single frequency as carried by a single photon (or a pure monochromatic wave of light) exists in principle. Therefore, it is possible that $\Phi(S,D,t,\{\nu\})>0$ for some set F consisting of a single frequency ν, and that $l(F) = 0$ at the same time. This follows from use of the usual measure of length on the frequency domain. If one redefines length on the frequency domain by adopting a Stieltjes measure, e.g., so that isolated single frequencies are given nonzero (usually unit) length (instead of the zero length we conventionally assigned them by the usual continuum measure) then (2) would hold on the electromagnetic and quantum levels too, and (5)-(8) remain as the source of the fundamental distinctions between the microscopic and the macroscopic views of light.

2.4 Irradiance and Radiant Emittance

We now turn to the task of defining the radiometric concepts used in radiative transfer in general and hydrologic optics in particular. The first two of these are the concepts of irradiance and radiant emittance. These concepts describe the flow of radiant energy per unit area across a surface. That is, they describe the area-density of radiant flux. Irradiance describes the flow *onto* a unit area; radiant emittance describes the flow *from* a unit area. From a strictly geometric point of view, this is the only distinction between the two concepts. However, radiant emittance occasionally has an additional physical connotation associated with it, namely that of a flow of radiant flux from a unit area of surface which encloses an emitting source of radiant flux, i.e., a region manufacturing radiant energy. However, within the operational definitions of these concepts, this additional connotation does not exist; the connotation exists only in the mind of the experimenter. We now turn to the detailed definitions of these concepts.

Definition of Irradiance

We begin with the concept of irradiance. Imagine a radiant flux meter transported to a point x in a natural hydrosol, or in the atmosphere. Let the collecting surface S of the meter be placed so that x falls within its small expanse, and orient the set D of directions of the meter as desired. A filter is fitted on the meter so as to pass monochromatic radiant flux of given frequency ν. Hence, the meter can be made to read P(S,D) directly (with ν and t and their units understood). Let "A(S)" denote the area of the collecting surface S. Then we shall write:

$$\text{"}H(S,D)\text{"} \quad \text{for} \quad P(S,D)/A(S) \tag{1}$$

and call H(S,D) the (*empirical*) *irradiance* over S within D. In full notation for the unpolarized context, we would write:

$$\text{"}H(S,D,t,F)\text{"} \quad \text{for } \Phi(S,D,t,F)/A(S) \tag{2}$$

or

$$\text{"}H(S,D,t,\nu)\text{"} \quad \text{for} \quad P(S,D,t,\nu)/A(S) \quad . \tag{3}$$

However, in most discussions of radiative transfer in hydrologic and meteorologic optics the light field is steady in time, and is studied frequency by frequency. Hence we shall until further notice hold t and ν (or F) fixed and so exclude their symbols and units from the notation, as in (1).

Next, we let S become smaller and smaller, such that it always contains the point x and such that the flow of radiant energy is onto S along the fixed set D of directions. Then we write:

$$\text{"}H(x,D)\text{"} \quad \text{for} \quad \lim_{S\to\{x\}} H(S,D) \quad . \tag{4}$$

The existence of this limit is guaranteed by the S-additive and S-continuity properties of Φ postulated in Sec. 2.3. The irradiance H(x,D) is the (*theoretical*) *irradiance* at x within D. The dimensions of both empirical and theoretical irradiance are *radiant flux per unit area* (per unit frequency interval); convenient units are *watts/(meter)*^{2} (per unit frequency interval).

It is of interest to see the meaning of H(x,D) in terms of the radiant flux function Φ of Sec. 2.2. Thus, from (4) and (3) (making ν and t explicit for the moment):

$$H(x,D,t,\nu) = \lim_{S\to\{x\}} P(S,D,t,\nu)/A(S)$$

From Sec. 2.3 this becomes:

$$H(x,D,t,\nu) = \lim_{S\to\{x\}} \left[\lim_{F\to\{\nu\}} \frac{\Phi(S,D,t,F)}{l(F)\ A(S)} \right]$$

$$= \lim_{S\to\{x\}} \left[\lim_{F\to\{\nu\}} \frac{H(S,D,t,F)}{l(F)} \right] \tag{5}$$

It follows from (3) above and a theorem of calculus that:

$$P(S,D,t,\nu) = \int_S H(x,D,t,\nu)\ dA(x) \tag{6}$$

and hence from (4) of Sec. 2.3 that:

$$\Phi(S,D,t,F) = \int_S \int_F H(x,D,t,\nu)\ dl(\nu)\ dA(x) \tag{7}$$

It is easy to see that these integrals can be generalized to the case where D in $H(x,D,t,\nu)$ may vary with x, and we shall understand that (6) and (7) hold in such cases.

In actual practice, the size of the collecting surface S, which serves to accept, diffuse, and transmit the incident flux on to the photoelectric element of the meter, ranges from the size of a pinhead to that of a dinner plate. These extremes are not intended to be precise limits; rather they are representative of the extremes that may be encountered in natural radiometric environments under ordinary working conditions. The lower limit cited above begins to approach the size where, for very sensitive photoelectric elements, effects of diffraction may be noticeable. For example, an ordinary household stickpin or a human hair held in a pencil-thin shaft of sunlight will cast a shadow with a diffraction pattern clearly discernable by the unaided human eye. Hence a very small radiant-flux meter collecting-surface can pick up such irradiance variations over the shadow. The upper limit cited above is dictated by the fact that natural lighting variations become noticeable over such relatively large areal extents: changes of lighting with depth in ponds or oceans, shadows cast by leaves or fish, edges of dense cloud shadows on the ground or a sea surface, etc. Hence by staying within these limits and choosing the size of S accordingly, good empirical estimates of irradiance can usually be made using the definition (1).

The Meaning of 'Irradiance'

It is occasionally helpful in both theoretical and experimental considerations to keep in mind the various meanings of 'radiant flux' discussed in Sec. 2.2. These may be applied directly to the concept of irradiance. Thus H(S,D) may be imagined as proportional to the number of lines of flux incident per unit area over S and whose directions at their points of intersection with S lie within the set D. Further, using the Poynting vector interpretation of radiant flux, we see that the dimensions of the vector are precisely those of irradiance. Finally, H(S,D) may be viewed as a measure of the number of photons per unit area per unit time on S, funneling down onto S along the directions of D. In particular, using (2) above and (2) of Sec. 2.2 for a monochromatic set of n photons over a small collecting area, and incident within a small set D of directions normally on S, we have:

$$H(S,D,t,\{\nu\}) = \frac{\Phi(S,D,t,\{\nu\})}{A(S)} = h\nu v n \Omega(D) \qquad (8)$$

A further insight into the concept of irradiance is obtained by considering some of the typical magnitudes of irradiance encountered in natural environments. Table I lists some of these values. They are order-of-magnitude estimates and are not to be used beyond establishing an intuitive feeling for the meaning of irradiance (see also Sec. 1.2).

TABLE 1

ENVIRONMENT	TYPICAL ORDER OF MAGNITUDE OF IRRADIANCE
At sea level, on surface S normal to sun's rays, clear day*	10^3 watt/m^2
At sea level, slightly overcast days, horizontal surfaces	10^2 watt/m^2
At sea level, heavily overcast day, horizontal surface S (sunset)	10 watt/m^2
Lighted interiors: walls, ceilings, floors	1 watt/m^2
At sea level, clear night, high full moon, horizontal suface S	10^{-3} watt/m^2
At sea level, clear night, flux from 1st magnitude (highly visible) star, on surface S normal to star's rays	10^{-9} watt/m^2

As another base for comparison and also to extend our intuitive feeling for irradiance and its connection with the photon picture of light, let us calculate the number of photons per unit volume, of wavelength λ, required to produce H watt/m^2 at a point of some surface. To fix ideas, suppose a thin pencil of photons arrives at each point x of a surface S in the direction of its inward normal ξ, and that each pencil is of the same density comprised of photons of a single frequency ν. It follows that the photon density $n(x,\xi,t,\nu)$ has the form

$$n(x,\xi',t,\nu') = n_0(x,t)\ \delta(\xi'-\xi)\ \delta(\nu'-\nu)$$

where δ is the Dirac delta function and where ξ is the inward normal to S, and ν is the frequency associated with λ. When used in (1) of Sec. 2.2, this equation yields:

*According to Moon, Ref. [185], at sea level, for sun zenith, clear dry air, the irradiance is nearly 1200 watt/m^2. See also [296] for a survey of solar irradiation measurements.

$$h\nu v\int_D\int_S\int_F n_o(x,t)\delta(\xi'-\xi)\delta(\nu'-\nu)\ dA(x)\ d\Omega(\xi')\ dl(\nu')$$

$$= h\nu v\int_S n_o(x,t)\ dA(x) = h\nu v n_o A(S)$$

as the radiant flux crossing S normally at time t. Hence $h\nu v n_o$ is the irradiance produced by each pencil. Setting

$$h\nu v n_o = H \text{ watt/m}^2$$

we have:

$$n_o = \frac{H}{h\nu v}\ \frac{\text{photons}}{\text{m}^3}$$

or

$$n_o v = \frac{H}{h\nu}\ \frac{\text{photons}}{\text{sec}\times\text{m}^2}$$

or

$$n_o v = \frac{H\lambda}{hv}\ \frac{\text{photons}}{\text{sec}\times\text{m}^2}$$

For example, setting $H = 1$ watt/m², $\lambda = 550$ mμ, and recalling that $h = 6.6 \times 10^{-34}$ Joule sec/photon and $v = 3 \times 10^8$ m/sec, we have

$n_o v$ = number of photons of wavelength 550 mμ per sec. normally incident per square meter which produce one watt

$$= \frac{5.5 \times 10^{-7}}{6.6 \times 10^{-34} \times 3 \times 10^8}$$

$$= 2.8 \times 10^{18}$$

From Table 1 we see that the normal irradiance of a first magnitude star is on the order of 10^{-9} watt/m². If we assume this flux to be comprised of photons all of wavelength $\lambda = 550$ mμ, then the number of associated photons producing this irradiance is $2.8 \times 10^{18} \times 10^{-9} = 2.8 \times 10^9$ photons per second normally incident per square meter $= 2.8 \times 10^5$ photons per second normally incident per square centimeter. Now a human eye's pupil is on the order of a tenth of a square centimeter in area. Hence when our eyes are directed toward a first magnitude star such as the present one, about 2.8×10^4 photons per second enter each eye to produce the visual sensation of brightness in the brain.

Terrestrial Coordinate Systems

Irradiance measurements and other radiometric measurements of hydrologic and meteorologic optics during careful experimental investigations are usually made with respect to either one of two *terrestrially-based frames of reference*. Each reference frame uses the usual euclidean three-dimensional coordinate system with its familiar xyz-axes. The two terrestrially-based reference frames are primarily distinguished by the way they anchor the directions of the x and z axes in each case. See Fig. 2.3. The *sun-based frame* directs the plane determined by the x-axis and z-axis (the xz *plane*) so as to contain the center of the sun. (The *north-based frame* directs the xz plane so as to lie in the plane of the local meridian circle on the earth.) In each frame the z-axis is parallel to the local vertical direction, (i.e., the local gradient of the gravitational field). In meteorologic optics z is measured as increasing in the upward direction, i.e., the unit vector **k** along the z-axis. In hydrologic optics it is more convenient to measure z as increasing in the downward direction -**k**, as shown in Fig. 2.3. In meteorologic optics, "z" (or other symbols) denotes *altitude*, in hydrologic optics, "z" (or other symbols) denotes *depth*, when specific reference to terrestrial coordinate frames is made.

The concept of direction within a reference frame established for a natural optical medium such as the atmosphere or the sea is of central importance in hydrologic optics and ranks equally in importance with the notion of location. In view of this importance it will be well to define with care precisely what is meant by "direction", and to develop some of the more frequently occurring concepts associated with it.

Now, to locate an object within a terrestrially-based reference frame, it suffices to give the x,y and z coordinates in terms of meters, say. Thus, in the hydrologic optics reference frames, the triple of numbers (1, 10, 100) locates a point in a natural hydrosol by going one meter along the direction **i** from the origin, then 10 meters along the direction **j**, and then 100 meters vertically downward. (Recall that in natural hydrosols, z is measured positive in the downward direction, i.e., in the direction -**k**.) Now this point obviously lies in a well defined "direction" from the origin of the reference frame. We observe that this "direction", however, has nothing to do with the distance of (1, 10, 100) from the origin. Indeed, the points (1/2, 5, 50) and (2, 20, 200) which are, respectively, half and twice as far from the origin as the original point, all lie in the same "direction" from the origin. A convenient measure of this common "direction" of all three points then would be established if we chose a point some standard fixed distance from the origin and which shares the same "direction" as they do. The obvious choice is the point a unit distance from the origin. Thus, if (x,y,z) is a point in a terrestrial frame of reference, then $(x,y,z)/(x^2+y^2+z^2)^{1/2}$ is a point a unit distance from the origin. We call this latter point the *direction* of (x,y,z) from the origin.

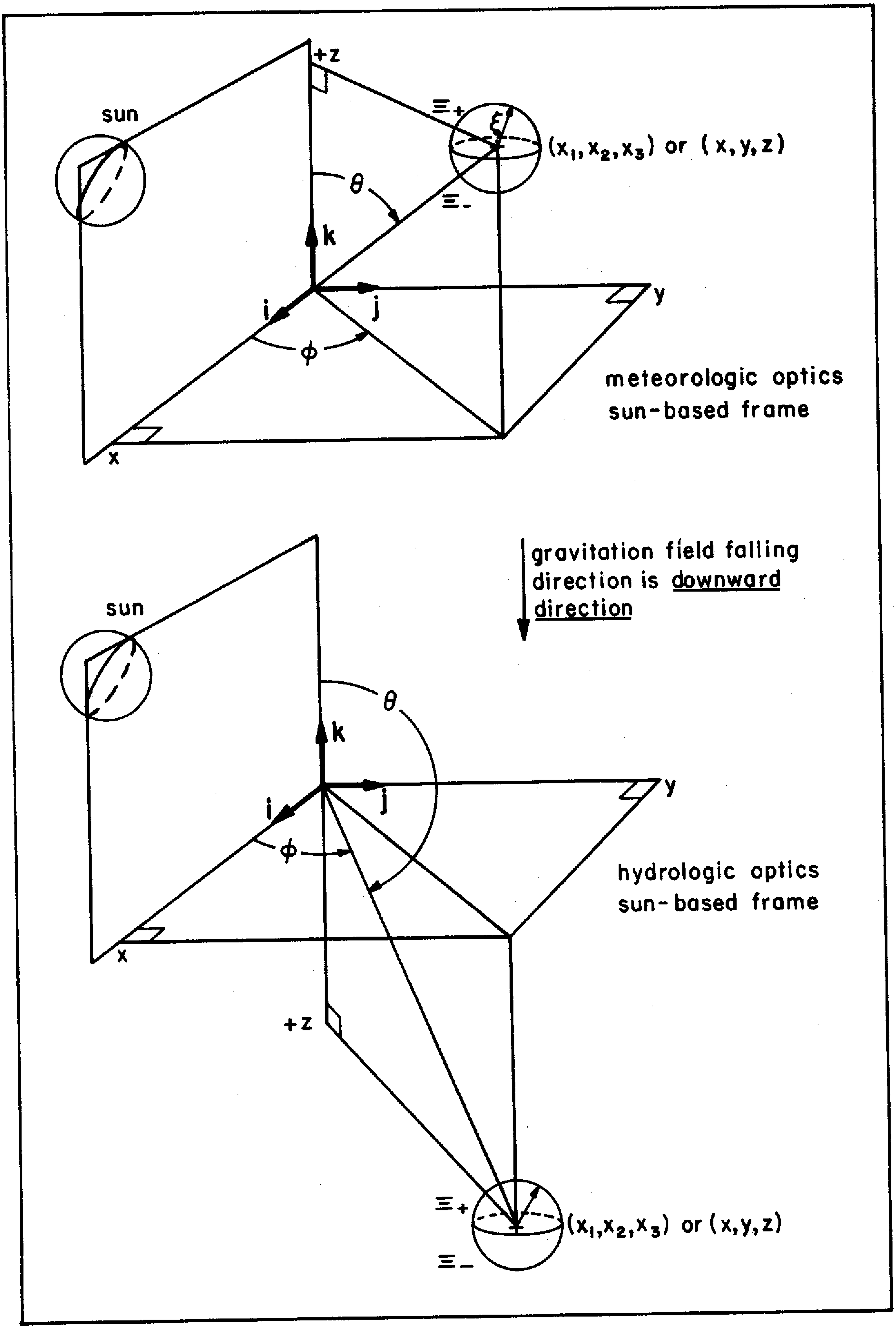

FIG. 2.3 Sun-based terrestrial frames of reference for meteorologic optics and hydrologic optics.

In many of our discussions we shall not need to specify explicitly the three coordinates of a point. In such cases we will simply write:

$$\text{"x"} \quad \text{for} \quad (x,y,z)$$

or

$$\text{"x"} \quad \text{for} \quad (x_1,x_2,x_3)$$

where the ordered triplets are the three coordinates of point x. Further we shall correspondingly write:

$$\text{"}\xi\text{"} \quad \text{for} \quad (x,y,z)/(x^2+y^2+z^2)^{1/2}$$

or

$$\text{"}\xi\text{"} \quad \text{for} \quad (x_1,x_2,x_3)/(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2}$$

Hence, throughout this work the letter "x" (in either lightface or boldface type, as emphasis requires) is generally to designate a location and the letter "ξ" is generally to designate a direction. The denotation of the components of x and ξ will vary so as to permit simplicity and clarity of expression. We have already used the three special directions **i**, **j**, **k**, which we have agreed to be the points (1,0,0),(0,1,0), and (0,0,1), respectively.

We will also wish to consider collections of directions in addition to single directions. For example, certain sets D were already encountered in our discussions above. In particular, let us denote by "Ξ" the *set of all directions about the origin.* Clearly Ξ is a sphere of unit radius with origin as center. Observe that we use an upper case Greek Xi (the Greek counterpart to the English letter "X") to designate the set of all directions. There are two more sets of directions of very frequent occurrence in practice. First, there is the *set of all upward directions*, i.e., the set of all directions ξ such that ξ and **k** make an angle of less than ninety degrees. We shall designate this set by "Ξ_+". Second, there is the *set of all downward directions*, i.e., the set of all directions ξ such that ξ and **k** make an angle of greater that ninety degrees. We shall designate this set by "Ξ_-". The "+" and "-" are convenient mnemonics which help distinguish one set from the other. The reader may recall from vector analysis at this point that if ξ is in Ξ_+ then $\xi\cdot\mathbf{k} > 0$, i.e., the dot (or scalar) product of the vectors ξ and **k** is a positive number; and that if ξ is in Ξ_-, then $\xi\cdot\mathbf{k}$ is a negative number. This is the reason for the plus and minus signs in the names "Ξ_+" and "Ξ_-". Indeed, it would be well to recall that for every direction ξ,

$$\xi\cdot\mathbf{k} = \cos\theta$$

where θ is the angle between the lines along which ξ and **k** lie. See Fig. 2.3. For convenience we reproduce below the definition of the dot product of two unit vectors ξ_1 and ξ_2. Suppose we have written:

$$"\xi_1" \quad \text{for} \quad (a_1,b_1,c_1)/(a_1{}^2+b_1{}^2+c_1{}^2)^{1/2}$$

$$"\xi_2" \quad \text{for} \quad (a_2,b_2,c_2)/(a_2{}^2+b_2{}^2+c_2{}^2)^{1/2}$$

Then we write:

$$"\xi_1 \cdot \xi_2" \quad \text{for} \quad \frac{a_1a_2+b_1b_2+c_1c_2}{(a_1{}^2+b_1{}^2+c_1{}^2)^{1/2}(a_2{}^2+b_2{}^2+c_2{}^2)^{1/2}}$$

From analytic geometry it is known that:

$$\xi_1 \cdot \xi_2 = \cos \vartheta$$

where ϑ is the angle between ξ_1 and ξ_2.

The representation of a unit vector ξ as an ordered triple of numbers takes on deeper meaning when we observe the following geometric fact. Let "(a,b,c)" denote the ordered triple representation of ξ. Then compute the dot product of ξ with **i**, **j**, and **k** in turn. By the cosine law cited above we have:

$$\xi \cdot \mathbf{i} = \cos \vartheta_1$$

$$\xi \cdot \mathbf{j} = \cos \vartheta_2$$

$$\xi \cdot \mathbf{k} = \cos \vartheta_3$$

where ϑ_1, ϑ_2, and ϑ_3 are, respectively, the angles between ξ and the positive x, y and z axes. Using the ordered triple representations of ξ, **i**, **j**, and **k**, and the definition of the dot product, we have:

$$\xi \cdot \mathbf{i} = a$$

$$\xi \cdot \mathbf{j} = b$$

$$\xi \cdot \mathbf{k} = c$$

Hence the components a,b,c of the direction ξ are simply the cosines of the angles that ξ makes with the positive x,y and z axes, i.e.:

$$a = \cos \vartheta_1$$

$$b = \cos \vartheta_2$$

$$c = \cos \vartheta_3$$

This leads to the representation:

$$\xi = (\cos \vartheta_1, \cos \vartheta_2, \cos \vartheta_3) \ .$$
$$= \mathbf{i} \cos \vartheta_1 + \mathbf{j} \cos \vartheta_2 + \mathbf{k} \cos \vartheta_3 \quad .$$

where we have written:

"**i**" for (1,0,0)

"**j**" for (0,1,0)

"**k**" for (0,0,1)

There is an alternate mode of representation of a unit vector ξ. This alternate mode attains its greatest utility in actual calculation. This is the representation of ξ by two especially constructed angles θ and ϕ, found as follows. By studying the schematic representation of Ξ in Fig. 2.4 it is clear that since each direction of Ξ has fixed known length (namely a unit length) it suffices to uniquely specify ξ by the angle it makes with the z-axis and the angle the plane determined by ξ and **k** makes with the xz plane. Suppose we designate the former angle by "θ", the latter by "ϕ", and agree to set $\theta = 0$ when $\xi = \mathbf{k}$. Further, we agree to have θ increase to π when $\xi = -\mathbf{k}$. Further, we agree to set $\phi = 0$ when ξ is in the xz plane and to have ϕ increase to $\pi/2$ as the plane of ξ and **k** rotates from the xz plane to the yz plane. We let ϕ increase in like manner through the next three quadrants, and finally have it measure 2π radians after one complete turn in this manner. To summarize this alternate mode

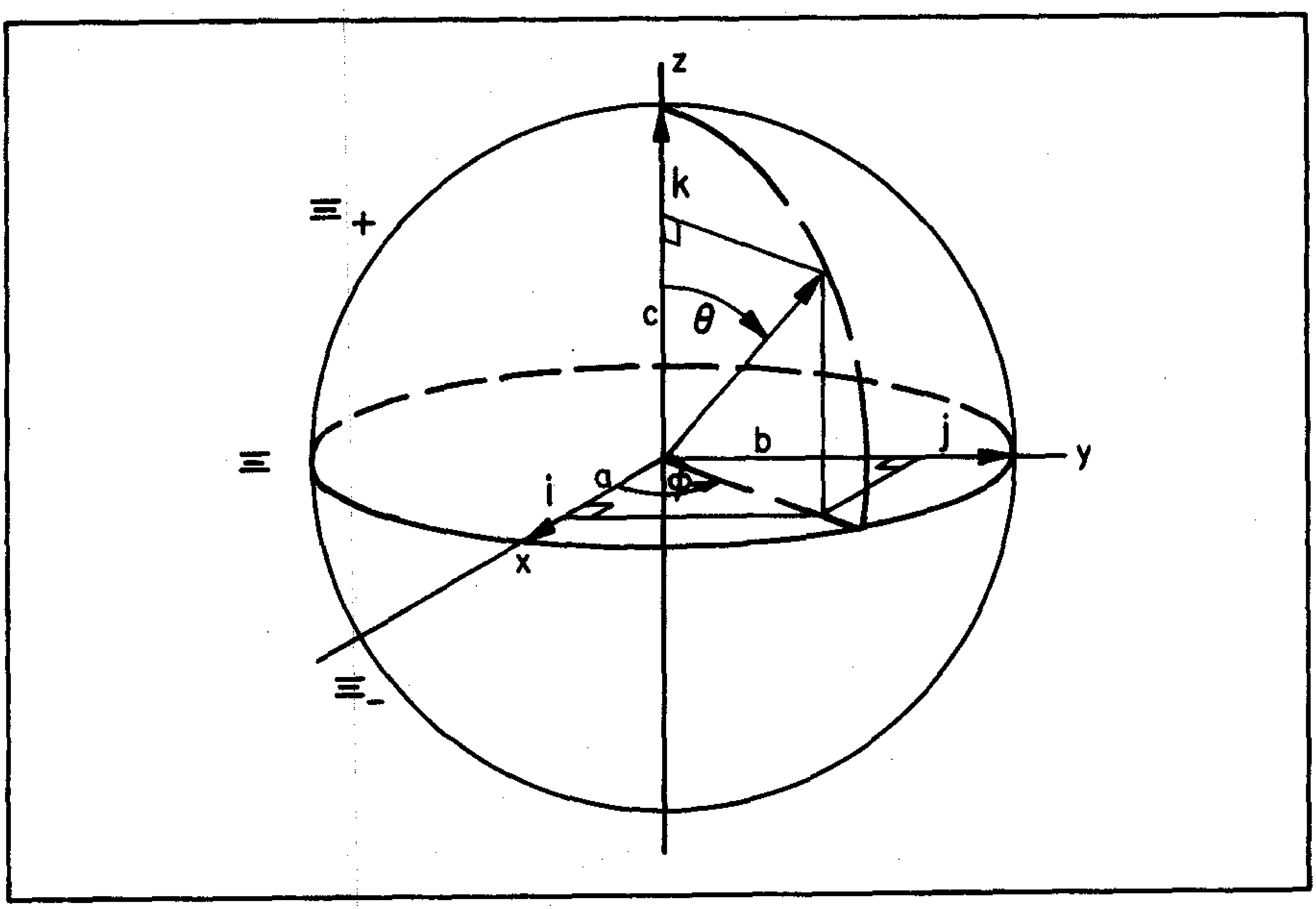

FIG. 2.4 Angle and direction definitions

of representation of ξ, we agree to write:

$$"(\theta,\phi)" \quad \text{for} \quad \xi$$

whenever ξ is in Ξ, and whenever $\xi = (a,b,c)$, $\theta = \text{arc cos } c$, and $\phi = \text{arc tan } b/a$, and where the quadrant containing ϕ is fixed by the signs of a and b. The angle θ is the *polar (or zenith) angle* of ξ, and ϕ the *azimuthal angle* of ξ.

Representation of Irradiance in Terrestrial Frames

Let us return now to apply these geometrical results to the task of specifying irradiance in natural optical media such as the atmosphere or the sea. It has become clear after much theoretical and experimental work in natural aerosols and hydrosols that the type of irradiance which is used most often in practice is the irradiance on a *horizontal* surface S at point x with a set D of directions which constitutes either the *hemisphere* Ξ_+ or Ξ_- of the unit sphere Ξ. To specify such irradiances, we return to the definition in (4), and replace "D" first by "Ξ_+" and then by "Ξ_-" (or by "+" and "-"). Thus $H(x,\Xi_+)$ (or $H(x,+)$) is the irradiance at point x induced by upward flowing radiant energy in the directions of Ξ_+, and $H(x,\Xi_-)$ (or $H(x,-)$) is the irradiance at point x induced by downward flowing radiant energy in the directions of Ξ_-.

A further specialization in notation can take place when the medium is stratified. Now, a natural optical medium (or a light field) with a terrestrially-based reference frame (Fig. 2.3) is said to be *stratified* if and only if the optical properties of the medium (or light field) as functions of coordinates x,y,z, are independent of the coordinates x and y. Thus for stratified light fields we may, for brevity and without loss of information, replace the general point name "x" in $H(x,\Xi_+)$ by "z", the depth-parameter name. Thus, let us agree henceforth in stratified natural optical media to write:

$$"H(z,+)" \quad \text{for} \quad H(x,\Xi_+) \tag{9}$$

and

$$"H(z,-)" \quad \text{for} \quad H(x,\Xi_-) \tag{10}$$

We call $H(z,+)$ the *upward irradiance* and $H(z,-)$ the *downward irradiance*.

The next most frequently occurring type of irradiance $H(x,D)$ after the types $H(z,\pm)$, is that for which D is an arbitrarily oriented hemisphere. Thus, let us denote by "$\Xi(\xi)$" that part of Ξ consisting of all unit vectors ξ' such that ξ' and ξ subtend an angle less than ninety degrees. Hence, after adapting definition (4) to the case where D is $\Xi(\xi)$, we have $H(x,\Xi(\xi))$ as the irradiance at point x on a collecting surface S with unit inward normal ξ, such that the irradiance is produced by radiant flux incident on S at x along the directions within $\Xi(\xi)$. See Fig. 2.5. Observe that the irradiance $H(x,\Xi(\mathbf{k}))$ is simply $H(x,\Xi_+)$ considered earlier, since $\Xi(\mathbf{k}) = \Xi_+$; and similarly $H(x,\Xi(-\mathbf{k})) = H(x,\Xi_-)$. Now a useful fact about such sets of directions as $\Xi(\xi)$ is that they are

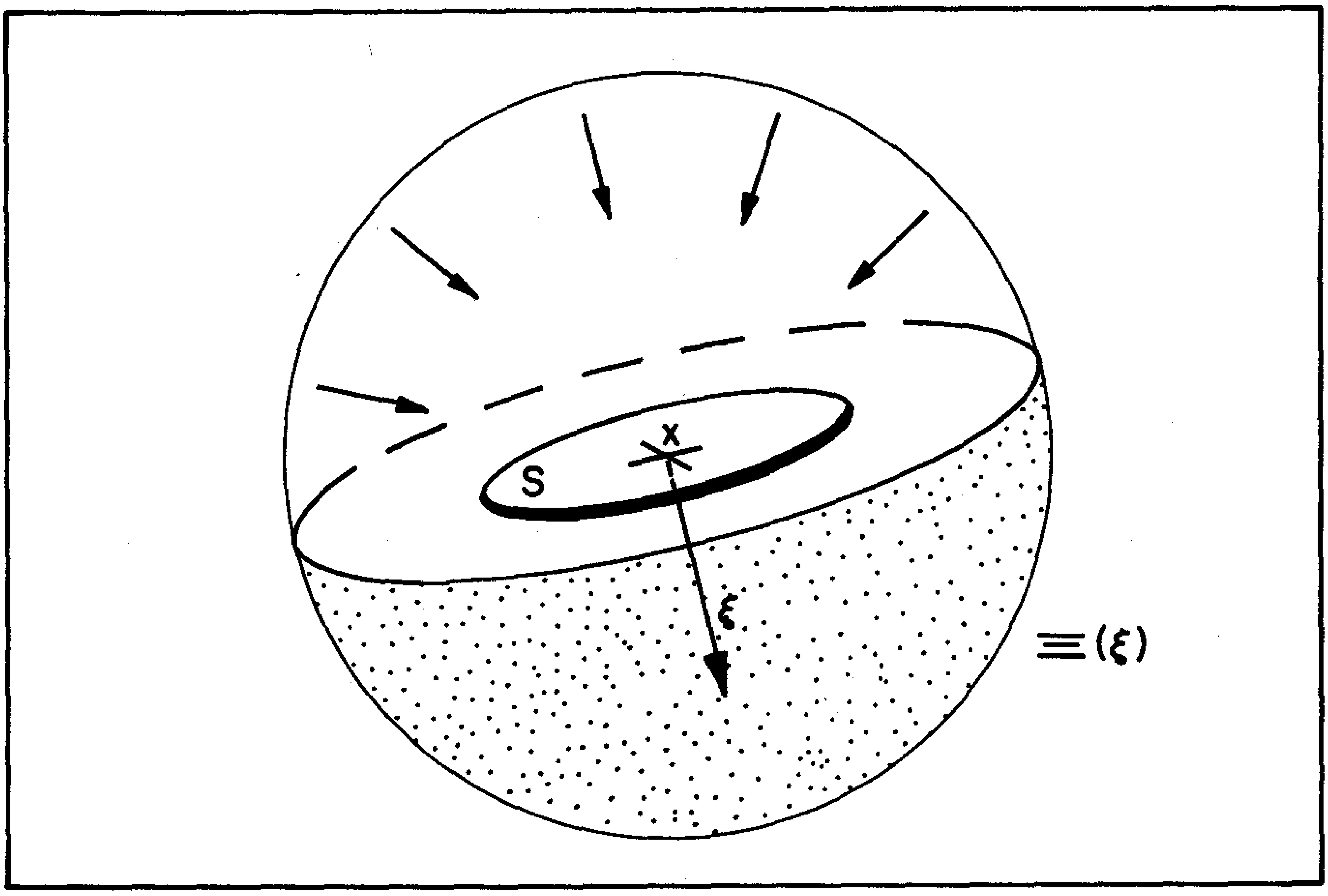

FIG. 2.5 Defining the hemisphere $\Xi(\xi)$ determined by the direction ξ

uniquely specified by giving the single vector ξ. We take advantage of this observation to shorten the irradiance notation by agreeing henceforth in (4), for the case $D = \Xi(\xi)$, to write:*

$$\text{"}H(x,\xi)\text{"} \quad \text{for} \quad H(x,\Xi(\xi)) \tag{11}$$

If we restrict attention to a fixed point x, then the totality of all values $H(x,\xi)$ as ξ varies over Ξ is called the *irradiance distribution* at x. If the light field is stratified we further agree to write:

$$\text{"}H(z,\xi)\text{"} \quad \text{for} \quad H(x,\xi) \tag{12}$$

Thus in stratified light fields, one may specify irradiances by giving a depth z and the *unit inward* normal ξ to a (hypothetical or real) collecting surface at that depth.

If one prefers to use the mode of representation of ξ by means of polar and azimuthal angles θ and ϕ, then it will be agreed to write:

---

*Whenever wavelength dependence and time dependence is to be shown explicitly we would use "$H(x,\Xi(\xi),\lambda)$", or "$H(x,\Xi(\xi),t)$" or "$H(x,\Xi(\xi),t,\lambda)$" as the case may be, and in contracted ξ-notation, as desired.

$$\text{"}H(x,\theta,\phi)\text{"} \quad \text{for} \quad H(x,\xi) \tag{13}$$

or

$$\text{"}H(z,\theta,\phi)\text{"} \quad \text{for} \quad H(z,\xi)$$

when the light field is stratified. It should be re-emphasized that the direction ξ (and hence (θ,ϕ)) refers to the unit *inward* normal to the collecting surface S in the operational definition of (13) *and that the flow of photons is onto* S *at* x *along the directions of* $\Xi(\xi)$. This is the convention we shall adopt when discussing irradiance measurements by collecting surfaces on a theoretical level; for the transport equations for $H(z,\pm)$ to be introduced later (Chapter 8) are written down in an intuitively natural manner if this convention is adopted. The convention may be altered if need be for empirical discussions. However, it is perhaps needless to point out that the fewer such conventions actually adopted for radiometers, the smaller will be the chance of conceptual chaos in practice.

One final definition, and then we shall be ready for a discussion of the cosine law for irradiance. We agree to write:

$$\text{"}\overline{H}(x,\xi)\text{"} \quad \text{for} \quad H(x,\xi) - H(x,-\xi) \tag{14}$$

and call $\overline{H}(x,\xi)$ the *net irradiance* at x in the direction ξ.

The Cosine Law for Irradiance

We now consider the property of irradiance which is its most important and most frequently used theoretical property. This is the cosine law for irradiance. The law is based on the simple geometric fact that the apparent area of a small plane surface at a fixed distance along one's line of sight varies as the cosine of the angle between the line of sight and the normal to the surface. If now we direct a swarm of photons along the line of sight toward the small surface then, all other things being equal, the area will intercept a number of photons proportional to the apparent area, i.e., proportional to the cosine of the angle between the direction of the beam of photons and the surface's normal. Hence the area density, i.e., the irradiance of the photons on the surface will vary as the cosine of this angle. The formal statement of this observation is the *cosine law* for irradiance. We now translate this verbal derivation of the cosine law into symbolic form.

In Fig. 2.6 a small plane surface is denoted by "S". An amount $P(S,D)$ of radiant flux is incident on S and arrives at each point of S through a very narrow fixed conical solid angle D such that the central direction of D is normal to S. Since the radiant flux is limited to a relatively narrow bundle of directions, essentially all the lines of flux are confined to a cylindrical volume C in the immediate neighborhood of S. Let "S'" denote a section of C generated by a plane whose normal makes an angle ϑ with the axis of C and such that

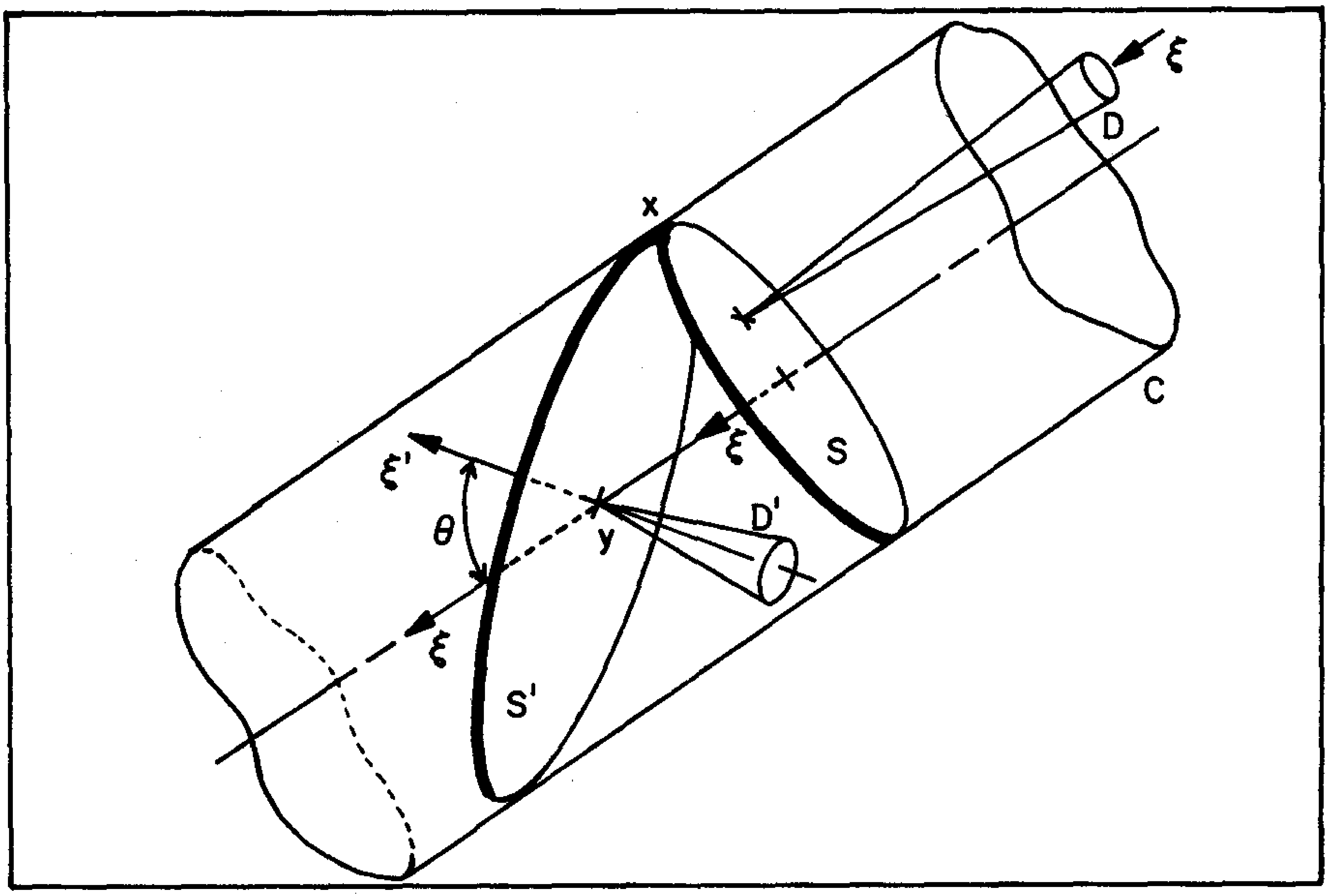

FIG. 2.6 Geometry of the cosine law for irradiance.

the plane goes through some point x on S. The area A(S') of S' is clearly related to the area A(S) of S by the relation:

$$A(S') = A(S) \sec \vartheta$$

Assuming no intervening sources or sinks of radiant flux in the region of C between S and S' the flux P(S,D) then also crosses S'. Thus we can write:

$$P(S,D) = P(S',D) \quad .$$

By definition, the area density H(S',D) of radiant flux across S' is:

$$H(S',D) = P(S',D)/A(S') \ .$$

In view of the preceding flux conservation statement and the geometric relation between A(S') and A(S) we can write:

$$H(S',D) = P(S,D)/(A(S) \sec \vartheta)$$

By definition H(S,D) is P(S,D)/A(S) and we therefore arrive at the statement:

$$\boxed{H(S',D) = H(S,D) \cos \vartheta} \qquad (15)$$

This is the empirical form of the *cosine law for irradiance*. A theoretical form of the law is obtained by letting $S \to \{x\}$ (and hence $S' \to \{x\}$). The result is:

$$H(x,\xi') = H(x,\xi)\ \xi \cdot \xi' \tag{16}$$

Here we have used the fact that D was sufficiently narrow so that in the limit H(S,D) goes to $H(x,\xi)$ as S goes to the set $\{x\}$ consisting of point x. Further, H(S',D) goes to $H(x,\xi')$ as S goes to $\{x\}$. Of course (16) is to be understood to apply to a set D of directions with a *small* but finite solid angle. The limiting case for $D \to \{\xi\}$ can be handled naturally only after the concept of radiance has been introduced. Further we have replaced "cos ϑ" by "$\xi \cdot \xi'$" in going from (15) to (16). After the introduction of the concept of vector irradiance (Sec. 2.8), (16) can readily be generalized to the case where the set of incident directions D is arbitrary.

Radiant Emittance

We close this section with a few comments on the concept of radiant emittance. As already noted in the introductory remarks to this section, the concept of radiant emittance is nearly identical to that of irradiance, differing from the latter geometrically only by the sense of flow of the radiant energy across a surface S. Fig. 2.7 schematically depicts the geometrical distinction between irradiance and radiant

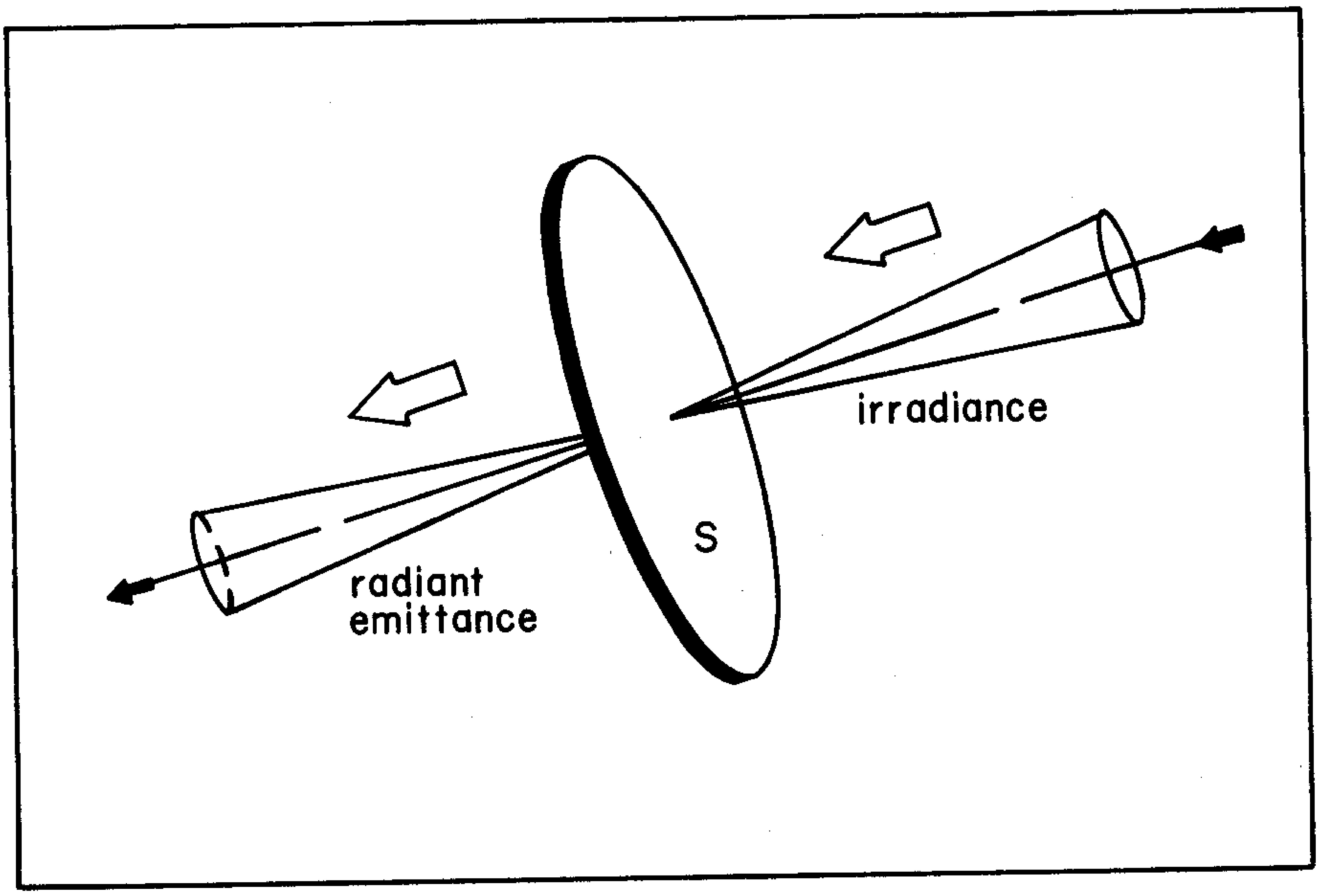

FIG. 2.7 Conceptual distinction between irradiance and radiant emittance.

emittance; a given parcel of radiant energy flowing *onto* a surface S generates *irradiance on* S: the same parcel flowing from the surface S *generates radiant emittance of* S. To emphasize this distinction and to have appropriate notation available when needed, we need only write "$\Phi^-(S,D)$" to denote radiant flux *onto* S and to write "$\Phi^+(S,D)$" for radiant flux *from* S. Then we extend this notation to radiant flux by means of "$P^-(S,D)$" and "$P^+(S,D)$". Thus, the definition (1) of empirical irradiance may be written as:

$$\text{"}H(S,D)\text{"} \quad \text{for} \quad P^-(S,D)/A(S) \tag{17}$$

for emphasis of the "onto" interpretation of the flux; and we now go on to write:

$$\text{"}W(S,D)\text{"} \quad \text{for} \quad P^+(S,D)/A(S) \tag{18}$$

for contrast of the two notions. We call W(S,D) the (empirical) *radiant emittance* over S within D. From consideration of Fig. 2.7 it is clear that in the context of that figure:

$$\boxed{P^+(S,D) = P^-(S,D)} \tag{19}$$

so that

$$\boxed{W(S,D) = H(S,D) .} \tag{20}$$

Another distinction between W(S.D) and H(S,D) for a given S and D lies on the physical rather than the geometric level. Indeed, it is on this level that the concept W(S,D) was originally conceived and arose in connection with the derivation of the complete (or Planckian) radiator wherein radiant flux is generated within a body and then emitted through its boundary. This interpretation will be used in Sec. 2.12 during the transition from radiometry to photometry.

We conclude by observing that every auxiliary geometric definition and geometric law considered above for irradiance now holds analogously for radiant emittance. We shall henceforth apply the analogous notation for W(S,D) (such as W(x,D), W(x,ξ), etc.) without further explicit definitions. Thus for example we write:

$$\text{"}W(x,D)\text{"} \quad \text{for} \quad \lim_{S \to \{x\}} W(S,D) \tag{21}$$

and

$$\text{"}W(x,\xi)\text{"} \quad \text{for} \quad W(x,\Xi(\xi)) \tag{22}$$

and so on.