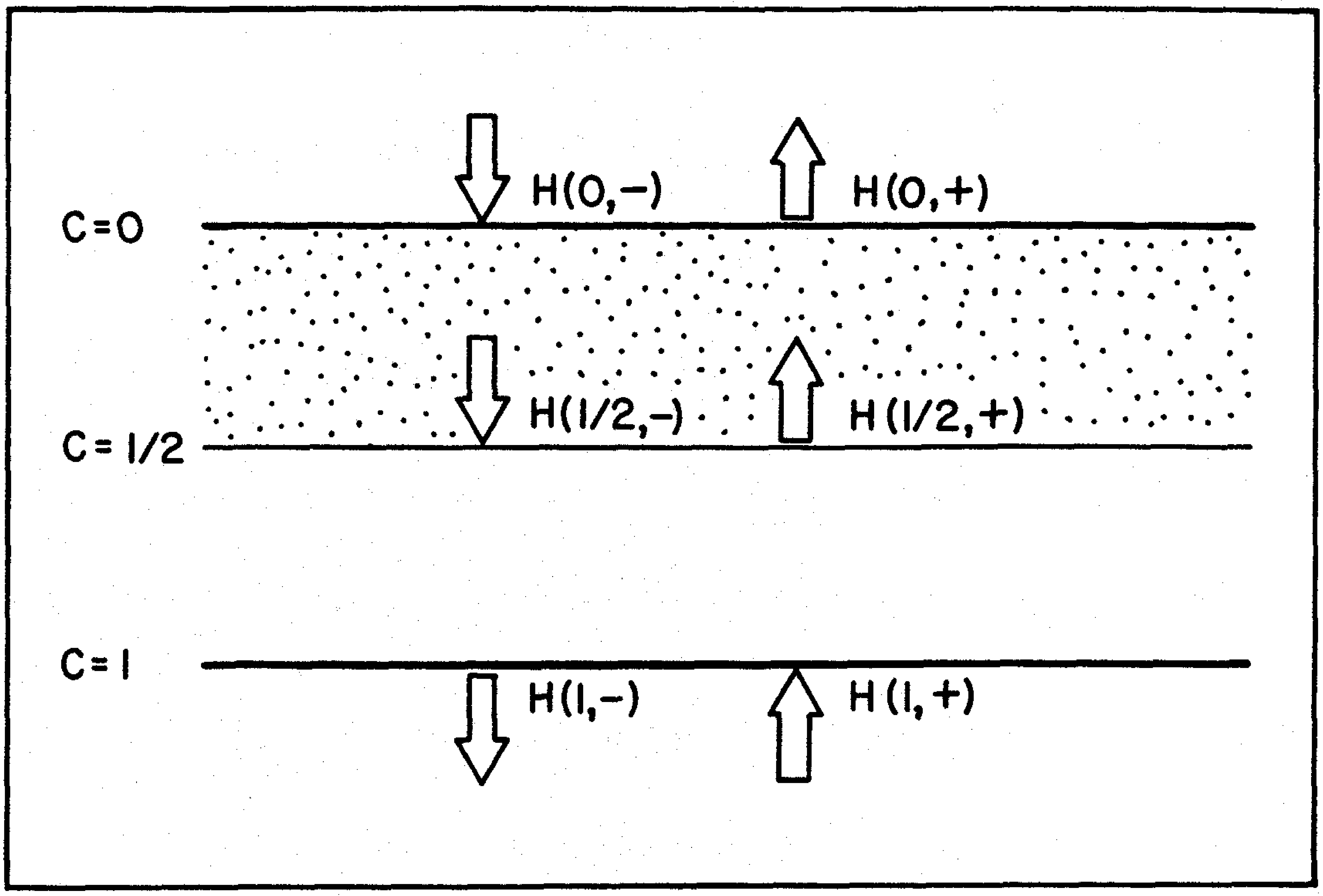


FIG. 1.46 Using invariant imbedding ideas to calculate internal irradiances from boundary irradiances.

Here we know H(1,+), and we have more relations for H(1/2,±) with another unknown H(1,-). But now we have four equations in four unknowns which we can solve for H(1/2,±), and rearrange as follows.

$$H(1/2,-) = H(0,-)\mathcal{T}(0,1/2,1) + H(1,+)\mathcal{R}(1,1/2,0) \quad (35)$$

$$H(1/2,+) = H(0,-)\mathcal{R}(0,1/2,1) + H(1,+)\mathcal{T}(1,1/2,0) \quad (36)$$

where for the present example:

$$\mathcal{R}(0,1/2,1) = \mathcal{R}(1,1/2,0) = \frac{R(1/2)T(1/2)}{1-R^2(1/2)} = .006 \quad (37)$$

$$\mathcal{T}(0,1/2,1) = \mathcal{T}(1,1/2,0) = \frac{T(1/2)}{1-R^2(1/2)} = .600 \quad (38)$$

Hence:

$$H(1/2,-) = 500 \times .600 + 100 \times .006$$

$$= 301 \text{ watts/m}^2$$

$$H(1/2,+) = 500 \times .006 + 100 \times .600$$

$$= 63 \text{ watts/m}^2$$

Equations (35)-(38) are special cases of the important invariant imbedding relations we shall study in many contexts later. If the reader has understood the deductions in this example, he will have no difficulty with the deductions in the remainder of this work concerning invariant imbedding concepts, for they are merely elaborations of the present simple example to general geometries and radiometric quantities. It suffices to observe here that the $\mathcal{R}$ and $\mathcal{T}$ factors are the *complete reflectances* and *complete transmittances* for the medium X(0,1) partitioned at level 1/2. More general partitions generally yield four such numbers.

With the preceding numerical examples in mind, we may now apply (26), (27) to the following situation which generalizes the setting of Fig. 1.46. Thus, being guided by Fig. 1.47 in which all depths are optical depths we have, for the medium X(a,b):

$$H(a,+) = H(a,-)R(b-a) + H(b,+)T(b-a) \tag{39}$$

$$H(b,-) = H(a,-)T(b-a) + H(b,+)R(b-a) \tag{40}$$

Applying (26), (27) again, now to X(b,c):

$$H(b,+) = H(b,-)R(c-b) + H(c,+)T(c-b) \tag{41}$$

$$H(c,-) = H(b,-)T(c-b) + H(c,+)R(c-b) \tag{42}$$

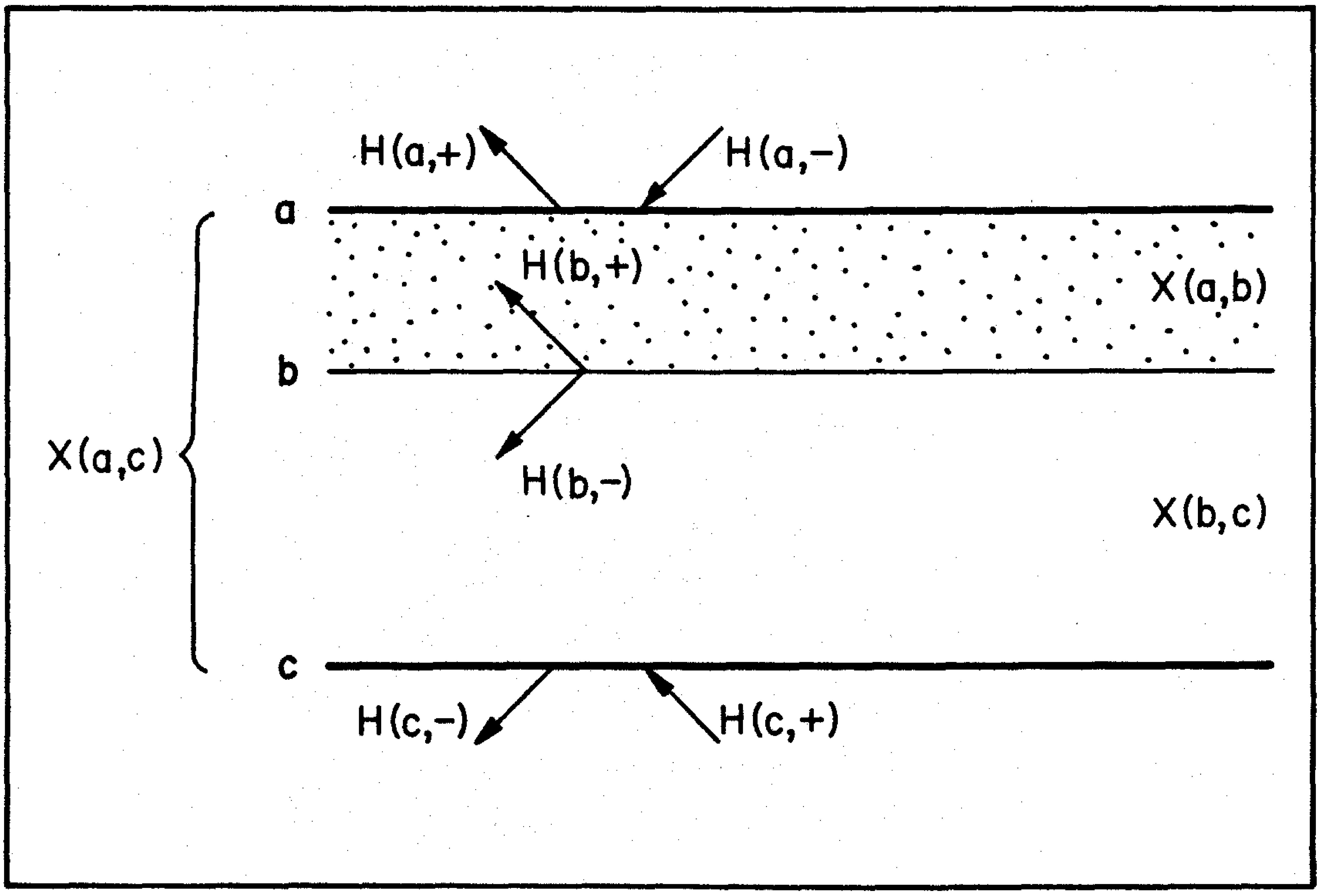


FIG. 1.47 General arrangement for calculating internal irradiances at level b from given irradiances at levels a and c, using the invariant imbedding relation.

Solving these four equations for H(b,±), we have, analogously to (35), (36):

$$H(b,-) = H(a,-)\mathcal{T}(a,b,c) + H(c,+)\mathcal{R}(c,b,a) \quad (43)$$

$$H(b,+) = H(a,-)\mathcal{R}(a,b,c) + H(c,+)\mathcal{T}(c,b,a) \quad (44)$$

where:

$$\mathcal{R}(a,b,c) = \frac{T(b-a)R(c-b)}{1-R(b-a)R(c-b)} \quad (45)$$

$$\mathcal{T}(a,b,c) = \frac{T(b-a)}{1-R(b-a)R(c-b)} \quad (46)$$

$$\mathcal{R}(c,b,a) = \frac{T(c-b)R(b-a)}{1-R(b-a)R(c-b)} \quad (47)$$

$$\mathcal{T}(c,b,a) = \frac{T(c-b)}{1-R(b-a)R(c-b)} \quad (48)$$

which are the general versions of (37), (38). These may be evaluated using the R and T values tabulated above.

Equations (43), (44) together constitute the *invariant imbedding relation* for the medium X(a,c) which is partitioned at level b, $a \le b \le c$ and in which irradiance is the radiometric concept of interest. It may be put into compact matrix form as follows:

$$(H(b,+),H(b,-)) = (H(c,+),H(a,-))\begin{bmatrix}\mathcal{T}(c,b,a) & \mathcal{R}(c,b,a)\\ \mathcal{R}(a,b,c) & \mathcal{T}(a,b,c)\end{bmatrix} \quad (49)$$

If the one-D model proves inadequate to predict or describe a given radiometric condition in a natural hydrosol, it may be that a more general and flexible model is required. The hierarchy of successively more refined irradiance models that may be tried after the present one is as follows: the *decomposed one*-D *model*, the *undecomposed two*-D *model*, the *decomposed two*-D *model*; these are studied in Chapter 8.

For a reversal of the preceding procedures in which the light field in real media may be measured so as to predict R(τ) and T(τ), see Examples 1, 2, in Sec. 13.10.

A Theoretical Basis for the Law: $N_*(z,\theta) = N_*(0,\theta)e^{-kz}$

In our derivation of the simple model for radiance leading to (14) of Sec. 1.3 we assumed that the path function N_* decreased exponentially with depth, as indicated in (13) of Sec. 1.3. We shall now do away with the assumption and deduce this form of N_* with the help of the two-flow model. This will place (13) of Sec. 1.3 on a sounder basis and also show how the simple models occasionally may be used to help each other attain their full descriptive powers.

Now the path function value $N_*(z,\theta)$, as defined, gives the amount of radiance generated by scattering at depth z, per unit length along the direction ξ of a path in a hydrosol, as shown in Fig. 1.39. What is scattered is the radiance at depth z impinging on the path in all directions ξ'. Just how much of the stream in the direction ξ' is scattered into the direction ξ, at depth z is given by means of the volume scattering function values $\sigma(z;\xi';\xi)$. Thus

$$N_*(z,\xi) = \int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\, d\Omega(\xi') \tag{50}$$

where the notation "Ξ" and "$d\Omega(\xi')$" is explained in Sec. 1.1. We shall carefully define σ and derive (50) from first principles in Chapter 3. For the present we can understand it on simple intuitive grounds, as just explained.

The two-flow model assumes that the radiance distribution $N(z,\cdot)$ has an arbitrary fixed shape in the upper and lower hemispheres Ξ_+, Ξ_- of the unit sphere of directions Ξ. If we assume in particular that for every depth z

$N(z,\cdot)$ on Ξ_+ has the constant value $\overline{N}(z)$

and that

$N(z,\cdot)$ on Ξ_- has the constant value $\underline{N}(z)$

then (50) yields up the following necessary form of N_*:

$$\begin{aligned} N_*(z,\xi) &= \int_{\Xi_+} N(z,\xi')\sigma(z;\xi';\xi)\, d\Omega(\xi') \\ &\quad + \int_{\Xi_-} N(z,\xi')\sigma(z;\xi';\xi)\, d\Omega(\xi') \\ &= \overline{N}(z)\int_{\Xi_+}\sigma(z;\xi';\xi)\, d\Omega(\xi') + \underline{N}(z)\int_{\Xi_-}\sigma(z;\xi';\xi)\, d\Omega(\xi') \end{aligned} \tag{51}$$

In the two-flow model adopted in this chapter, the medium is assumed isotropic and homogeneous.* Further, the light field is such that the path direction ξ can be characterized by a single angle θ as shown in Fig. 1.39. We can therefore write, *ad hoc*:

$$\text{"}s(\theta)\text{"} \quad \text{for} \quad \int_{\Xi_-} \sigma(z;\xi';\xi)\, d\Omega(\xi') \tag{52}$$

We shall also write:

$$\text{"}s\text{"} \quad \text{for} \quad \int_{\Xi} \sigma(z;\xi';\xi)\, d\Omega(\xi') \tag{53}$$

so that,

$$\int_{\Xi_+} \sigma(z;\xi';\xi)\, d\Omega(\xi') = s - s(\theta). \tag{54}$$

The quantity s is simply the volume total scattering function introduced during the derivation of the simple model for radiance. The portion of the scattering lobe used in finding $s(\theta)$ is shown unshaded in Fig. 1.39. In view of these conventions, we can write:

$$N_*(z,\theta) = \underline{N}(z)s(\theta) + \overline{N}(z)(s - s(\theta)) \tag{55}$$

From (11) of Sec. 1.1 is is clear that:

$$H(z,-) = \pi\underline{N}(z) \tag{56}$$

$$H(z,+) = \pi\overline{N}(z) \tag{57}$$

and from (57), (56) and (5) we have:

$$\overline{N} = \underline{N}\, R_\infty \tag{58}$$

Therefore:

$$N_*(z,\theta) = \underline{N}(z)[s(\theta) + R_\infty(s - s(\theta))] \tag{59}$$

We are essentially finished, because by (8), (11) of Sec. 1.1 we have (setting ξ equal to **k**)

$$\begin{aligned} h(z) &= h(z,+) + h(z,-) \\ &= 2H(z,+) + 2H(z,-) \\ &= 2\pi(R_\infty + 1)\underline{N}(z) \end{aligned} \tag{60}$$

Using this in (59) and recalling (7), we find:

**Homogeneity* means σ is independent of z; *isotropy* means σ depends only on $\xi \cdot \xi'$.

$$N_*(z,\theta) = \frac{h(z)[s(\theta) + R_\infty(s-s(\theta))]}{2\pi(1+R_\infty)}$$

$$= N_*(0,\theta)e^{-kz} \tag{61}$$

where:

$$N_*(0,\theta) = \frac{h(0)[s(\theta) + R_\infty(s-s(\theta))]}{2\pi(1+R_\infty)} , \tag{62}$$

which is the desired result. In practice we can therefore use the theoretical k and the empirical K interchangeably. This derivation also shows how, using (50), one can generalize the construction of $N_*(0,\theta)$ to quite realistic angular dependences using existing light fields at or somewhat below the air-water surface. The unshaded region of the σ-lobe in Fig. 1.39 shows the portion of the three dimensional surface of $\sigma(z;\xi';\xi)$ over which the integration takes place to obtain $s(\theta)$, $0 \leq \theta \leq \pi$. Observe that if $s(\theta)$ is a surface of revolution (as it is in practice) then:

$$s(\theta) + s(\pi-\theta) = s \tag{63}$$

whence:

$$s(\pi/2) = s/2$$

and in particular:

$$s(0) + s(\pi) = s$$

As an example of the use of (62) we observe that in some Pacific coastal waters (cf. [300]) as measured in the wavelength band of a Wratten 57 filter, we have

$$s(0) = .001/m$$

$$s(\pi) = .013/m$$

Observe that $s(0)$ acts like a backward scattering function for *collimated* flux, whereas $s(\pi)$ acts like a forward scattering function for *collimated* flux, so that by (63)

$$s(0) + s(\pi) = s = .014/m$$

This water was also found to have a corresponding volume absorption coefficient of $a = .104/m$, and hence the medium has an $\alpha = .118/m$. Such water is highly forward scattering and also relatively highly absorptive, and will therefore force the simple models to work hard in their descriptive tasks. Since the present medium is highly absorptive, the downward scattered daylight light near the surface will be relatively highly collimated. Accordingly we assume a relatively small distribution factor D, say $D = 1.1$. Since the medium is highly forward scattering, we shall estimate the backward scattering coefficient b for the *scattered* flux field to be .002/m. It follows from the one-D two-flow model ((9) of

1.3 and (9) of 1.4) that $aD = .114$, and that $k = .114/m$ along with $\gamma = .99$; so that $R_\infty = .01$. Let $h(0) = 500$ watts/m^2 just below the surface.

These assumed conditions allow us to illustrate the path function formula (62). We have, for the downward path function just below the surface:

$$N_*(0,\pi) = \frac{h(0)[s(\pi) + R_\infty s(0)]}{2\pi(1+R_\infty)}$$

$$= \frac{500[.013 + .01 \times .001]}{6.28 \times (1 + .01)}$$

$$= 1.03 \text{ watts/m}^3\text{sr}$$

$$= 1.03 \text{ herschels/m}$$

Further, for the horizontal path function:

$$N_*(0,\pi/2) = \frac{h(0)[1+R_\infty]s}{4\pi[1+R_\infty]} = \frac{sh(0)}{4\pi}$$

$$= \frac{500 \times .014}{4 \times 3.14} = .557 \text{ herschels/m}$$

Finally, for the upward path function:

$$N_*(0,0) = \frac{h(0)[s(0)+ R_\infty s(\pi)]}{2\pi[1+R_\infty]}$$

$$= \frac{500[.001+.01 \times .013]}{6.28(1+.01)}$$

$$= .080 \text{ herschels/m.}$$

Computing Radiances from the Simple Model

Some illustrations of the computation of radiances using (14) of Sec. 1.3 will help fix in mind the typical orders of magnitudes of radiance values in natural waters. Let us begin with the case of a horizontal path of sight some given depth z_0 below the surface. Then in (14) of Sec. 1.3, we set $\theta = \pi/2$, and that equation becomes:

$$N_r(z_0,\pi/2) = N_0(z_0,\pi/2)e^{-\alpha r} + \frac{N_*(z_0,\pi/2)}{\alpha}\left[1-e^{-\alpha r}\right] \qquad (64)$$

which we can write quite simply as:

$$N_r = N_o e^{-\alpha r} + \frac{N_*}{\alpha}\left[1-e^{-\alpha r}\right]$$

provided the depth and direction of the path are understood. (The right is reserved to disinter the depth and direction variables at any time.) For infinitely long horizontal paths, i.e., for the case $r = \infty$, this formula yields:

$$\boxed{N_q = \frac{N_*}{\alpha}} \tag{65}$$

for the observable horizontal radiance N_q at a given depth in any laterally extensive stratified optical medium. Observe that in such media the N_q defined above does not change with location along the path. For this reason we denote the observable radiance as "N_q" and call it the *equilibrium radiance*.

An estimate of N_q for shallow depths in Pacific coastal water around the blue-green part of the spectrum can be made on the basis of the preceding example, wherein we found that $N_*(0,\pi/2) = .557$ herschels/m. In such waters, for example, $\alpha = a+s = .104 + .014 = .118$/m. Hence:

$$N_q(0,\pi/2) = .557/.118 = 4.72 \text{ herschels}$$

is the equilibrium radiance just below the surface. At a depth of 5 meters, it follows from (61) that

$$\begin{aligned} N_q(5,\pi/2) &= \frac{N_*(5,\pi/2)}{\alpha} = \frac{N_*(0,\pi/2)}{\alpha}\, e^{-5k} \\ &= N_q(0,\pi/2)e^{-5k} \\ &= 4.72 \times e^{(-5 \times .115)} \\ &= 4.72 \times .560 = 2.64 \text{ herschels} \end{aligned}$$

where we have used the k for the water of the preceding example.

As another example of the use of the radiance model, we set $\theta = \pi$, and $\theta = 0$ in (14) of Sec. 1.3, to find that, at depth z at the lower end of a vertical path of length r:

$$\boxed{N_r(z,\pi) = N_o(z_o,\pi)e^{-\alpha r} + \frac{N_*(z,\pi)}{\alpha-K}\left[1-e^{-(\alpha-K)r}\right]} \tag{66}$$

and similarly at depth z, at the upper end of a vertical path of length r:

$$\boxed{N_r(z,0) = N_o(z_o,0)e^{-\alpha r} + \frac{N_*(z,0)}{\alpha+K}\left[1-e^{-(\alpha+K)r}\right]} \tag{67}$$

The reader is reminded of the standing convention that $N_r(z,\pi)$ is the apparent radiance at depth z flowing in the downward direction and to see it, one must direct his eye or radiance meter upward (cf. Fig. 1.39). We persist in using this form of radiance (i.e., surface radiance) because it simplifies the dynamics of photons in scattering-absorbing media.

Suppose the medium is infinitely deep, so that we can set $r = \infty$ in (67) and still keep the path within the medium. Then (67) becomes:

$$N(z,0) = \frac{N_*(z,0)}{\alpha + K}$$

which is the radiance one would see at depth z looking straight down into the infinite deeps. Suppose $z = 0$, then our preceding example lets us estimate that:

$$N(0,0) = \frac{N_*(0,0)}{\alpha + K}$$

$$= \frac{.080}{.118 + .115}$$

$$= .344 \text{ herschels.}$$

Let the zenith radiance as seen just above an air-water surface be 80 herschels in a given band width of the blue-green part of the spectrum, say at 480 ± 64 mμ, and suppose that $h(0) = 500$ watts/m² just below the surface. If the surface is calm, then just below it, by virtue of the n^2-law for radiance, (Sec. 1.1), we would have

$$N_o(0,\pi) = 80 \times \left(\frac{4}{3}\right)^2 = 142 \text{ herschels}$$

where 4/3 is the index of refraction of water. This radiance value is to be modified slightly if surface transmittance and reflectance effects are to be included. These corrections are of secondary importance and so we shall not include these effects at the moment. Now, to the present task: we can estimate $N(z,\pi)$ for $z = 5$ meters, by means of (66) in which we set $z_o = 0$m, $z = 5$m, and use $\alpha = .118$/m, $k = .115$/m. Thus, with the help of our estimate of $N_*(0,\pi)$ above:

$$N_5(5,\pi) = N_o(0,\pi)e^{-.118\times 5} + \frac{N_*(0,\pi)e^{-.115\times 5}}{.118-.115}\left[1-e^{-(.118-.115)5}\right]$$

$$= 142 \times .554 + (1.03 \times .560) \times 5$$

$$= 78.6 + 2.88$$

$$= 81.5 \text{ herschels}$$

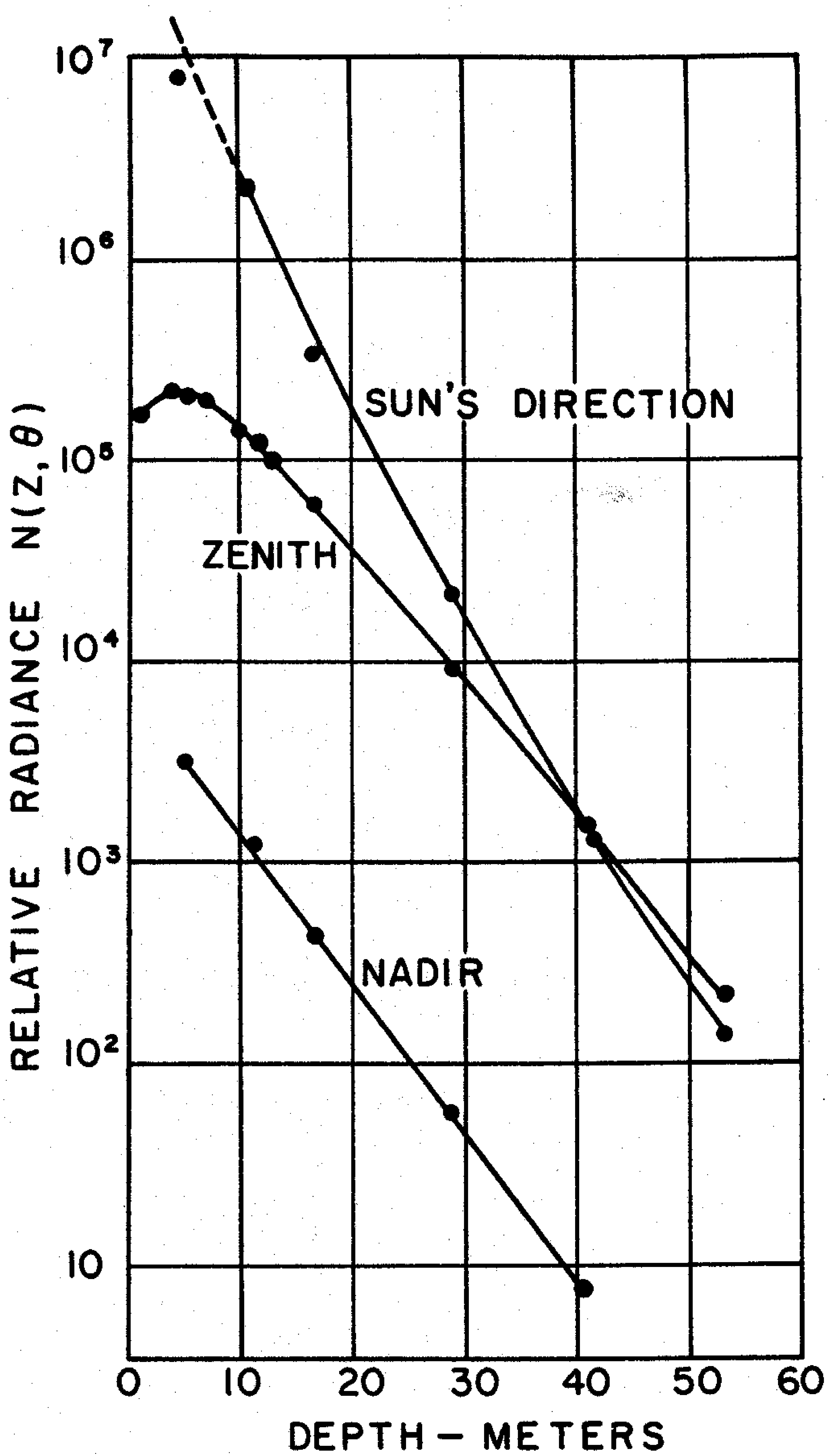


FIG. 1.48 Experimental verification of the simple model for radiance, as measured by Tyler in Lake Pend Oreille, Idaho, April 1957. (From [298], by permission)

The next to last equality shows that at a depth of 5 meters, 78.6 herschels are transmitted from the original 142 just below the surface, and that 2.88 herschels are added by the process of scattering over the 5 meter path.

Figure 1.48, which is based on the work in [298], shows the observed radiance distribution in Lake Pend Oreille and its associated predicted values using (14) of Sec. 1.3 for three important directions. The solid curve is computed from the model, the dots denote measured radiances.

A word or two may be in order here on the rather unintuitive-seeming jump by the radiance function as the flux crosses the air-water surface. We saw in the example above how it jumped from 80 to 142 herschels. To simplify matters suppose for the moment that there are no losses by reflection as the flux crosses the surface. Fig. 1.49 depicts the flux

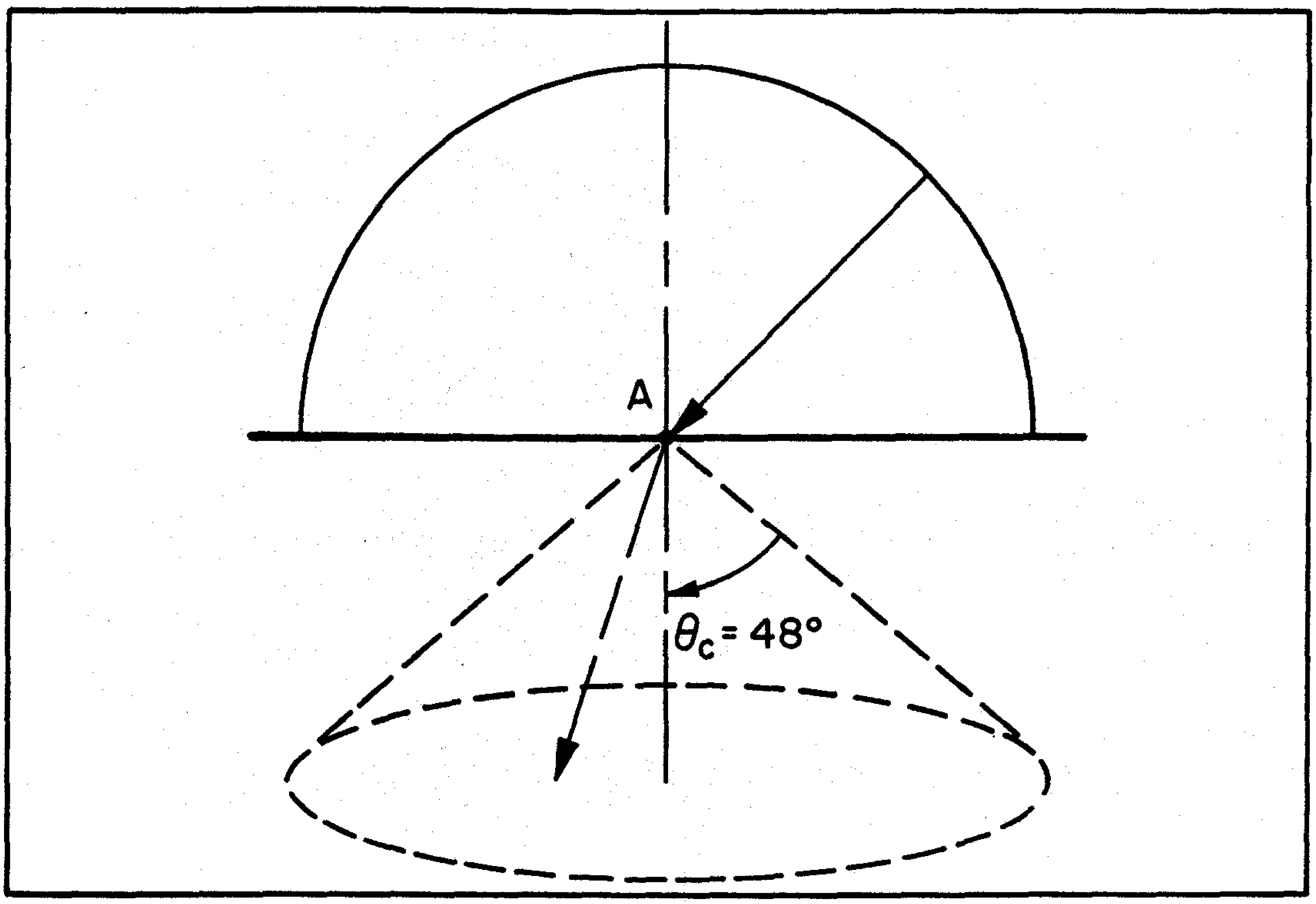


FIG. 1.49 To show that the irradiance conservation law holds despite the n^2 effect on radiance at the air-water level.

incident at a point A on the surface, flowing in from all directions in a hemisphere. The refracted rays below the surface do not fan out in a full hemisphere, but are limited to a right circular cone of half angle $\theta_c = 48°$, or more precisely,

$$\theta_c = \arc\sin\left(\frac{3}{4}\right) \quad .$$

Let the incident radiance distribution be of constant magnitude N. Then the irradiance on the air-water surface, by (11) of Sec. 1.1, is simply πN. Let us compute the irradiance just below the surface produced by the refracted incident flux of radiance $(4/3)^2 N$. By (1) of Sec. 1.1 we now find:

$$H = \int_{\theta'=0}^{\theta_c} \int_{\phi'=0}^{2\pi} \left[\left(\frac{4}{3}\right)^2 N\right] \cos\theta' \sin\theta' \, d\theta' \, d\phi'$$

$$= \left(\frac{4}{3}\right)^2 2\pi N \int_{\theta'=0}^{\theta_c} \cos\theta' \sin\theta' \, d\theta'$$

$$= \pi N$$

This shows that, despite the rather odd buildup of refracted radiance across the air-water surface, this buildup is of such a magnitude, and takes place over such a restricted set of directions, that, as expected, energy conservation is observed. The argument just given can readily be extended to ideal transmitting surfaces bounding media of arbitrary index of refraction. When, in addition, reflection processes are to be taken into account, the more extensive calculations discussed in Sec. 12.2 are to be used.

Derivation of the Contrast Transmittance Law and the Radiance Difference Law

The contrast transmittance law:

$$C_r = C_o e^{-(\alpha+K \cos \theta)r}$$

for an inclined path of sight of length r in a homogeneous optical medium was first encountered experimentally (in the special instances of vertical and horizontal directions) as explained in the discussion leading up to (12) of Sec. 1.2. It is now our purpose to show how this law may be deduced from the simple model for radiance (14) of Sec. 1.3, and under what conditions it is expected to hold.

Let the hydrosol X be infinitely deep and consider a path in X as shown in (a) of Fig. 1.50, where the observer is at depth z and the apparent radiance ${}_tN_r(z,\theta)$ of an object of inherent radiance ${}_tN_o(z_t,\theta)$ is observed. The angle θ is such

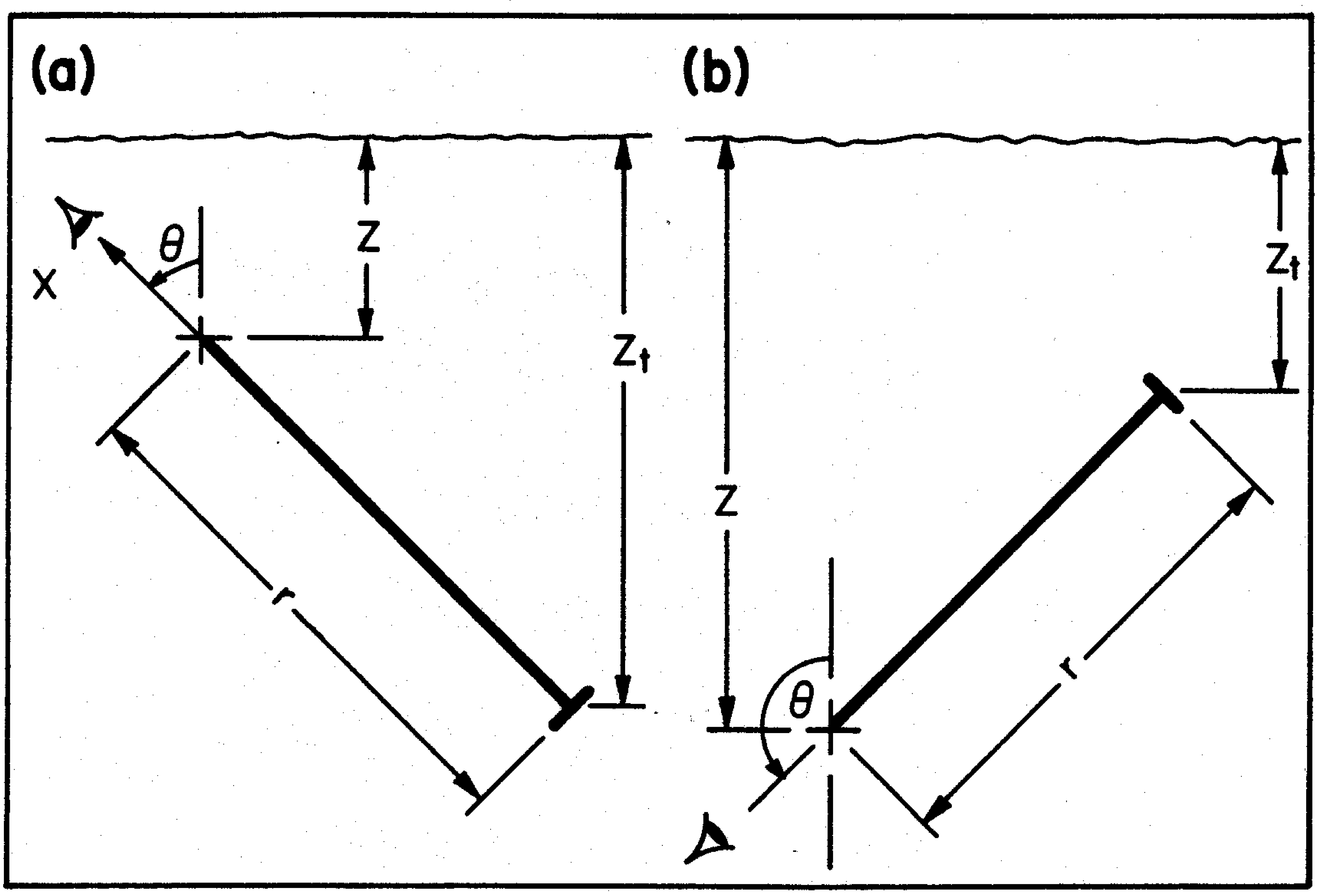


FIG. 1.50 Setting for a theoretical proof of the exponential law for apparent contrast. (cf. Figs. 1.29, 1.30)

that $0 \leq \theta \leq \pi/2$. Recall that θ is the angle from the vertical at which the photons are flowing, as shown by the arrow in the figure.

Now the background apparent radiance ${}_bN_r$ for the present path of length r is obtained from (14) of Sec. 1.3 by setting $r = \infty$ in that general equation:

$$ {}_bN_r(z,\theta) = \frac{N_*(z,\theta)}{\alpha + K \cos\theta} \quad . \tag{68} $$

This is the apparent radiance of the background of the target as seen at a range r from the target. The apparent contrast $C_r(z,\theta)$ of the object against its background (recall (11) of Sec. 1.2) is:

$$ C_r(z,\theta) = \frac{{}_tN_r(z,\theta) - {}_bN_r(z,\theta)}{{}_bN_r(z,\theta)} , \tag{69} $$

wherein we have:

$$ {}_tN_r(z,\theta) = {}_tN_o(z_t,\theta)e^{-\alpha r} + \frac{N_*(z,\theta)}{\alpha + K \cos\theta}\left[1 - e^{-(\alpha + K\cos\theta)r}\right] . \tag{70} $$

Observing that:

$$ C_o(z_t,\theta) = \frac{{}_tN_o(z_t,\theta) - {}_bN_o(z_t,\theta)}{{}_bN_o(z_t,\theta)} , \tag{71} $$

it follows from the preceding four relations, by straightforward substitution of (68) and (70) into (69), and a reduction using (71), that:

$$ \boxed{C_r(z,\theta) = C_o(z_t,\theta)e^{-(\alpha + K\cos\theta)r}} \tag{72} $$

which was to be shown. The quotient C_r/C_o is called the *contrast transmittance.* Equation (72) is the requisite contrast transmittance law. The quantity $(\alpha + K \cos\theta)^{-1}$ is called the *attenuation length* L_θ of the medium along the given path. For $\theta = \pi/2$, $L_{\pi/2} = 1/\alpha$, a basic property of the medium, while $L_o = 1/(\alpha + K)$ is associated with secchi disk readings (cf. (84) below). The quantity $4L_\theta$ is mainly of historic interest and is the *hydrologic range* for the given path of sight. Its plot is an ellipse vs θ (cf. Sec. 1.9).

This simple derivation cannot be repeated in its entirety when the photons are streaming in from a nearby boundary, such as depicted in (b) of Fig. 1.40. In this case (68) must be replaced by the full form of (14) of Sec. 1.3. However, by using (69) and (71), which are general definitions of apparent and inherent contrasts, along with (14) of Sec. 1.3

once again, it follows readily by a similar calculation, that quite generally:

$$C_r(z,\theta) = \left[\frac{{}_bN_o(z_t,\theta)}{{}_bN_r(z,\theta)} e^{-\alpha r}\right] C_o(z_t,\theta) \qquad (73)$$

The reader may show that this formula holds for both situations depicted in Fig. 1.50, i.e., for $0 \leq \theta \leq \pi$. It reduces to (72) when $0 \leq \theta \leq \pi/2$, i.e., when (14) of Sec. 1.3 reduces to (68). A fuller discussion of contrast and contrast transmittance is given in Chapter 9.

Figures 1.51 and 1.52 illustrate two experimental checks of the contrast transmittance law for the cases of $\theta = 90°$, and $\theta = 58.8°$. The radiometric quantity used was apparent luminance B_r, and the medium (Lake Winnipesaukee, N.H.) had an α of .490/m for Fig. 1.51 and for Fig. 1.52 the medium had $\alpha = .585$/m and $K = .350$/m. These optical properties therefore pertain to averages of α, K over the visible spectrum. The observation point in each case was about a meter below a calm air-water surface and when the skies were overcast or early

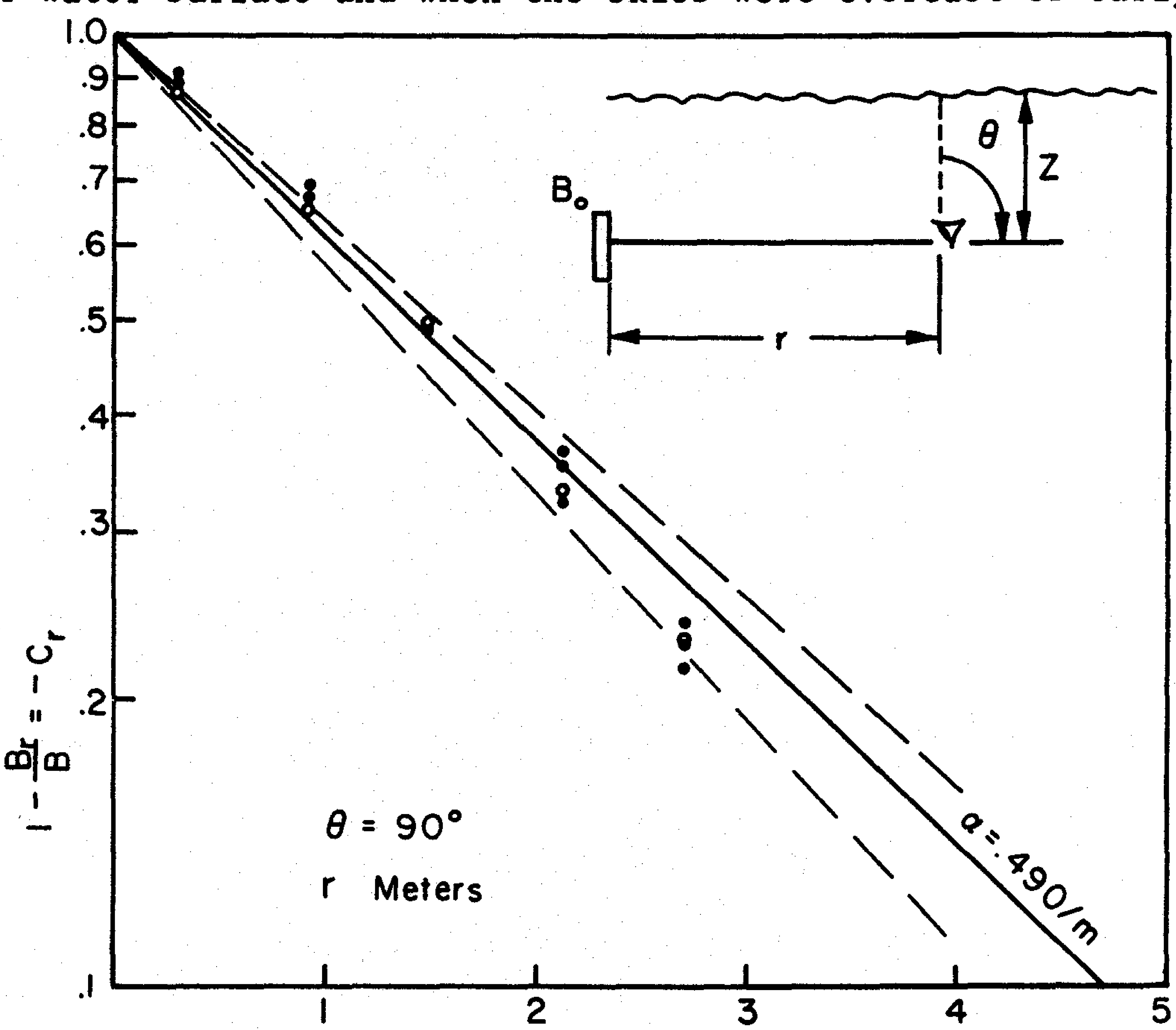


FIG. 1.51 Experimental checks of the exponential law for apparent contrast (cf. Fig. 1.50) by Duntley, Tyler, and Taylor, Lake Winnipesaukee, N.H., Summer 1958.

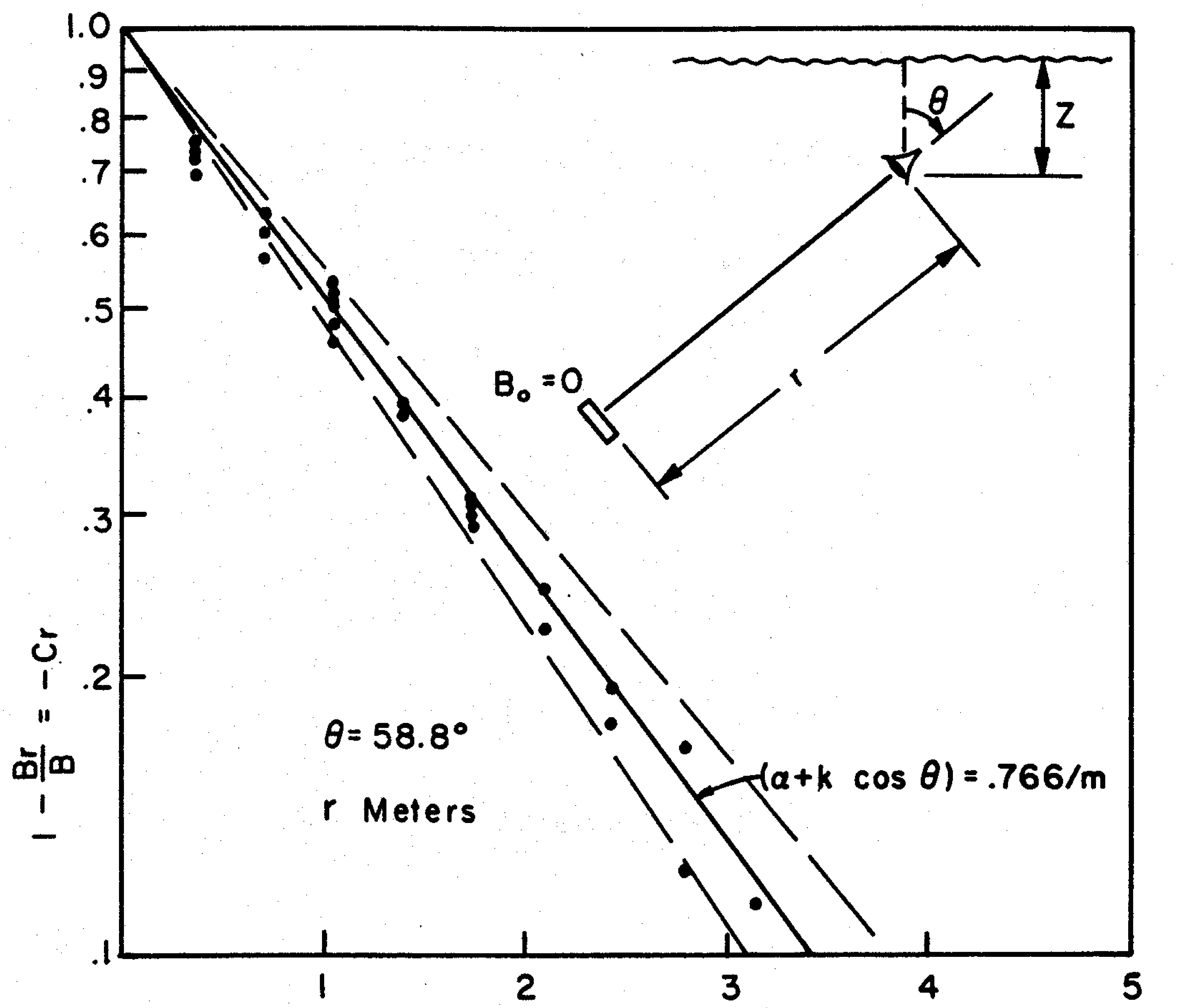


FIG. 1.52 Experimental checks of the exponential law for apparent contrast (cf. Fig. 1.51).

morning skies in each case. Further details may be found in [83].

We conclude with the observation of a useful corollary of (73), namely that *radiance differences propagate exactly according to the exponential law.* Thus

$$\left[{}_bN_r(z,\theta) - {}_tN_r(z,\theta)\right] = \left[{}_bN_o(z_t,\theta) - {}_tN_o(z_t,\theta)\right] e^{-\alpha r} \qquad (74)$$

Contrast Transmittances for General Backgrounds

It should be observed explicitly that formula (73) is of such generality that the apparent contrast C_r of an object need not be with respect to a water background. Rather, if ${}_bN_o$ in (73) is the inherent radiance of a background (as in Fig. 1.53) for a target of inherent radiance ${}_tN_o$ (shaded in the figure), then by computing ${}_bN_r$ according to (14) of

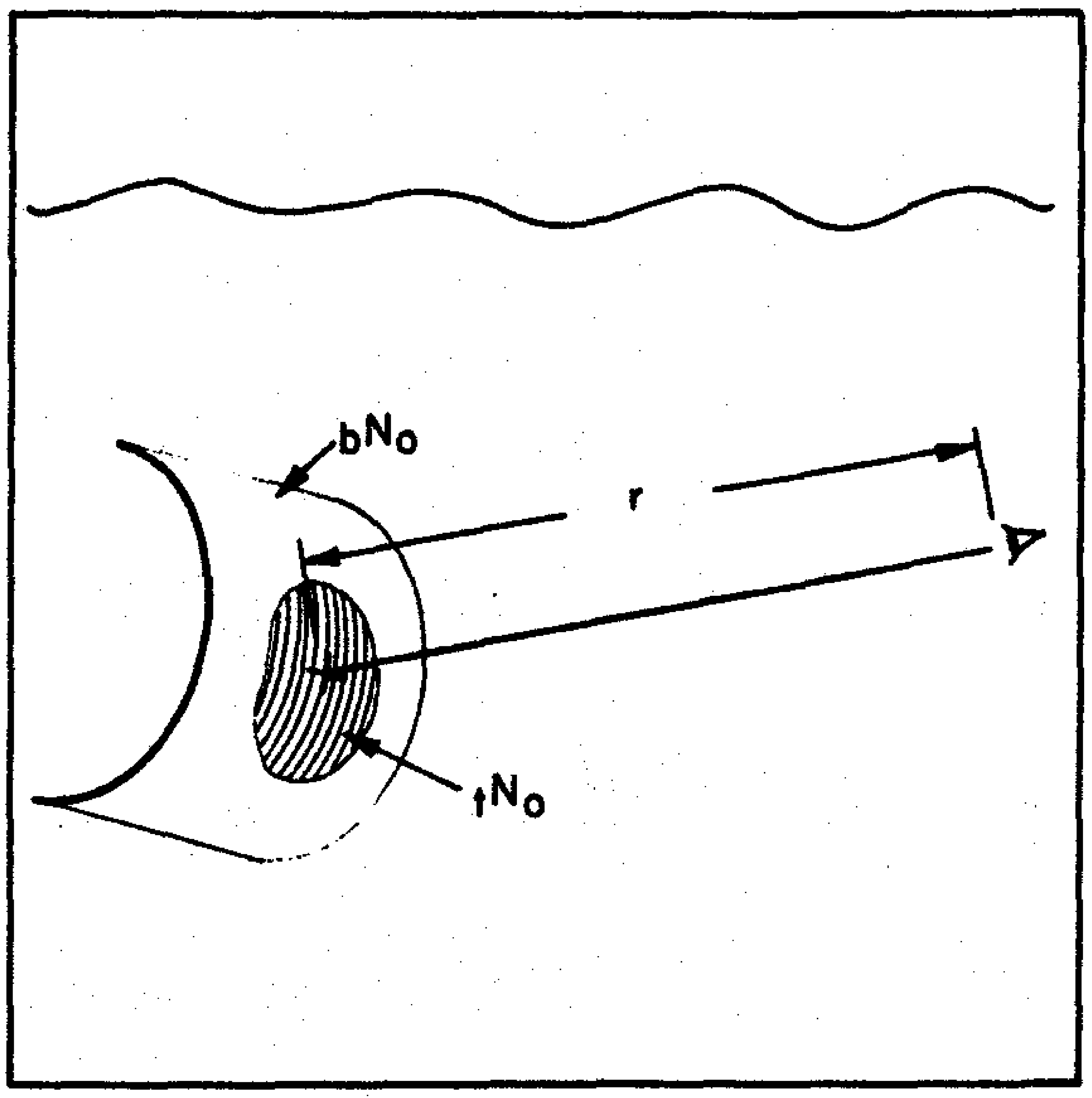


FIG. 1.53 Contrast transmittance for paths with arbitrary backgrounds.

Sec. 1.3 and using this in (73), the requisite apparent contrast transmittance C_r/C_o is determinable.

As a special case of (74) under these circumstances, let the line of sight be horizontal, then it is easy to see that:

$$C_r/C_o = \frac{1}{1 + \frac{N_q}{{}_bN_o}\left(e^{\alpha r} - 1\right)}, \qquad (75)$$

where N_q is the equilibrium radiance for the given horizontal path. Observe that the contrast transmittance of the given path uses only the radiances ${}_bN_o$ and N_q, i.e., the radiances making up the immediate background of the target. Of course in real media N_q is somewhat affected by both ${}_tN_o$ and ${}_bN_o$ (and conversely) so that the classically simple formula (75) does not rigorously hold. But within the framework of the present simple models and for paths of sight under ordinary lighting conditions, (75) is a quite useful and adequate formula.

The Multiplicative Property of Contrast Transmittance

If we take a still closer look at the contrast transmittance law (73), we find a most interesting property held by contrast transmittance in general, whether it be for paths of sight within the sea, or within the atmosphere, or even for paths partly in the sea *and* partly in the atmosphere!

To facilitate our discussions let us write:

$$"\mathcal{T}_r" \quad \text{for} \quad C_r/C_o$$

and when necessary we include location and direction variables with "$\mathcal{T}_r$". Now observe that $e^{-\alpha r}$ in (73) can be written as

$$e^{-\alpha r} = \frac{{}_bN_r^o(z,\theta)}{{}_bN_o(z_t,\theta)}$$

where ${}_bN_r^o$ is the *residual radiance* coming directly from the target background over the path of length r. (It is what is left of ${}_bN_o$ after scattering and absorption have taken their toll; cf. (24) of Sec. 1.3). Then we see that (73) can be cast into the form:

$$\mathcal{T}_r = \frac{{}_bN_r^o}{{}_bN_r} .$$

On this basis, we can work solely with the background radiance of a target when discussing beam transmittance of a path along which it is viewed. Hence we need no longer carry the reminder "b" before the radiance symbol. In other words we find that for a *general path of length* r *in a general hydrosol, the contrast transmittance* C_r/C_o *of the path is given by*

$$\mathcal{T}_r = \frac{N_r^o}{N_r} \tag{76}$$

This situation is summarized schematically in (a) of Fig. 1.54.

Next, suppose we have two paths of length r, s, end to end, as shown in (b) of Fig. 1.54. Let the inherent radiance at the far end be N_o. Then at the end of the first path segment of length s, we have, according to the preceding rule:

$$\mathcal{T}_s = \frac{N_s^o}{N_s} = \frac{N_o e^{-\alpha s}}{N_s} \tag{77}$$

where N_s is the apparent radiance associated with N_o, and N_s^o the residual radiance associated with N_o over the path of length s, both reckoned via (12) of Sec. 1.3, for example. The apparent radiance N_s now acts as did the initial radiance N_o, and N_s is transferred over the second segment of length r to give rise to an observed residual radiance $N_s e^{-\alpha r}$ and the apparent radiance N_{r+s} associated with N_s. Hence:

$$\mathcal{T}_r = \frac{N_s e^{-\alpha r}}{N_{r+s}} \tag{78}$$

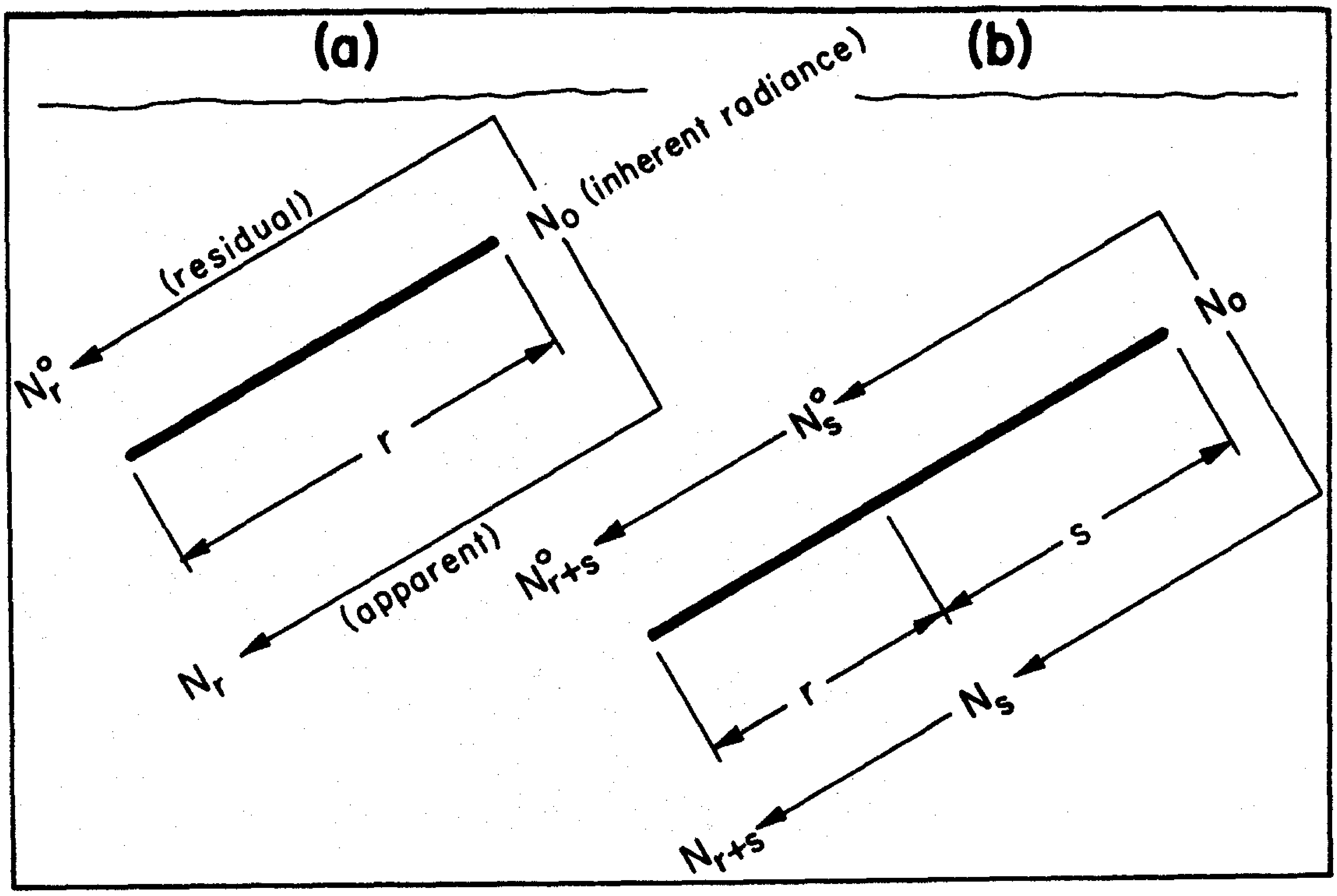


FIG. 1.54 Part (a): Deriving the contrast transmittance as the ratio of a residual radiance and an apparent radiance. Part (b): Deriving the multiplicative (or semigroup) property of contrast transmittance.

But looking at the path as a whole, we can also write:

$$\mathcal{T}_{r+s} = \frac{N^o_{r+s}}{N_{r+s}} \tag{79}$$

Comparing (77)-(79) we find:

$$\boxed{\mathcal{T}_{r+s} = \mathcal{T}_r \, \mathcal{T}_s} \tag{80}$$

This is the *multiplicative* (or *semigroup*) *property* of the contrast transmittance. The argument just used to derive (80) is readily extended without change of the form of (80) to arbitrary paths in air or water and across places where the index of refraction varies, provided in such cases we work with N/n^2 rather than N, where n is the index of refraction. For by the n^2-law for radiance, N/n^2 is invariant in transparent media with varying index of refraction (see Sec. 9.5).

As an obvious extension of (80), if a path consists of three contiguous, successive segments of arbitrary lengths r, s, t, then the contrast transmittance $\mathcal{T}_{r+s+t}$ of the composite path is simply a product of the three contrast transmittances of the segments:

$$\boxed{\mathcal{T}_{r+s+t} = \mathcal{T}_r \mathcal{T}_s \mathcal{T}_t} \qquad (81)$$

As an example of (81), consider a calm air-water surface. A line of sight of length t begins at a submerged object, is refracted at the air-water surface, and runs a length r in the air. Each of the three paths have an associated contrast transmittance. Errata While the path across the surface is of zero length, i.e., $s = 0$, there is a definite contrast reduction that takes place because of reflected sky light and reflected and transmitted underwater light occurring at the surface. The form of this *singular* contrast transmittance $\mathcal{T}_0$ is given in detail in (20), (23) of Sec. 12.2.

If, in addition, the air-water surface is in motion, then the above analysis must include an additional factor $\bar{\mathcal{T}}_0$ ($= \bar{C}/C$) associated with the time-averaged contrast reduction by refraction (cf. in (5) of Sec. 1.2). Hence now:

$$\bar{\mathcal{T}}_{r+t} = \mathcal{T}_r \mathcal{T}_0 \bar{\mathcal{T}}_0 \mathcal{T}_t \qquad (82)$$

gives the *time averaged* contrast transmittance $\bar{\mathcal{T}}_{r+t}$ for a path of length r in air, and going across a moving air-water surface and plunging a length t in a natural hydrosol (Fig. 1.55). The factors are as follows for a vertical line of sight in air and a small submerged target of half-angle subtense ψ as seen just below the surface:

$$\mathcal{T}_r = e^{-\alpha_1 r} \qquad \text{(in air)}$$

$$\mathcal{T}_t = e^{-(\alpha_2 + K)t} \qquad \text{(in water)}$$

$$\bar{\mathcal{T}}_0 = \left[1 - e^{-\frac{\tan^2\psi}{2\sigma^2}}\right] \qquad \text{(at the interface)}$$ Errata

Finally $\mathcal{T}_0$ is as given in (21) of Sec. 12.2 (wherein $N^0(x,\xi')$ is now the *time averaged* vertical upward radiance). The complete analysis of the time averaged radiance transmitted across the air-water surface is made in the latter half of Chapter 12, wherein the more or less intuitive type of factor analysis in (82) is bypassed in a direct, more general, but somewhat more difficult solution of the problem.

Theory of the Secchi and Duntley Disks

It is a part of almost everyone's experience to have thrown or dropped an object into deep water and to have watched it disappear into the depths. If the object is

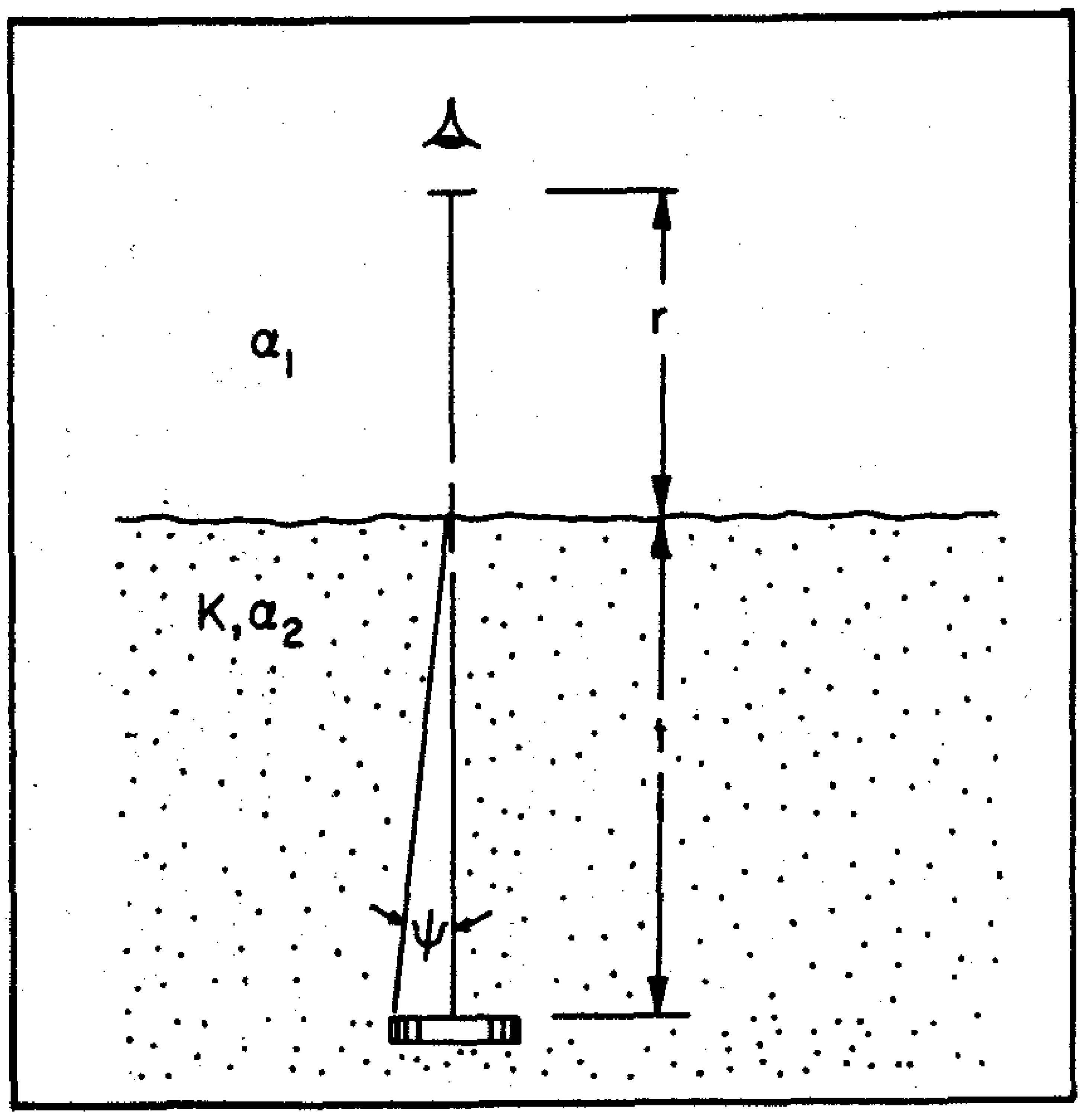


FIG. 1.55 Contrast reduction along a composite path through air, air-water surface, and water.

something bright or white, the eye can easily follow it down into the depths until it seemingly abruptly winks out, to be lost from sight from some depth onward. If the observer were of an inquisitive turn of mind, he may surmise that the general clarity of the water had something to do with the depth at which the object disappeared, and he may wonder if there were indeed a quantitative connection between the depth rate of decay of the light field in the water and also the depth rate of decay of the *whiteness* or *contrast* of the sinking object against the watery background, and maybe even the depth at which it seems to wink into obscurity.

Here is a hypothetical discussion about the radiometric problem of the sinking object, defined above, and which may occur on shipboard between a young eager theorist just learning the ropes and a seasoned experimenter in hydrologic optics just after one of them had accidently dropped a white glass coffee mug over the side of the oceanographic research vessel (which was moored in deep calm water at the time). Errata

Theorist: There you go, being careless with your design of experiments again. You didn't even note the sun altitude or what filter you were using.

Experimenter: I had an irresistable urge to see what would happen if I dropped it in.

Th. Good heavens man! Why the experiment? Have you forgotten Archimedes Law? On theoretical grounds, I predict that the mug will sink!

Ex. (Recovering from the accident) Look--it's turning bilious as it sinks deeper. What an interesting transformation of shades and hues. It looks like it's down 10 meters by now and I can still see it quite clearly!

Th. (Peering down over the railing) It must have reached terminal velocity by now and is surely sinking according to Stokes' law. (Looking at his watch, then a pause) At the sound of the tone it will be exactly 20 meters.

Ex. (Ignoring the other's babbling) There it goes. I lost track of it. There's no doubt about it, this is pretty clear water!

Th. What's the alpha and the kay for this water? Did you measure it again this morning?

Ex. It's the same as yesterday. The alpha's about a tenth per meter and the kay is about fifty thousandths per meter, both in the green. What are you doing?

Th. (Jotting something on a piece of paper so that the other can see it) I'll bet I can connect the mug's depth of disappearance with the alpha and kay of this water.

Ex. (Smiling wearily to himself, and then with a sigh): Here we go again. Take it easy, Einstein, my calculus is buried under a ton of barnacles.

Th. We really don't need it. Didn't you explain to me how it's known that the light level generally goes down exponentially with depth in deep water like this? I can use this fact to figure out how much light gets to the mug at each depth z. It would be (writing on the paper) $H_o e^{-Kz}$, correct?

Ex. Yes, and let's say that H_o is the irradiance on a horizontal surface just below the surface and K is the kay for this water, namely, .050/m. So you can figure out the irradiance on a horizontal surface at depth z. (Then feigning puzzlement) Where does that get you?

Th. Why, this lets you compute the inherent radiance of the mug at depth z, if you know its reflectance.

Ex. Do you know it?

Th. No, but let's just call it "R". Then (writing again) $R(H_o e^{-Kz})$ would give an estimate of the radiance reflected upward by the mug.

Ex. Hmm---Yes, but that's its *inherent* radiance down at depth z. Here we are on deck.

Th. I see what you mean. So we need the *apparent* radiance of the mug. But that'll mean knowing the path radiance generated by scattered light between us and the mug and also the effect of the air-water surface. Gosh, all that's pretty hard to come by isn't it?

Ex. Quite. But if you remember what I told you the other day about radiance *differences* . . .

Th. Radiance *differences*? Oh, of course! They are transmitted exactly according to the exponential law $e^{-\alpha r}$ for beamed light. Let's see, the radiance difference in this case will be between the inherent radiance of the mug at depth z and the inherent radiance of the background water at the same depth. Such a difference is easy to figure.

Ex. Is it? Again you don't know the reflectance of the water at the depth of the mug. At least I haven't measured it yet for this place.

Th. That's O.K. Let's call the reflectance of the water "R_∞". It could not be much different from .02 for all depths. I was looking over some of your old reports, and review articles yesterday. Everywhere you measured R_∞ you got something around .02 for green light, even some deep clear lakes and ponds, *n'est ce pas*?

Ex. (Gritting his teeth) I am afraid so. Very few surprises left there. Well, where are you leading me next with your paper and pencil?

Th. The average radiance of the water background at depth z is simply R_∞ times the downward irradiance at that depth. That is, we would have $R_\infty(H_o e^{-Kz})$. Right?

Ex. Yes, except for a factor of π--but they'll all cancel out anyway in the end. So don't worry about it now.

Th. (Looking up surprised) Say--how do you know that? Have you worked all this out before?

Ex. (With a straight face, looking out at the horizon) Not exactly. On with it--what is your next step?

Th. Well here is the radiance difference between the mug and the sea at depth z:

$$H_o R e^{-Kz} - H_o R_\infty e^{-Kz}$$

Ex. And then?

Th. And then at long last I can use the radiance difference law. That is I multiply this difference by $e^{-\alpha z}$ to transmit it up to just below the surface--where it'll be what we will actually see if we went there. Thus:

$$\left[H_o R e^{-Kz} - H_o R_\infty e^{-Kz}\right] e^{-\alpha z}$$

Ex. Can you simplify this mess?

Th. Sure, like this:

$$H_o(R-R_\infty)e^{-(\alpha+K)z}$$

Ex. Also I don't like to bother with absolute light levels. Can you take care of that, too?

Th. Yes, I suppose. Why not divide the whole thing by the amount of reflected radiance from the sea just below the surface? Like this:

$$\frac{H_o(R-R_\infty)e^{-(\alpha+K)z}}{H_o R_\infty}$$

Ex. That'll work fine. Now, what have you got for all your trouble?

Th. (A pause, and then) Why this looks like it could be a kind of contrast reduction formula...yes, it is...just let $H_o(R-R_\infty)/H_o R_\infty$ or simply $(R-R_\infty)/R_\infty$ be the inherent contrast of the mug against its background. It looks like this contrast is independent of the depth of the mug. That's fantastic! Is that right? Errata

Ex. (Blanching) Yes, go on...

Th. So if the apparent contrast of the mug at depth z as seen from just below the surface is C_z, then it looks like we have

$$C_z = C_o e^{-(\alpha+K)z}$$

Ex. (A little startled at the equation's quick appearance from an unexpected line of argument) Would you know how to use something like that?

Th. (After a while) Well, if we can agree that the mug disappears when C_z/C_0 is some small number, maybe like 1/50, and measure the z for such a ratio, then we can compute the corresponding $\alpha+K$. It's true we couldn't find α and K separately this way, but the sum is probably still a good index of water clarity.

Ex. (In mock anger) Incredible! Do you know what you've just done, boy?

Th. (Somewhat aghast) No, sir. But I do know that we haven't allowed for the surface effects yet. Is something wrong?

Ex. No, it's just that throughout this discussion I've seen several old friends in a new light. You did well. Now, you run along below and get me a fresh mug of coffee. And on the way back drop into the ship's library. I want to show you something in Sec. 1.4 of "Hydrologic Optics".

It wasn't long until the young theorist saw how to derive the contrast law in the orthodox way (see, e.g., (72)) and how to put in the contrast transmittance factors for the surface, as we have seen for ourselves in (82). It was also made clear to him how Secchi [283] had many years before, in 1865, devised an empirical procedure of just this type for finding a water clarity index which used the depth of disappearance of a standardized disk, and finally of how the meticulous care with which Secchi had stated his measuring procedures had generally been ignored or diluted by subsequent generations of users of his method.

In 1949 Duntley [82] examined the Secchi disk procedure and devised a simple alternative scheme whereby it would be less subject to the vagaries of individual experimenters and lighting conditions during the moment of disappearance of the disk. Duntley observed that one important seat of the difficulty of using Secchi disk readings lay in coping with the contrast transmittance factors $\mathcal{T}_0$ and $\bar{\mathcal{T}}_0$ in (82) (the factor $\mathcal{T}_r$ is essentially unity for work right above the surface).

Suppose then, Duntley reasoned, that *two disks were used,* one being white, the other gray. Suppose further that the two disks are lowered together, side by side into the water a meter or two or so below the water surface, say to depth z. An observer above the surface will see them side by side: a white and a gray disk--each a bit dimmer now, but their luminances still quite distinct. Then the white one is slowly lowered farther into the water, the other being held fixed. As it is lowered, the white disk becomes darker (the e^{-Kz} effect setting in) and soon, at some depth d below the gray disk, there appears to be a luminance match between the two disks (see Fig. 1.56). At this stage of the experiment, we see that by (72) and (82):

$$C_z = \left[C_0 e^{-(\alpha+K)z}\right] \mathcal{T}_0 \bar{\mathcal{T}}_0$$

for the gray disk, and that:

$$C'_{z+d} = \left[C'_0\ e^{-(\alpha+K)(z+d)}\right] \mathcal{T}_0 \bar{\mathcal{T}}_0$$

for the white disk, and indeed, that:

$$C_z = C'_{z+d} \quad .$$

(As they stand, either of these formulas for C_z or C'_{z+d} by itself comprises the theory of the Secchi disk.) By taking the ratio of these contrasts, we eliminate the troublesome contrast transmittances $\mathcal{T}_o$, $\bar{\mathcal{T}}_o$, to find:

$$1 = \frac{C_z}{C'_{z+d}} = \frac{C_o}{C'_o} e^{(\alpha+K)d} = \frac{R-R_\infty}{R'-R_\infty} e^{(\alpha+K)d} \qquad (83)$$

Hence

$$\boxed{\alpha+K = \frac{1}{d} \ln \left(\frac{R'-R_\infty}{R-R_\infty}\right)} \qquad (84)$$

Using the experimental fact that in green light R_∞ is on the order of .02 (but of course with some variation possible) for most natural hydrosols, and that the R of the gray disk and the R' of the white disk may be easily chosen much greater than the R_∞ of the water to be measured, (84) can be written very nearly as:

$$\alpha+K = \frac{1}{d} \ln \left(\frac{R'}{R}\right) \qquad (85)$$

Since the number ln (R'/R) is known and fixed for a pair of disks, a table can be made from which one can read off $\alpha+K$ directly from the match-depth-difference d.

Suppose further that someday an optical oceanographer equipped with a scuba and a light-weight pair of Duntley

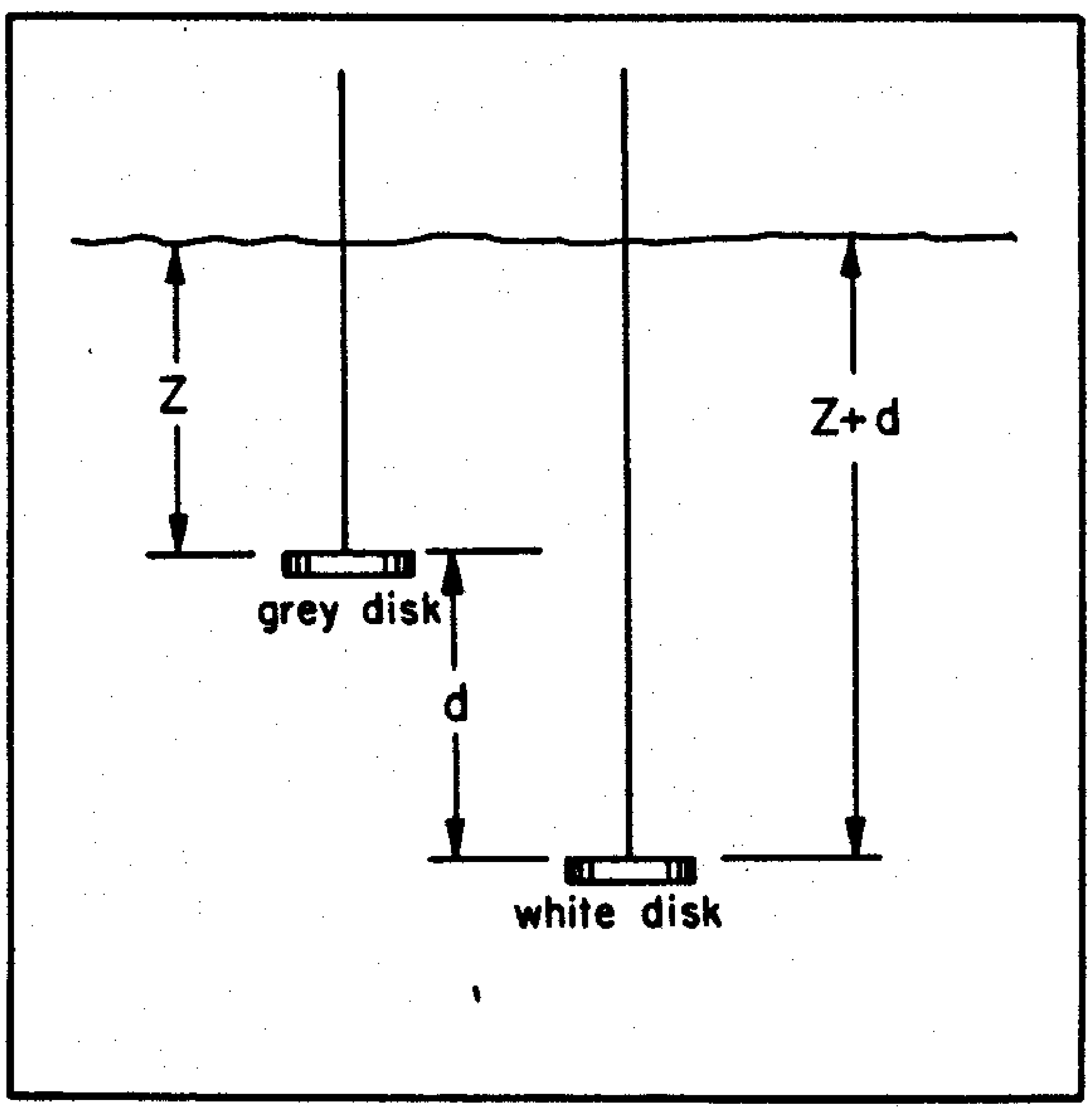


FIG. 1.56 The Duntley-disk procedure for measuring α+K.

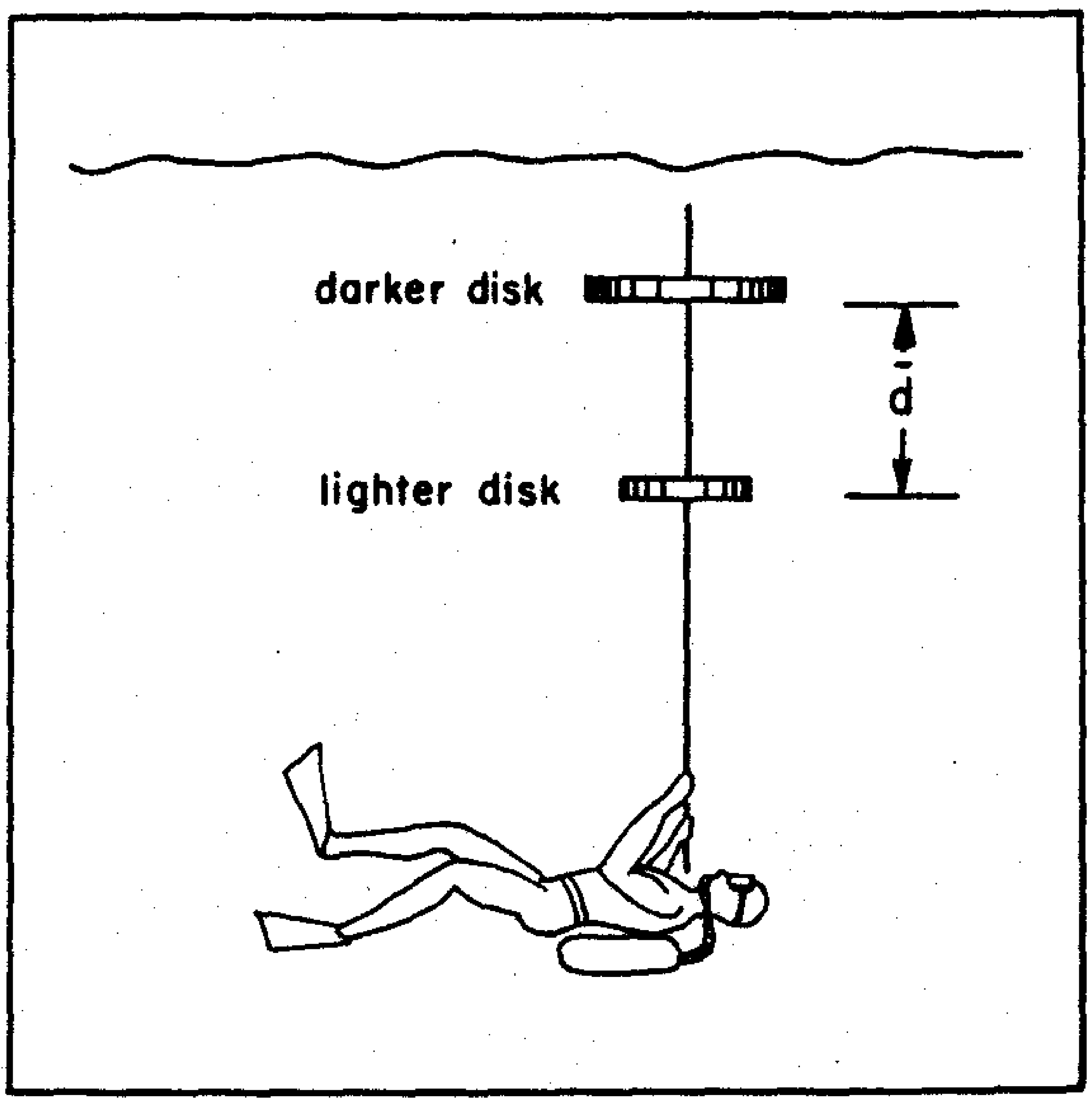


FIG. 1.57 The Duntley-disk procedure for measuring α-K.

disks on a rod (as in Fig. 1.57) will be able to measure the match-depth-difference $\bar{d}$ for a vertically upward line of sight. Then from an analysis based either on the kind of reasoning by the young theorist in the dialogue above or by simply appealing to (72) with θ = 180°, we could deduce that, analogously to (85):

$$\alpha - K = \frac{1}{\bar{d}} \ln \left(\frac{R'}{R} \right) \qquad (86)$$

Errata

From this and (85), we find:

$$\alpha = \frac{1}{2} \left(\frac{1}{d} + \frac{1}{\bar{d}} \right) \ln \left(\frac{R'}{R} \right) \qquad (87)$$

and

$$K = \frac{1}{2} \left(\frac{1}{d} - \frac{1}{\bar{d}} \right) \ln \left(\frac{R'}{R} \right) \qquad (88)$$

If such a device is used, it should have sectors (or perhaps annuli) on each disk of different whites and grays (when the diver looks upward the darker disk must be farther from him at match time). It is also suggested that the divers wear goggles which transmit in some given small band width of the spectrum around which the K and α values are to be determined. A readily used band width would be centered on the blue-green or yellow-green peaks of transmittance of most natural waters. Some care must also be given to the adaptation of the diver's eyes to the general level of illumination in which the visual match is best made. The importance of levels of illumination in underwater visibility tasks will be illustrated as a matter of course in Sec. 1.9.

Theory of Absorption Measurements in Natural Hydrosols

It is probably a continual source of fascination for highway patrolmen to examine the daily tallies of vehicles that pass over certain continuous road segments on superhighways or relatively desolate roads located between consecutive toll houses, and occasionally to be rewarded with a positive net influx of cars across a given segment. That is, when they subtract from the recorded number of vehicles entering the segment for each day the number of vehicles leaving the segment that same day they occasionally find a *positive difference*! From a purely phenomenological point of view, this means that some vehicles have been absorbed in their passage through the given stretch of highway! Of course, if the tally is correct, this could mean for example that there exist stalled vehicles somewhere along the segment, and a patrol is usually dispatched to investigate.

The principle of detection of the absorption of photons in a given layer of a natural hydrosol is exactly analogous to the toll house tally procedure for wayward vehicles described above. In Fig. 1.58 a laterally extensive layer of water between two levels y and z in a stratified optical medium is monitored by irradiance meters measuring $H(y,\pm)$ and $H(z,\pm)$. The total influx of irradiance to the layer is $H(y,-)+H(z,+)$, and the total efflux is $H(y,+)+H(z,-)$. Therefore the net influx of irradiance is

$$[H(y,-)+H(z,+)] - [H(y,+)+H(z,-)] = \bar{H}(y,-)-\bar{H}(z,-)$$

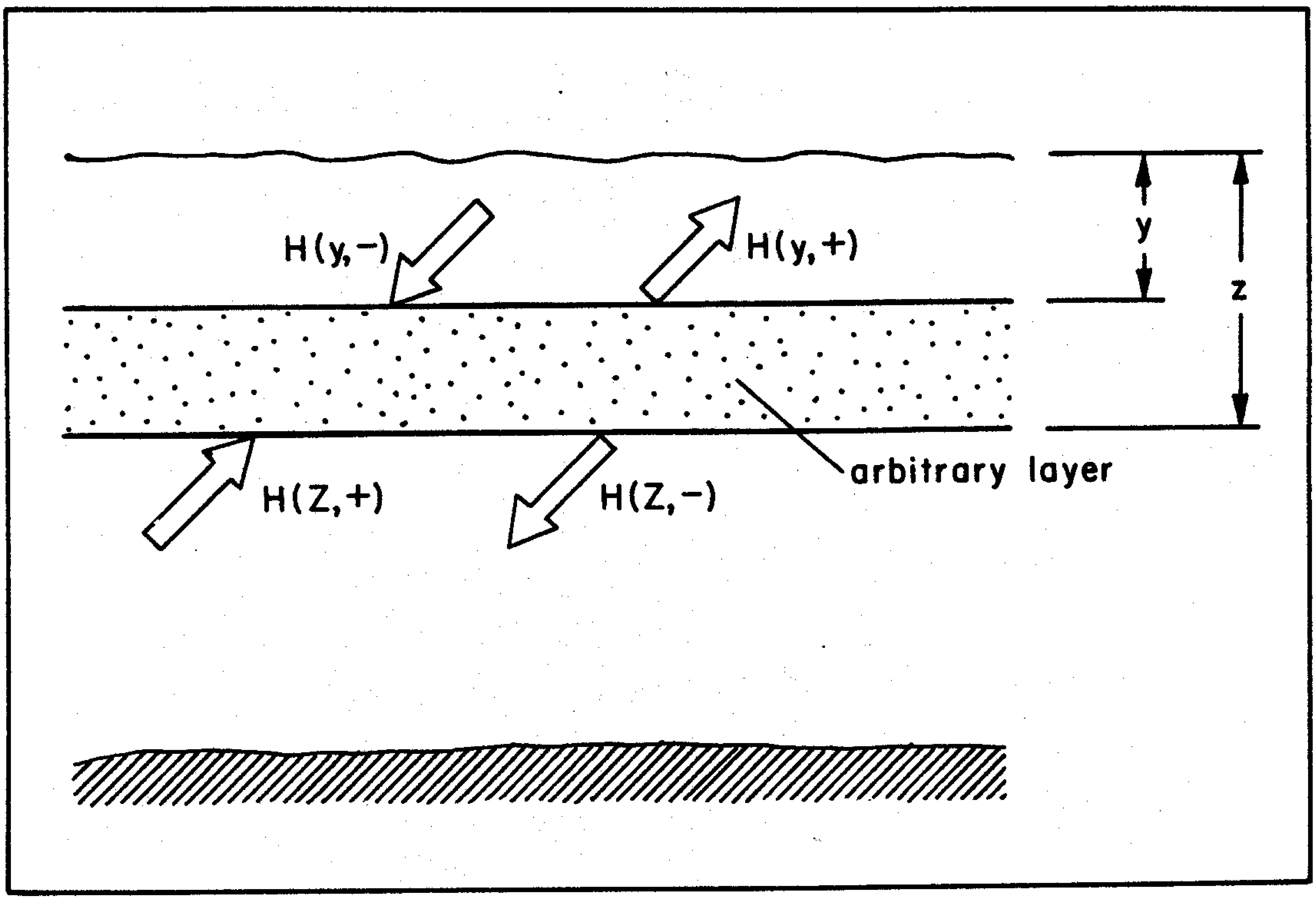


FIG. 1.58 The principle for determining light absorption in a layer of natural hydrosol.

By the same general reasoning leading to (2) we find that (as in the case of the one-D model) in all real natural hydrosols wherein there are no internal sources of radiant flux, this difference is positive, indicating that some fraction of the entering photons is continually being absorbed within the layer. The nature of the absorption is two-fold: if a tally is being kept only of photons of a given color (frequency) then the absorption in this case includes scattering with change in *color*. Secondly, absorption could mean the transformation of radiant energy into non-radiant energy. In practice both of these mechanisms are operative. The difference $\bar{H}(y,-)-\bar{H}(z,-)$ is a direct measure of the amount of radiant flux absorbed by a column of unit horizontal cross section bounded by the upper and lower planes of the layer of water. If a is the volume absorption coefficient of the (homogeneous) layer, then this quantity is directly measurable by means of the relation:

$$a = \frac{\bar{H}(y,-) - \bar{H}(z,-)}{\int_y^z h(z')\,dz'} \tag{89}$$

provided a probe is sent down to find the values h(z') of the scalar irradiance between depths y and z. The reader may check that (89) follows directly from (16) of Sec. 1.3. Hence (89) is an exact formula for homogeneous media with a stratified light field. A discussion of (89) and a systematic derivation of the related formulas below is given in Sec. 13.8.

A local version of (89) comes from (16) of Sec. 1.3 directly:

$$a(z) = \frac{1}{h(z)} \frac{d\bar{H}(z,+)}{dz} \tag{90}$$

To use (90), one need only measure h(z) at depth z, and also $\bar{H}(z,+)$ in a small neighborhood of depths about depth z, so as to be able to compute the derivative of $\bar{H}(z,+)$ at that depth. This method is exact for all inhomogeneous stratified media. An instrument to measure a, and which is based on the principle represented by (90), has been devised by Tyler [299] at the Visibility Laboratory.

It is important to notice two essential features of (90). First, observe that scalar (rather than ordinary irradiance) is used to normalize the derivative; second, the *net* irradiance is used in the derivative. Now it turns out that of these two features, it is the first that is of critical importance and which gives the formula its distinctive power in natural hydrosols. To see this, recall from the preceding discussions that the reflectance R_∞ for green light is quite small in clear deep media, the kind found in most oceanic work, for example, Hence in:

$$\bar{H}(z,+) = H(z,+) - H(z,-)$$

and in:

$$h(z) = h(z,+) + h(z,-)$$

we can ignore with a fair measure of impunity the terms H(z,+) and h(z,+). In that case, (90) becomes:

$$a(z) \cong \frac{-1}{h(z,-)} \frac{dH(z,-)}{dz} \quad .$$

Furthermore, by virtue of the distribution factors $D_{\pm}$ defined in the two-flow model, we can write:

$$h(z,-) = D_{-}H(z,-) \quad . \tag{91}$$

In addition, if we estimate H(z,-) by means of the exponential law:

$$H(z,-) = H(0,-)e^{-Kz}$$

(where K is obtained either via the one-D model, as in (9) of Sec. 1.3, or empirically), then (91) yields:

$$a \cong \frac{-1}{D_{-}H(z,-)} \cdot [-KH(z,-)]$$

$$= \frac{K}{D_{-}} \tag{92}$$

This points up the critical importance of the *scalar* irradiance h(z) in (90); for if we used H(z,+)+H(z,-) in its place, then we would have (90) yield up the estimate

$$a \approx K \quad \text{(wrong)}$$

which is clearly false. Indeed, the factor D_{-} is often on the order of 1.0-2.0 in natural optical media with values clustering about 1.3 for blue-green light, so the use of H rather than h to normalize the derivative in (90) could lead to errors anywhere from 0 to 100 percent in the estimate of a(z), but mostly on the order of 30 percent.

From (90) we can also obtain a crude but occasionally useful estimate of the *rate of absorption of radiant energy per unit volume of a layer of water.* First:

$$\frac{d\bar{H}(z,+)}{dz} = a(z)\ h(z) \tag{93}$$

is the exact formula for the required depth rate of absorption, i.e., of net influx of irradiance to a unit layer at depth z. It is simply the product of a(z) at depth z with h(z) at depth z. Now if we again drop off H(z,+) and h(z,+) as being small compared to H(z,-) and h(z,-), we have:

$$\frac{dH(z,-)}{dz} \cong -a(z)\ h(z,-)$$

$$= -a(z)D_-H(z,-)$$

$$\cong -KH(z,-) = -KH(0,-)e^{-Kz}$$

as the depth rate of absorption of radiant flux per unit volume at depth z. The last approximation comes from (92) and by means of the exponential law for irradiance. It should be noted that (93) is exact only for stratified media. If one wishes to compute exactly the rate of absorption of a small volume of water in a general light field in a generally inhomogeneous optical medium he may use (1) of Sec. 13.8 and the general instructions given there.

As an illustration of (90) as a means of estimation of the volume absorption coefficient, consider the sample light field given in Table 1.

TABLE 1

Irradiance and Scalar Irradiance in Lake Pend Oreille, Idaho. (Relative values)

z(meters)	H(z,-)	H(z,+)	h(z,-)	h(z,+)
4.24	721,000	15,500	899,000	41,900
10.42	329,000	6,040	413,000	16,500
16.58	109,000	2,230	141,000	6,190
28.96	13,100	298	17,200	830
41.30	1,660	39	2,190	108
53.71	221	5	289	14

These data were obtained by Tyler, Richardson, and Holmes from radiance distribution measurements in Lake Pend Oreille, Idaho [306]. Radiance filters were centered on 480 ± 64 mμ. Observe first that D_- at 4.24 meters is 1.25, and that its value at 53.71 meters is 1.31. This shows, incidentally, the general magnitude of D_- found in most natural waters for blue-green light. Similar values may be found at the other depths. By computing the slope of the $\bar{H}(z,+)$-plot derived from the tabulations above, and using the computed h(z) values, it was found via (90) that the lake was essentially homogeneous with an a on the order of .117/m. The K for this medium was found to be .169/m, and α = .442/m.

We can invert the formulas (89) and (90) to find the rate of absorption of radiant energy in a given medium, given the volume absorption function and some radiometric samplings of the medium. For example in infinitely deep media in which scalar irradiance decreases according to the exponential law, we can estimate the total rate of absorption as follows. In (89) set $y = 0$ and $z = \infty$, so that $\bar{H}(\infty,-) = 0$. This leaves:

$$\bar{H}(0,-) = a\int_0^{\infty} h(z')\, dz'$$

Using the exponential law for $h(z)$:

$$\bar{H}(0,-) = ah(0)\int_0^{\infty} e^{-Kz'}\, dz'$$

That is:

$$\bar{H}(0,-) = \frac{a}{K}\, h(0) \qquad (94)$$

This formula holds actually for any depth z below the surface. (Simply multiply each side by e^{-Kz}.) If z is used in place of 0, then $\bar{H}(z,-)$ in (94) is a measure of the radiant flux absorbed by the entire medium *below* the level z.

As an illustration of (94), suppose that $h(0) = 250$ watts/m² on some sunny day just below the surface, for the wavelength band 480 ± 64 mμ. The total rate of absorption throughout the lake per square meter of lake surface is therefore:

$$\bar{H}(0,-) = \frac{.117}{.169} \times 250$$

$$= 173 \text{ watts/m}^2 \quad .$$

The remaining power, namely 250-173 = 77 watts/m² goes on to initiate and sustain the scattered light field within the body of the lake.

As another illustration of (94), suppose that measurements of $\bar{H}(y,-)$ and $h(y)$ are made at some depth y in a deep homogeneous medium, and also that K is known for the same wavelength interval. We can then estimate a as follows. From (94):

$$a = K\,\frac{\bar{H}(z,-)}{h(z)} \qquad (95)$$

For example, from Table 1, at depth 28.96 meters, we have

$$\bar{H}(28.96,-) = 13{,}100 - 298$$

$$= 12{,}802 \text{ watts/m}^2$$ Errata

Also,

$$h(28.96) = h(28.96,+) + h(28.96,-)$$

$$= 830 + 17{,}200$$

$$= 18{,}030 \text{ watts/m}^2$$ Errata

Hence:

$$a = .169 \quad \frac{12,802}{18,030}$$

$$= .120/m \quad .$$

which agrees to within .003/m with the estimate .117/m for a obtained by light field measurements using (90).

We conclude with some observations on the radiant energy content of natural hydrosols, a concept which is closely related to the absorption concept presently under discussion. Recall the general relation between scalar irradiance h(z) and radiant density u(z) as given in (5) of Sec. 1.1:

$$u(z) = \frac{1}{v} h(z) \tag{96}$$

Here v is to the speed of light in homogeneous water:

$$v = 2.25 \times 10^8 \text{ m/sec} \quad .$$

By integrating h(z) from the surface (z = 0) down to depth z in an infinitely deep medium we find:

$$U(z) = \int_0^z u(z)\ dz = \frac{1}{v}\int_0^z h(z)\ dz = \frac{h(0)}{vK}\left[1-e^{-Kz}\right] \tag{97}$$

provided h follows the exponential law. This gives the amount of radiant energy U(z) in a vertical column of unit horizontal cross section with upper end at the surface and lower end at depth z. Observe that by (89) this also can be written

$$U(z) = \frac{\overline{H}(0,-) - \overline{H}(z,-)}{va} = \frac{h(0)}{vK}\left[1-e^{-Kz}\right] \tag{98}$$

For very shallow media, (98) yields

$$U(z) \cong \frac{h(0)z}{v} \tag{99}$$

For very deep media (98) yields

$$U(\infty) = \frac{h(0)}{vK} \tag{100}$$

In the present medium, (Lake Pend Oreille) which is very deep, with K = .169/m and h(0) = 250 watts/m² (say), we find

$$U(\infty) = \frac{250}{2.25 \times 10^8 \times .169}$$

$$= 6.6 \times 10^{-6} \text{ joules/m}^2$$

Hence over a region of one square kilometer (10^6 m^2) the present medium contains below the surface about 7 joules of radiant energy in the blue-green wavelength interval in scattered or directly transmitted form. Observe by (98) that nearly 95% of this radiant energy is stored within the first three diffuse attenuation lengths below the surface, i.e., within $3/K = 3/.169 = 17.7$ meters of the surface. Equation (98) shows how $U(z)$ can be estimated if the net influx of radiant energy over the depth interval $[0,z]$ is known, along with the volume absorption coefficient a. Further discussion of light storage phenomena in natural waters is given in Sec. 5.13.

<< Table of Contents << Title Page >> Next Page >>

1.5 Some Properties of Artificial Light Fields in Natural Waters

Artificial light fields in seas and lakes are produced by men seeking to illuminate natural underwater environs to carry out search or detection procedures, to study biological processes, or to establish techniques of underwater communication by means of residual and scattered radiant flux. To facilitate these activities some knowledge is desirable of the general quantitative relations between the optical properties of a medium and the light fields produced in that medium by various artificial sources. Such sources commonly range from those that produce highly collimated beams to those that produce conical beams of varying spread, up to uniform point sources. In this section we shall discuss several interesting empirical relations developed for artificial light fields.

Useful models of artificial light fields, which can completely elucidate the empirical findings presented below, may be based on the diffusion models discussed in Chapter 6, in particular in Secs. 6.5-6.7. However, we shall concentrate in this brief survey of artificial light fields only on the diffusion model (27) of Sec. 1.3, as it affords a simple yet adequate base on which to rest the empirical formulas.

The Pure Absorption Case

To see what the difficulties are in describing artificial light fields in the sea, suppose for the moment that sea water or any other natural hydrosol only absorbed radiant flux, and therefore did not scatter it. Suppose that a spherical source S of radius r_o, as in Fig. 1.59, has a uniform inherent surface radiance N_o. Then the apparent radiance N_r of this source's surface is:

$$N_r = N_o e^{-ar} \tag{1}$$

where a is the volume absorption coefficient of the medium. The radiant flux output P_o of the source is:

$$P_o = (4\pi r_o^2)\pi N_o \tag{2}$$