2.6 An Invariance Property of Radiance

In this section we shall discuss a property of the concept of radiance which is of central importance from the point of view of radiative transfer phenomena. This is the so-called n^2-law of radiance which states that the quotient N/n^2 does not change along a path of sight through a transparent medium in which there is a generally variable index of refraction n. The importance of this law rests in the base line it establishes for comparison of the behavior of N/n^2 along lines of sight in *non*-transparent media, i.e., media that scatter and absorb radiant energy such as the atmosphere and the seas, and other natural optical media. The law also indicates a measure of success in our attempt to simulate the sensation of brightness by means of a simply defined radiometric concept. For it is a matter of daily experience that as one approaches or recedes from an object along a line of sight through a very clear homogeneous stretch of atmosphere (so that n is constant), the "brightness" of the object does not appear to change. For example, the brightness of a small part of a desk blotter does not change as we move away from it in a room, keeping attention constantly directed toward the patch. Of course, the *total flux* entering the eye and originating from the patch falls off rapidly with distance (very nearly as the square of the distance, as we shall eventually show); however, the *brightness* of the patch does not change with the observer's distance. This phenomenon is reproduced in the special form of the n^2-law where n is constant over the path of sight. We now show how the n^2-law for radiance follows from the definition of radiance. We shall divide the discussion into two main parts. The first part considers the important case in which n is constant along the path of sight. The second part considers the general case of a variable index of refraction.

The Radiance-Invariance Law

We begin the derivation of the n^2-law for the special case where n is constant along a line of sight through a transparent optical medium. This special case is of sufficient importance to be given a special name, the *radiance-invariance law*. We shall prove the radiance invariance law twice: first in as simple a way as possible so as to reveal the geometrical essence of the law; then the derivation will be repeated in slightly more detail, filling in steps and giving more explanations on the way.

The setting for the simple derivation is shown in Fig. 2.13. Two holes S and S' of arbitrary shape and about the size of collecting surfaces used in radiant flux meters are cut out of two large pieces of opaque cardboard. The pieces are then mounted so that they are parallel and separated a distance r which is large compared with the linear dimensions of the holes. Light is then directed through S which flows along straight lines in the transparent space between the cardboards and then on through S'. The holes are arranged so

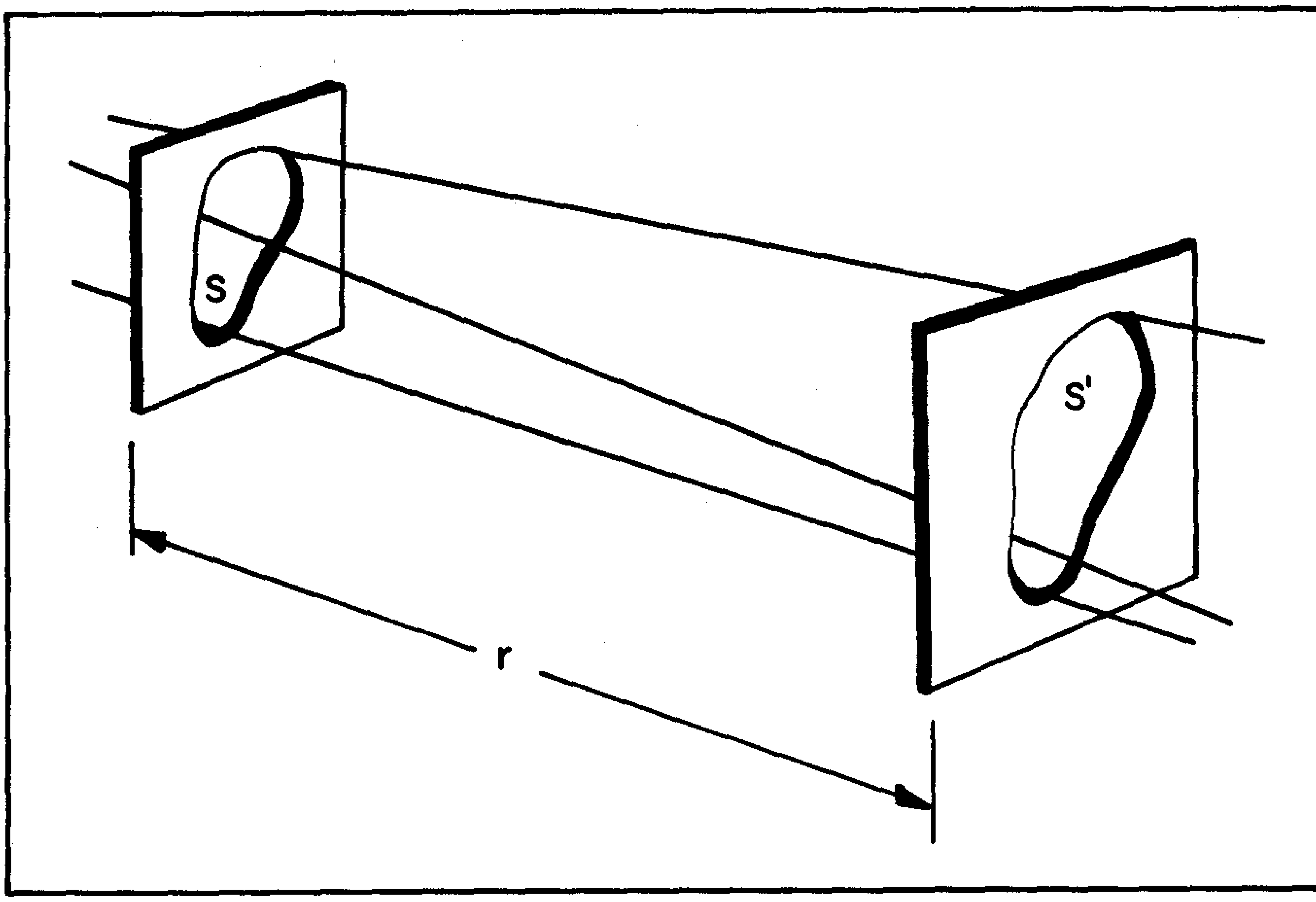

FIG. 2.13 Illustrating the invariance of the radiance of a narrow bundle of light rays in a vacuum.

that for the most part, the lines of flux through both openings are nearly perpendicular to the planes of the holes. The observation is now made that the amount P of radiant flux across S, associated with the common bundle of lines of flux through S and S', is the same as that across S'. Thus the same number of lines of flux go through both S and S'. With this in mind we consider the number:

$$\frac{P}{\frac{A(S)A(S')}{r^2}}$$

in two ways. First as:

$$\frac{P(S,D)}{A(S)\frac{A(S')}{r^2}} ,$$

and then as:

$$\frac{P(S',D')}{\frac{A(S)}{r^2}\,A(S')} .$$

In the first case we observe that $A(S')/r^2$ is essentially the solid angle $\Omega(D)$ subtended by S' at each point of S. In the second case $A(S)/r^2$ is the solid angle $\Omega(D')$ subtended by S at each point of S'. "P(S,D)" and "P(S',D')" both denote the common radiant flux P, but now in an obviously suggestive way by recalling the meanings of S, S', D, D'. Therefore we have:

$$\boxed{N(S,D) = N(S',D')} \qquad (1)$$

This is the empirical form of the *radiance-invariance law*. The form of the law is "empirical" because it is couched in terms of empirical radiances--radiances directly measurable by real radiance meters.

A somewhat more detailed derivation of the radiance-invariance law will now be given. Part (b) of Fig. 2.14 depicts a radiance meter G directed at a surface S at the end

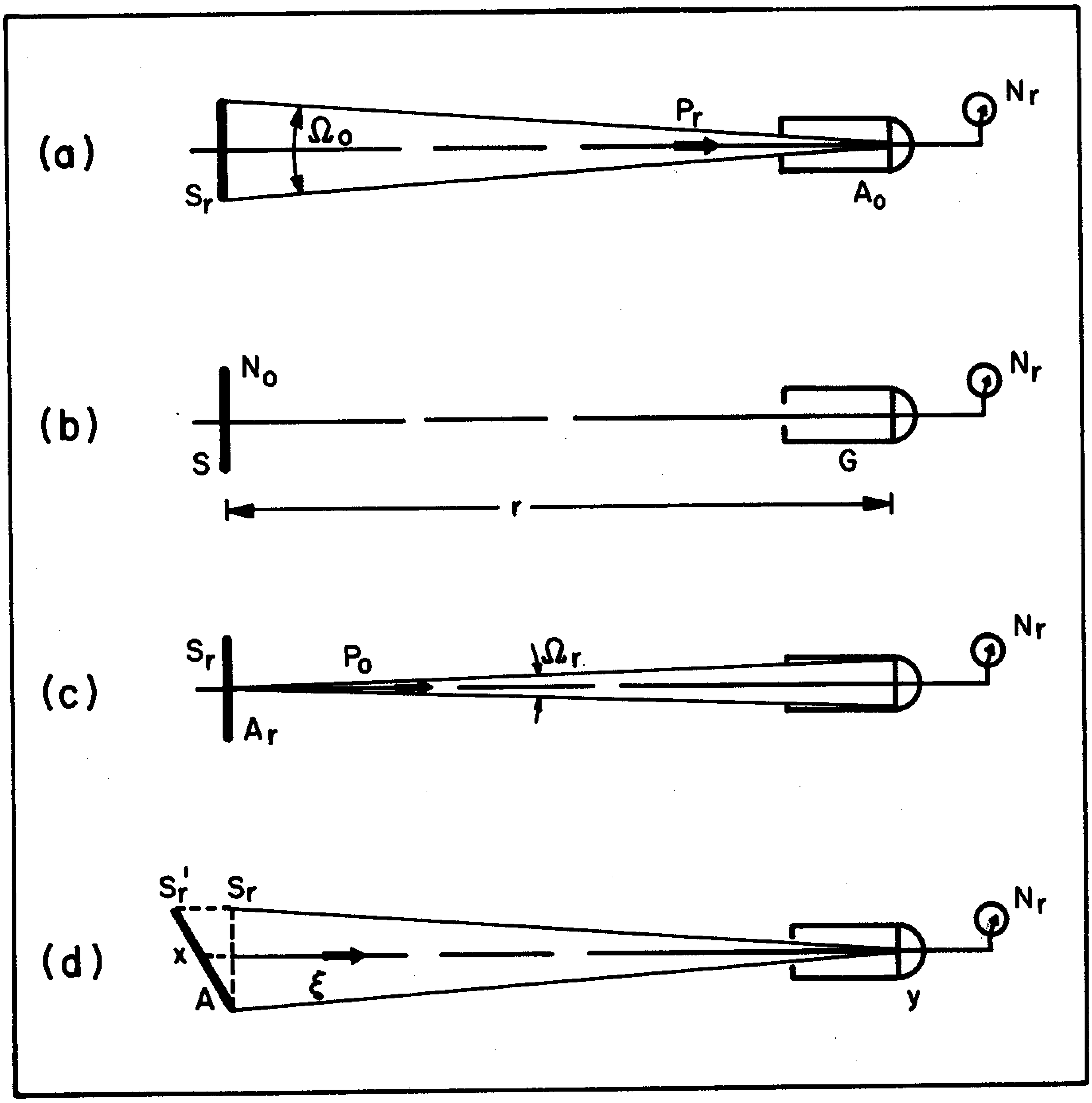

FIG. 2.14 A more detailed, and operational, study of the radiance-invariance law

of a clear path of sight of length r. The surface S is normal to the line of sight and has a uniform surface radiance N_o over its extent in the direction of G. The meter G has its field of view completely filled by S. The resultant radiance reading is N_r. We will show that, under these conditions we have, for every r, $N_o = N_r$. The basic idea of the proof is to examine the same diagram (b) from two distinct points of view. These points of view are schematically depicted in parts (a) and (c) of Fig. 2.14. We consider part (a) first. Here the radiance meter's reading N_r is seen to be the quotient $P_r/A_o\Omega_o$, where P_r is the radiant flux originating on S and incident on the collecting surface of area A_o, and which has funneled through the solid angle of magnitude Ω_o, defined as shown. On the other hand, part (c) views this flux as an amount P_o sent to area A_o in G and as emitted from those points of S_r within G's field of view. The emitting surface S_r comprising G's field of view is of variable magnitude A_r, and the emitted flux from each point of S_r is within the bundle of directions of solid angle defined as shown. Hence, N_o is the quotient $P_o/A_r\Omega_r$. The definitions of P_o and P_r imply at once (as in the previous proof) that:

$$P_o = P_r \quad .$$

Further, we have the geometric observation that:

$$\frac{A_o A_r}{r^2} = \Omega_r A_r = A_o \Omega_o \quad .$$

On the basis of these two facts, we see that, by virtue of the defining equations:

$$N_r = P_r/A_o\Omega_o$$

$$N_o = P_o/A_r\Omega_r$$

we have:

$$\boxed{N_o = N_r} \qquad (2)$$

Observe how we have implicitly used the distinction between field and surface radiance and the connection (32) of Sec. 2.5 in order to interpret N_o operationally at surface S, which then is N_r for $r = 0$.

The Operational Meaning of Surface Radiance

One final matter must be resolved before the radiance-invariance law is fully established. This is the matter of assigning a meaning to the surface radiance of a surface at a point x in a direction other than the normal direction to S at x. Observe that this problem does not arise with field radiance, since field radiance is defined by the convention of using the fixed collecting area of a radiance meter, which is assembled so that it is perpendicular to the axis of the meter. In the case of surfaces such as a portion of the earth's surface, a desk top or a wall, or a given cloud boundary, how shall we assign a surface radiance to radiant flux leaving a point on such surfaces in directions other than the perpendicular direction to the surface at that point? The path to the answer is guided by the manner in which such surface-radiance information is first of all to be interpreted and secondly how it is to be used. In the first case we really have very little choice as to the manner of interpretation of the radiance information. We have already committed ourselves to work solely with operational concepts: measurable fluxes, areas and solid angles. Hence, if we heard someone say: "The surface radiance of flux of wavelength 550 mμ at point x on wall A is 2 watts/($m^2 \times$ steradian) in every direction 30° from the normal to A at x", our first impulse, after this data has been mentally assimilated, would be to attempt a verification by directing a radiance meter toward x on A so that the axis of the meter makes an angle of 30° with the normal to A at x. If we were challenged to defend such a procedure, we would cite the argument leading to the radiance-invariance law above. However, if the challenger were particularly tenacious, he would point out that the argument establishing the law holds only for directions of sight *normal* to A at x. At this juncture we must concede that he is right.

The preceding objection to our justification for assigning an operational meaning to oblique surface radiance is logically unassailable. However, we have one more matter to consider which will add strength to the justification. We now consider the second aspect of the question posed above, namely: how is the information of oblique surface radiance to be used? The answer, based on considerable practical and theoretical experience, is that such oblique surface radiance information is to be used to calculate irradiances, scalar, vector, or of the ordinary variety, at points optically accessible to the surface which emits the surface radiance. Or again, the surface radiance information will be used to obtain path functions, and various attenuation functions used in hydrologic optics or meteorologic optics and these determinations will be made at points optically accessible to the surface. The pertinent fact that emerges as these uses of the surface radiance information are paraded before the mind's eye is the following: *without exception, the information used can always be in the form of field radiance values of the radiometric field in the direction of point* x *on surface* A. In short, surface radiance *per se* while of great conceptual and theoretical worth, is never really used in actual practical calculations--only the directly observable field radiance values are used in such calculations. We are therefore motivated to *assign an operational value of surface radiance to a surface* A *at* x *in the general outward direction* ξ *by means of the corresponding field radiance reading* $N(y,\xi)$ *obtained when the radiance meter is at some point* y *and is directed at* x *so that the unit inward normal to the collecting surface of the meter is* ξ. *The point* y *is to be anywhere along a clear path of sight from* x *in the direction* ξ *so that the radiance-invariance law holds.*

For conceptual definiteness in the preceding convention, one can imagine an hypothetical surface S_r normal to ξ as in (d) of Fig. 2.14, which is assigned the surface radiance $N(x,\xi)$, (as N_0 in the derivation of the radiance-invariance law). Now while the radiance-invariance law allows us to conclude that the radiance reading will remain unchanged as distance r varies from 0 at S_r to larger values from S_r, there still is a conceptual gap that must be filled between the surface radiance of S_r and the surface radiance of S_r', the projection of S_r onto the oblique surface under consideration (as, e.g., A above). And this gap, we have agreed, is to be filled by means of the preceding convention. Equation (2) and every result deduced from it, shall henceforth be interpreted with this convention implicitly understood.

The n^2-Law for Radiance

The intuitive basis for the n^2-law, to which we now turn, becomes clear upon consideration of Fig. 2.13. This figure shows a narrow bundle of lines of flux coursing through empty space. The two holes in the cardboard arrangement used above were so much inessential material scaffolding which can be removed now that the idea of the derivation has been explained. What is left after this is done is the concept of a narrow bundle of lines of flux coursing through space in such a way that at each section the product $A\Omega$ of the normal cross sectional area A of the bundle and solid angle Ω of the bundle is a fixed quantity. This invariance of $A\Omega$ is a purely geometric concept. Physical considerations enter subsequently at the point where we assert the invariance of the radiant flux through a variable section of the bundle of lines of flux. By combining these physical and geometric considerations, the desired radiance-invariance law is obtained for a light beam in a vacuum. We now inquire: how are these physical and geometrical considerations to be modified in the case of a light beam coursing through matter such as air or water? The physical considerations governing the radiant flux content of the beam must take into account the scattering and absorption phenomena all along the extent of the beam. These phenomena affect the radiant flux content of the beam in complex and subtle ways. The full study of these effects is reserved for the theory of Part Two of this work. We shall limit our present inquiry to sets of adjoining transparent media. Any alterations of the radiant flux content of the beam are then limited to the interfaces of these media. If we now repeat our query above for the case of contiguous transparent media which are distinguished from each other only by their various indices of refraction, then the answer to the query is given in the form of the n^2-law for radiance. The derivation of this law for the simplest case will now be given.

Figure 2.15 depicts a beam of radiant flux lines incident on the interface Y between two transparent optical media X_1 and X_2. Let us agree that the central axis of the beam is normally incident on the interface, that it arrives from medium X_1, and that the beam passes on through the interface Y and enters medium X_2. For example, X_1 may be a part of the

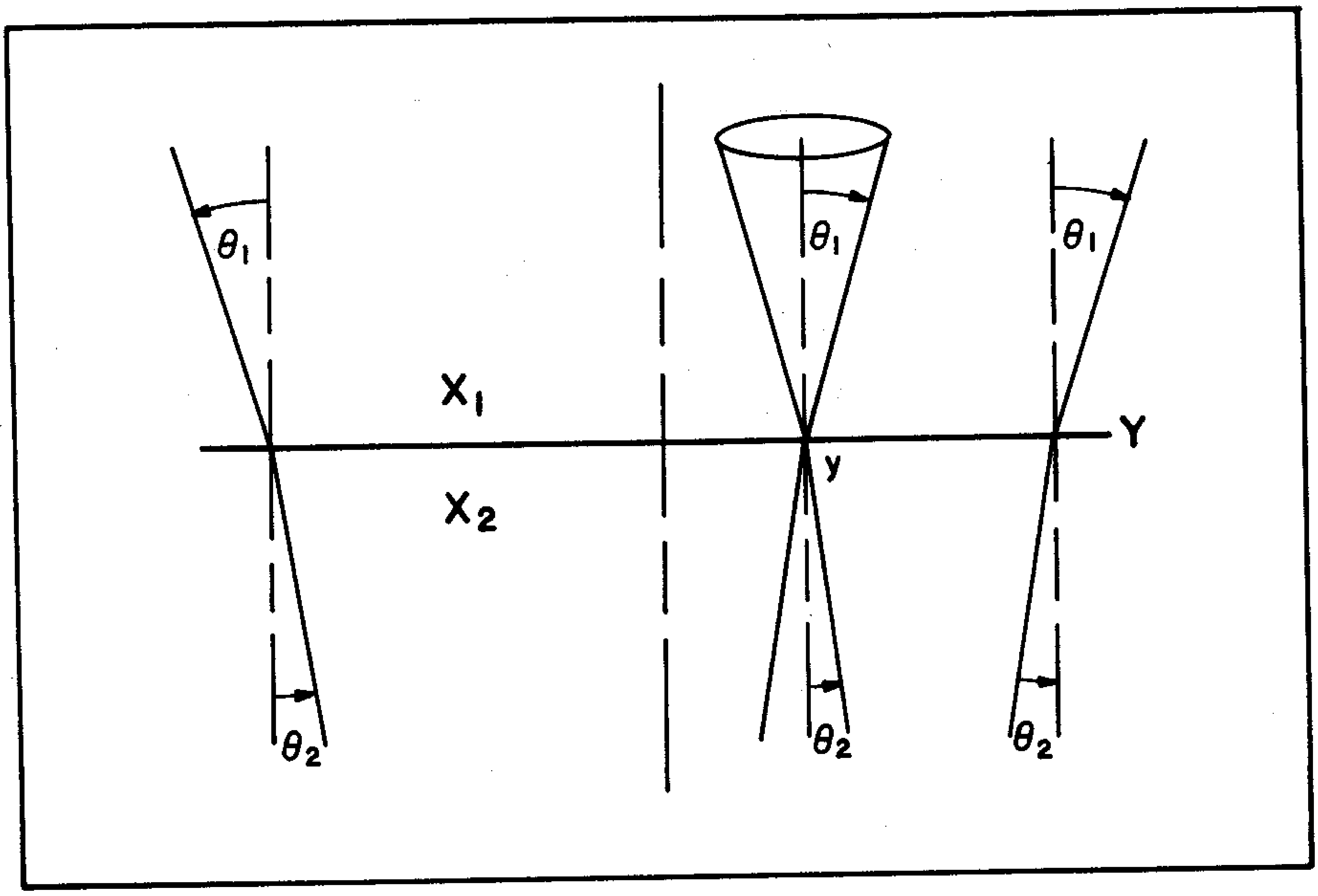

FIG. 2.15 When a bundle of light rays is suddenly squeezed into a narrower bundle--without changing its flux content--the radiance of the bundle increases proportionately. This essentially is what happens, e.g., at the air-water surface of natural hydrosols (flux losses to one side).

atmosphere, and X_2 a part of the hydrosphere, so that Y is the air-water interface. In general, X_1 has some index of refraction n_1 and X_2 an index of refraction n_2 in the immediate neighborhood of the interface. Our current goal is to relate the radiance of the beam in X_1 to the radiance of the beam in X_2 in the immediate vicinity of Y. Before going into the details of the derivation it is instructive to anticipate the result intuitively. We ask: which of the three main quantities P, A or Ω of the definition of radiance N will change from one side of Y to the other? Clearly, the radiant flux content P will be affected to some extent by reflection of some of the lines of flux back into X_1. The area A of the beam will remain essentially unchanged arbitrarily close to each side of Y. Finally, the solid angle magnitude Ω of the beam will change from one side to another because of refraction of the lines of flux transmitted across the surface Y. For the moment we ignore the change of P, this change yielding a relatively small change in N and one which will eventually vanish as the derivation proceeds to its final stages. Hence the principal change in N that is wrought on the radiant flux is traceable to the abrupt change in the Ω of the beam as it crosses Y. For example, if X_1 and X_2 are respectively, air and water, then as a glance at Fig. 2.15 would show, Ω_1, the solid angle magnitude of the beam in air, is greater than Ω_2, the solid angle magnitude of the transmitted beam in water. Since P and A are essentially unchanged during the passage of the flux across Y, the radiance N_2 of the beam in water exceeds its radiance in air. This yields the general observation that, aside from reflection losses, *as a narrow beam of radiant flux goes from a medium of smaller to greater index of refraction, its radiance changes from a smaller to a greater amount.* The reverse change occurs for a reverse traversal. The n^2-law gives a precise quantitative description of this observation.

Now to the particulars. At each point y of the interface Y within the beam, there is a cone of incident lines of flux on y. This cone has a small half angle θ_1. The associated solid angle magnitude Ω_1 of the incident cone is, by (12) of Sec. 2.5, $\pi\theta_1^2$. Similarly $\pi\theta_2^2$ is the solid angle magnitude Ω_2 of the beam of transmitted lines of flux. Snell's law ((5) of Sec. 12.1) gives the connection between θ_1 and θ_2 :

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad .$$

For small θ_1 and θ_2, this becomes, very nearly:

$$n_1\theta_1 = n_2\theta_2 \quad .$$

Squaring each side, we obtain:

$$n_1^2\theta_1^2 = n_2^2\theta_2^2 \quad .$$

Multiplying each side by π we obtain:

$$n_1^2(\pi\theta_1^2) = n_2^2(\pi\theta_2^2)$$

That is:

$$n_1^2\Omega_1 = n_2^2\Omega_2 \quad . \tag{3}$$

Now starting with the equal irradiances H_1 and H_2 of the beam on each side of Y, we have, for the reasons discussed above:

$$\frac{P_1}{A_1} = H_1 = H_2 = \frac{P_2}{A_2} \quad .$$

We now use (3) with this to get:

$$\frac{P_1}{A_1(n_1^2\Omega_1)} = \frac{P_2}{A_2(n_2^2\Omega_2)}$$

That is:

$$\boxed{\frac{N_1}{n_1^2} = \frac{N_2}{n_2^2}} \tag{4}$$

This is the desired form of the n^2-*law for radiance.*

It is now an easy matter to gradually generalize (4) to the following more general settings: (a) passage through an arbitrary finite number of transparent contiguous media and going on to the limit of continuously varying n; (b) oblique rather than normal incidence of the beam on Y; (c) inclusion of a transmittance factor to allow for scattering and absorption losses from P. Generalizations (a) and (b) result in no change in the form of (4). Generalization (c) results in a multiplicative factor T included on the left side of (4). This factor will be considered at great length in Sec. 3.10 in the discussions of beam transmittance. Specific suggestions for these generalizations are given in Sec. 12 of Ref. [251]. See also [98]. *Henceforth, whenever radiances are related within media of differing indices of refraction, it will be understood that* N/n^2 *rather than* N *will be used, even though "N" only appears in the equations.*

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2.7 Scalar Irradiance, Radiant Energy, and Related Concepts

The radiometric concepts studied in this section are those of scalar irradiance, radiant energy, and related radiometric concepts. The first of these concepts is designed to quantitatively describe the volume density of radiant energy in a way which is amenable to operational methods of determination. In addition to the notion of scalar irradiance, we shall develop in this section several closely related notions which together with scalar irradiance comprise a set of useful measures of the volume density of radiant energy. The first of these is radiant density.

Radiant Density

The notion of radiant density is one of several concepts designed to give a measure of the radiant energy per unit volume at a point. Consider a steady beam of radiant flux normally incident on surface S at point x at time t, as shown in Fig. 2.16. Let the field radiance of the beam at this instant be N, its cross sectional area be A, and its solid angle be Ω. The amount of radiant flux incident on S at time t is then NAΩ. An instant t later, the flux of the beam will have moved on a distance r = vt, and the flux will have swept out a cylindrical volume of magnitude V = Avt. During this time the beam has been steadily pouring an amount of radiant energy into the volume at the rate of NAΩ watts. Hence the radiant energy content of the beam is NAΩt, and its average content per unit volume is NAΩt/Avt = NΩ/v.

Suppose that point x were simultaneously irradiated at time t by an arbitrary finite number of narrow beams of radiance N_i, i= 1,..., n, and corresponding solid angles Ω_i. Then the radiant energy u(x,t) per unit volume at x is given at time t, by means of the D-additivity of Φ (equation (7), Sec. 2.3):