ideally as $\cos^2\psi$. Hence $N(\psi,0)$ reaches its ideal maximum of $N(0,0) = N$ at $\psi = 0$. In practice, however, $N(0,0) < N$, and so $T < 1$. It is now easy to establish that the practical counterpart to (7) is:

$$N = ({}_1N + {}_2N)/T \tag{9}$$

where ${}_1N$ and ${}_2N$ are measured with the same polarizer P as that used to obtain T in (8) and with the identical disposition of W as that used for (8), i.e., either W is in the tube and $\varepsilon = 0$, or W is removed entirely.

To summarize, if a radiance tube is fitted with attachments to allow the determination of the observable radiance vector $({}_1N, {}_2N, {}_3N, {}_4N)$, then the associated reading N of the meter without these accouterments is generally related to the vector components by means of (9), with T defined as in (8). In this way the radiant flux content N of **N** is established.

On the basis of relation (9) or its suitable generalizations, it is possible to use tabulated polarized radiance data to compute all the usual unpolarized radiance, scalar irradiance and vector irradiance quantities, etc. formulated in the preceding sections, simply by replacing "N" everywhere in those formulas by "$({}_1N+{}_2N)/T$". In this sense then we understand polarized radiance data to be more general than unpolarized radiance data, for it includes the latter as a special case.

2.11 Examples Illustrating the Radiometric Concepts

In this section, we conclude our discussion of geometrical radiometry and, before going on to the discussion of photometry, we consider some examples which may serve to illustrate in some depth the various radiometric concepts and the relations among them. The contents of this section are intended to serve a multiple purpose. First of all we take the opportunity of collecting together some worked examples in geometrical radiometry which illustrate the theory developed above; secondly, various special topics of only limited interest to hydrologic optics *per se* are considered on the basis of their intrinsic radiometric merits; and finally the section serves as a repository for certain special radiometric results needed as a matter of course in the later developments of this work.

Example 1: Radiance of the Sun and Moon

We illustrate the use of the empirical radiance definition (1) of Sec. 2.5 by using it to compute the empirical field radiances of the sun and moon. Now in (1) of Sec. 2.5, S is the area collecting the flux P(S,D) funneling down the set D of directions from either the sun or the moon. Hence S may be chosen at will and we fix it in this example as a

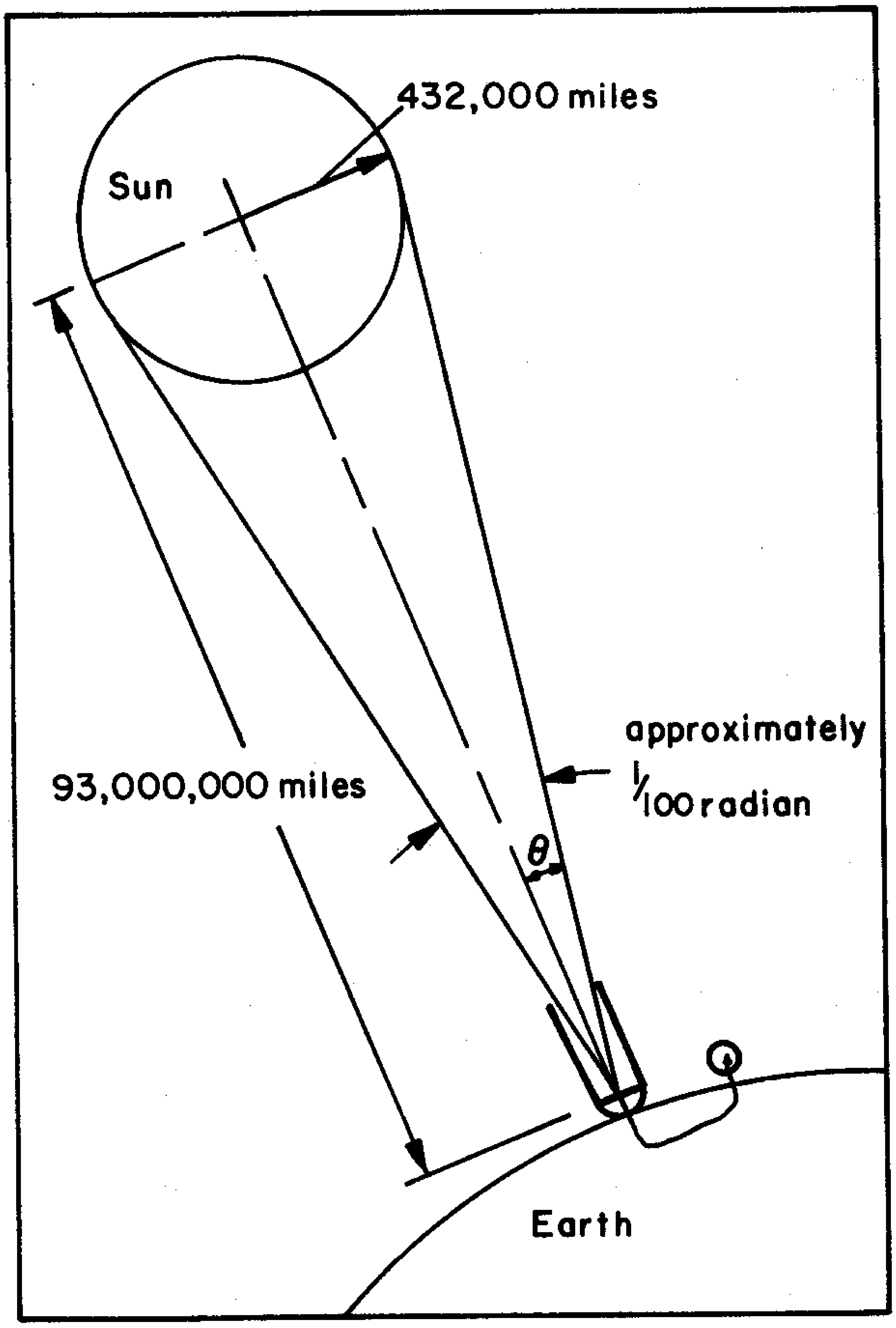

FIG. 2.28 Approximate angular subtense of the sun at the earth.

square meter of plane surface just outside the atmosphere and whose normal when extended goes through the center of the sun or moon. For the purpose of computing N(S,D) we choose D to be the solid conical set of directions from any point on the collecting surface to and within the limb of the sun or moon. See Fig. 2.28. We consider first the case of the sun.

The sun is a nearly spherical body with diameter nearly 864,000 miles and at a distance of about 93,000,000 miles from the earth. It follows that the half-angle subtense θ of the sun at the earth's surface is very nearly:

$$\theta = 4.32 \times 10^5 / 9.3 \times 10^7$$

$$= 4.65 \times 10^{-3} \text{ radians}$$

Hence, by (12) of Sec. 2.5, the solid angle subtense Ω(D) of the sun is:

$$\Omega(D) = \pi \times (4.65)^2 \times 10^{-6}$$
$$= 6.78 \times 10^{-5} \text{ steradians}$$

Now Table 1 of Sec. 2.4 gives an order of magnitude estimate of 10^3 watts/m^2 of the irradiance P(S,D)/A(S) produced by the sun's radiation over the whole spectrum and which is incident through D on a surface S normal to the sun's rays. A more accurate estimate of this irradiance produced outside the atmosphere is 1396 watts/m^2; see Ref. [128]. A still more meaningful alternate estimate of H(S,D) can be made for the interval of wavelengths in the visible spectrum (approximately 400 to 700 millimicrons). In this case H(S,D)-estimates vary from 542 watts/m^2 to 555 watts/m^2 (p. 31, Ref. [185]), see also Ref. [128]). Using the first estimate we obtain*:

$$N(S,D) = H(S,D)/\Omega(D)$$
$$= 5.42 \times 10^2 / 6.78 \times 10^{-5}$$

Errata

$$= 8 \times 10^2 \text{ watts/m}^2 \times \text{steradian} \qquad (1)$$

This radiance is the overall average radiance of the sun's disk as seen just outside the atmosphere and over the wavelengths of the visible spectrum. (Hence the set F of frequencies of Sec. 2.3 now consists of all frequencies from approximately 4×10^{14} to 7×10^{14}/sec.)

A good rule of thumb for remembering the angular subtense of the sun is that its entire disk subtends an angle of about 1/100 of a radian. The more exact estimate is given above. In other words the sun subtends about the same angle as a disk of a centimeter diameter at a meter's distance.

We turn now to the case of the moon. The geometric and radiometric principles are the same as in the case of the sun. And again, the crucial point of the calculation rests in the estimate of H(S,D). For this case we assume that the irradiance H(S,D) of the sun is on the order of 7×10^5 times that of the full moon over the visible spectrum. (See, e.g., Fig. 1.12 and Table 2 of Sec. 2.12.) In other words we assume that for the case of the moon, H(S,D) = 7.75×10^{-4} watts/m^2. Estimates of this ratio vary considerably. The one just chosen is an order of magnitude estimate only for the purposes of the present example.

The moon is a nearly spherical body with diameter nearly 2100 miles and at a distance of about 240,000 miles. It follows that the half-angle subtense θ of the moon at the earth's surface is very nearly:

$$\theta = (1.05 \times 10^3)/(2.4 \times 10^5)$$
$$= 4.38 \times 10^{-3} \text{ radians.}$$

*At sea level under a clean dry atmosphere, H(S,D) on the order of 472 watts/m^2. See also Table 2, Sec. 1.2.

Hence, by (12) of Sec. 2.5, the solid angle subtense $\Omega(D)$ of the moon is:

$$\Omega(D) = \pi(4.38)^2 \times 10^{-6}$$

$$= 6.00 \times 10^{-5} \text{ steradians} .$$

Using the adopted estimate we obtain:

$$N(S,D) = H(S,D)/\Omega(D)$$

$$= 7.75 \times 10^{-4}/6.00 \times 10^{-5}$$

$$= 13 \text{ watts/m}^2 \times \text{steradian} \qquad (2)$$

This may be used as an overall average radiance of the full moon's disk as seen at sea level on a clear night and over the wavelengths of the visible spectrum. An extensive literature exists with reference to lunar photometry and radiometry. See, e.g., [8].

In conclusion we note that the rule of thumb adopted for the angular size of the sun as seen from earth evidently also holds for the moon. For more detailed radiometric information on the radiant energy output of the sun, the reader may consult, e.g., Sec. 1.1 and Refs. [185] and [128]. Detailed discussion is made of the estimates of the solar irradiances in the latter references.

Example 2: Radiant Intensity of the Sun and Moon

The present example illustrates the use of the concept of radiant intensity as defined in (1) of Sec. 2.9.

We begin by computing the radiant intensity of the hemisphere S of the sun visible from the earth. Let ξ be the unit vector pointing from the center of the sun to the center of the earth. Then the radiant intensity $J(S,\xi)$ of S in the direction ξ is given by (14) of Sec. 2.9, where $N(x,\xi)$ is the surface radiance of the sun in the direction ξ at a point x on S. In Example 1 we estimated the field radiance of S for radiant flux in the wavelength interval from 400 to 700 millimicrons. Now, by the radiance invariance law (2) of Sec. 2.6, the estimate of Example 1 may be taken as the surface radiance of the sun over S, the radiance $N(x,\xi)$ being sensibly independent of x on S. Then if "N" denotes this fixed surface radiance, (14) of Sec. 2.9 yields:

$$J(S,\xi) = N\int_S \xi\cdot\xi'(x)\, dA(x)$$

$$= NA(S')$$

where $A(S')$ is the area of the projection S' of S on a plane perpendicular to ξ. The area $A(S')$ is readily determinable. From the data in Example 1, we have:

$$\begin{aligned} A(S') &= \pi(4.32)^2 \times 10^{10}(\text{miles})^2 \\ &= \pi(4.32)^2 \times (1.6)^2 \times 10^6 \times 10^{10} \ (\text{meters})^2 \\ &= 1.5 \times 10^{18} \ (\text{meters})^2 \end{aligned}$$

Using this estimate of A(S') and the estimate of N(S,D) for the sun given in Example 1, we have:

$$\begin{aligned} J(S,\xi) &= 8 \times 10^6 \times 1.5 \times 10^{18} \\ &= 1.2 \times 10^{25} \text{ watts/steradian} \qquad (3) \end{aligned}$$

as the radiant intensity of a hemisphere of the sun facing the earth and over the visible spectrum.

The sun is radiometrically a point source (Sec. 2.9) with respect to points on the earth and may thus be imagined to be compressed to its center x. Furthermore, we may evidently assume that $J(S,\xi)$ is independent of ξ. Hence (17) of Sec. 2.9 is applicable, and we can estimate the total radiant flux output of the sun over the visible spectrum to be:

$$\begin{aligned} P(x) &= 4\pi J(S,\xi) \\ &= 1.5 \times 10^{26} \text{ watts.} \end{aligned}$$

Turning now to the case of the moon, we have a slightly more interesting geometrical situation arising from the possible phases of the moon. Fig. 2.29 depicts this situation. If "S'" now denotes the projection of a lunar hemisphere on a plane normal to the direction ξ, then we have by means of (14) of Sec. 2.9:

$$J(S,\xi) = \frac{NA(S')}{2}(1 + \cos\theta)$$

where N is the surface radiance of the lighted hemisphere of the moon, as estimated*, e.g., in Example 1, and θ is the phase angle of the moon as described in Fig. 2.29. Thus at full moon, $\theta = 0$ and $J(S,\xi)$ is in particular NA(S). To estimate this product we first compute:

$$\begin{aligned} A(S') &= \pi(1.05)^2 \times 10^6 \ (\text{miles})^2 \\ &= \pi(1.05)^2 \times (1.6)^2 \times 10^6 \times 10^6 \ (\text{meters})^2 \\ &= 8.9 \times 10^{12} \ (\text{meters})^2 \quad . \end{aligned}$$

---

*The precise analysis of the gradation of the radiance distribution over the sunlit hemisphere of the moon is a delicate problem. The estimate here is deliberately kept simple in order to first emphasize the radiometric geometry essentials. A source reference on radiometry of the moon and planets is [8].

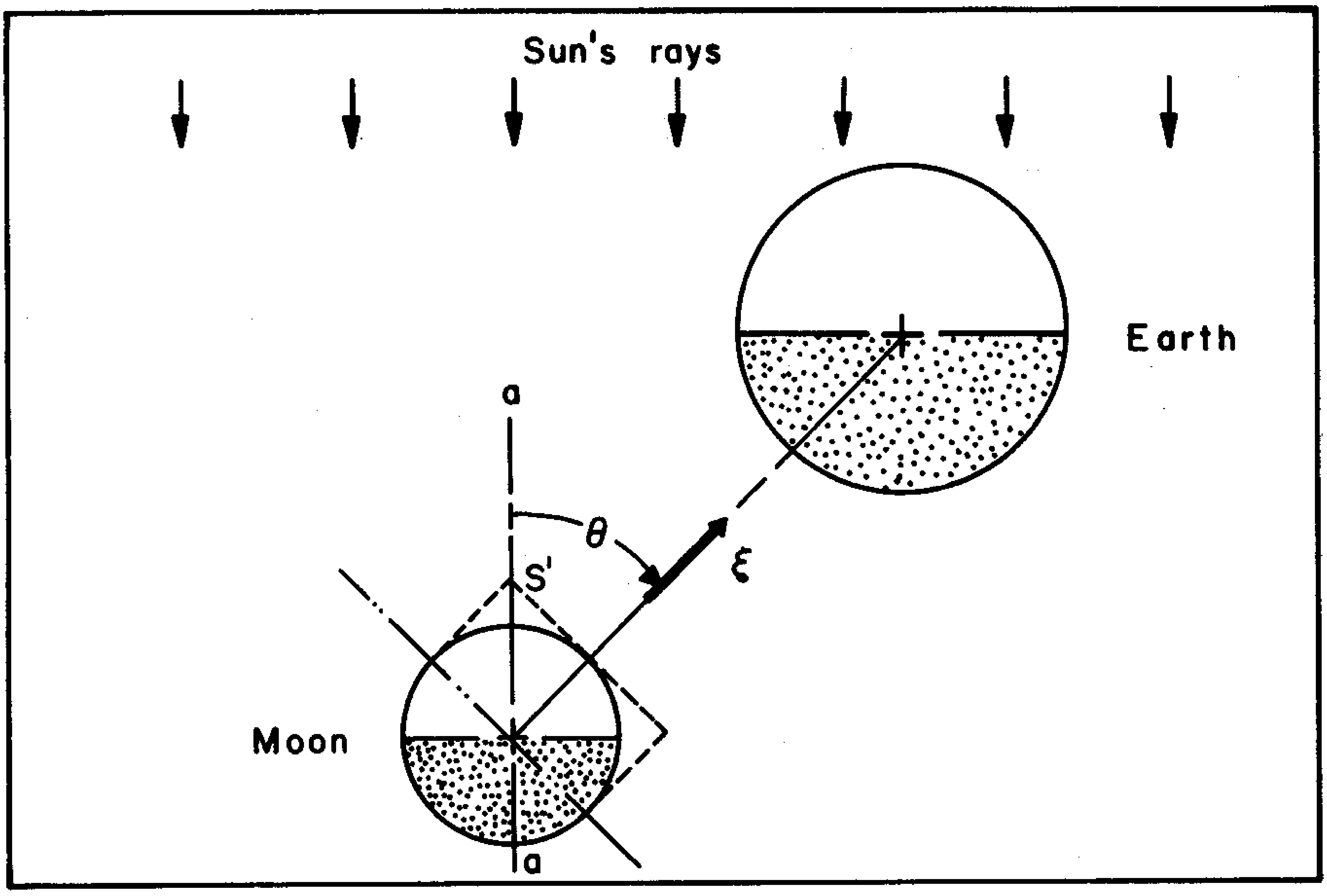

FIG. 2.29 Simple phase diagram for the earth-moon system

Using 13 watts/(m^2 × steradian) for N (justified by means of the radiance invariance law) we have:

$$NA(S') = 1.3 \times 10 \times 8.9 \times 10^{12} = 1.2 \times 10^{14} \text{ watts/steradian}$$

as the radiant intensity of the surface of the full moon over the visible spectrum. Hence for any phase θ, the corresponding radiant intensity of the lighted surface S of the moon in direction ξ (Fig. 2.29) is:

$$J(S,\xi) = 0.6 \times 10^{14}(1 + \cos\theta) \text{ watts/steradian} \qquad (4)$$

We conclude this example by computing the total radiant flux content of the reflected radiant flux from the moon, over the visible spectrum. Using the radiant intensity estimate just made, and assuming N to be independent of direction S, and the moon to be a point source at its center x as seen from the earth, we then integrate J(x,ξ) over all directions to obtain the requisite radiant flux, according to (15) of Sec. 2.9. Thus if "x" denotes the center of the moon and D' is now Ξ, Equation (15) of Sec. 2.9 becomes:

$$P(x) = \int_{\Xi} J(x,\xi)\, d\Omega(\xi)$$

$$= \frac{NA(S')}{2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (1+\cos\theta) \sin\theta \, d\theta \, d\phi = 2\pi NA(S)$$

To perform this integration, a "lunar based" polar coordinate system was used with θ measured as shown in Fig. 2.29 and ϕ measured from 0 to 2π around the axis a -a in a plane perpendicular to the page. We might have expected this relation on intuitive grounds: the radiant flux output of an entire sphere lighted uniformly all over, should be just twice that given off by one hemisphere. Thus P(x) is 2π times NA(S) instead of 4π times NA(S'). Hence:

$$P(x) = 2\pi \times 1.2 \times 10^{14}$$
$$= 7.5 \times 10^{14} \text{ watts}$$

is the radiant flux output of the moon over the visible spectrum.

Example 3: Radiant Flux Incident on Portions of the Earth

In this example, Equations (7) of Sec. 2.4 and (8) of Sec. 2.5 will be illustrated. Now, from Example 1, we find that at each point x just outside the atmosphere of the earth we have $H(x,D,t,F) = 542$ watts/m^2 funneling down a narrow cone D from the disc of the sun and with wavelengths over the visible spectrum F. Suppose S is some portion of the earth's surface accessible to the sun's rays, as in Fig. 2.30. To compute $\Phi(S,D,t,F)$, we establish a polar coordinate system as depicted in the figure. We first deduce that:

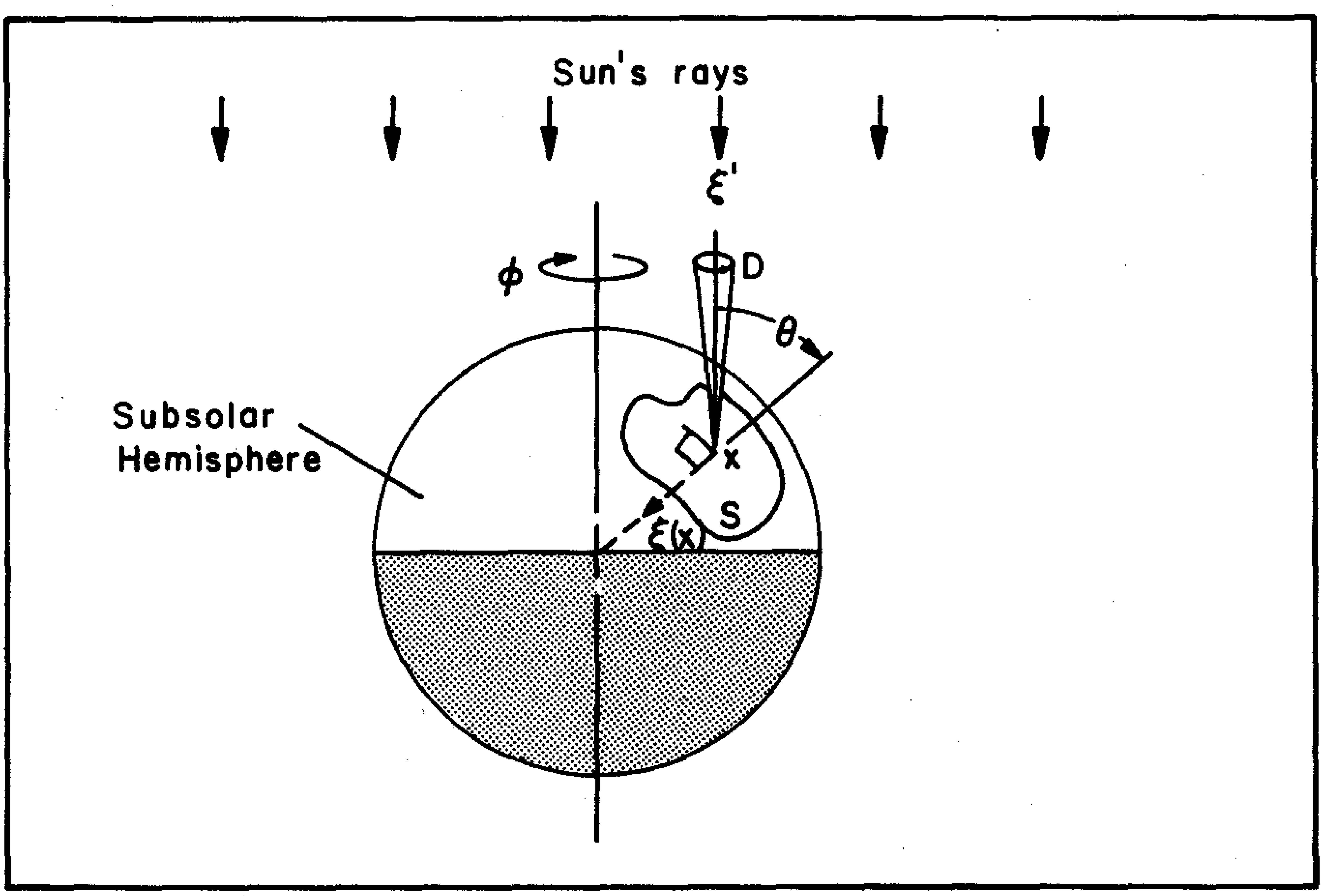

FIG. 2.30 Geometry for solar irradiation calculations

$$H(x,D,t,F) = \int_F H(x,D,t,\nu)\ dl(\nu) \tag{5}$$

which follows from (5) of Sec. 2.4 by means of a theorem of elementary calculus. Then by (7) of Sec. 2.4 we have:

$$\Phi(S,D,t,F) = \int_S H(x,D,t,F)\ dA(x)$$

Next, from (8) of Sec. 2.5:

$$H(x,D,t,F) = \int_D N(x,\xi',t,F)\xi\cdot\xi'\ d\Omega(\xi')$$

where we now explicitly use the fact that wavelengths are over the visible spectrum F. Since D is small and the sun's field radiance is uniform of magnitude N over D we can estimate H(x,D,t,F) fairly accurately by means of the equality:

$$H(x,D,t,F) = N\xi'\cdot\xi(x)\Omega(D) \quad ,$$

where N and $\Omega(D)$ were estimated for the sun in Example 1. Furthermore, $\xi(x)$ is the unit inward normal to the earth's surface at x, and ξ' is the direction from the center of the sun to the center of the earth. Using this representation of H(x,D,t,F) in the preceding integral for $\Phi(S,D,t,F)$, we arrive at the expression:

$$\Phi(S,D,t,F) = N\Omega(D) \int_S \xi'\cdot\xi(x)\ dA(x)$$

$$= NA(S')\Omega(D) \tag{6}$$

where A(S') is the area of the projection S' of S on a plane normal to the direction ξ' of the sun's rays.

As a specific example, we use N and $\Omega(D)$ as in Example 1, and let S be the sub solar hemisphere of the earth. Then:

$$A(S') = \pi(4)^2 \times 10^6\ (\text{miles})^2$$

$$= \pi(4)^2 \times (1.6)^2 \times 10^6 \times 10^6$$

$$= 1.3 \times 10^{14}\ (\text{meters})^2$$

Hence:

$$\Phi(S,D,t,F) = 8 \times 10^6 \times 1.3 \times 10^{14} \times 6.78 \times 10^{-5}$$

$$= 7 \times 10^{16}\ \text{watts} \tag{7}$$

over the visible spectrum. The corresponding radiant flux $\Phi(S_1,D,t,F)$ incident on any proper portion S_1 of the entire subsolar hemisphere is simply obtained by finding $A(S_1')/A(S')$, where now S_1' is the projection of S_1 on a plane normal to the sun's rays, and then multiplying 7×10^{16} by this fraction; or alternatively, 542 watts/m^2 by $A(S_1')$. Of course these estimates are somewhat crude, and serve only to illustrate the correct mathematical use of the geometric radiometry formulas deduced above. The present estimate of $\Phi(S,D,t,F)$ omits, e.g., the effect of the atmosphere which at each point subtly attenuates and augments the solar influx by permitting absorption, scattering, and interreflections with the earth below.

Example 4: Irradiance Distance-Law for Spheres

In this and several of the examples below we shall explore some interesting consequences of the irradiance integral (8) of Sec. 2.5.

We begin the investigations by considering a spherical surface S of radius a with uniform radiance distribution of magnitude N at each point. Suppose that S is viewed at a point x a distance r from the center y of S. The lines of sight lie in a vacuum and the background radiance of S is zero. See Fig. 2.31 (a). We ask: what is the irradiance $H(x,\xi)$ at point x? Here ξ is the direction from y to x.

Equation (8) of Sec. 2.5 is readily applied to the present situation. For the present case we may use the radiance invariance law to say that $N(x,\xi') = N$ for every ξ' in the conical set D of directions subtended by S at x. Hence (8) of Sec. 2.5 becomes:

$$
\begin{aligned}
H(x,\xi) &= N\int_D \xi\cdot\xi' \, d\Omega(\xi') \\
&= N\int_0^{2\pi}\int_0^{\theta} \cos\theta' \sin\theta' \, d\theta' \, d\phi' \\
&= 2\pi N\int_0^{\theta} \cos\theta' \sin\theta' \, d\theta' \\
&= \pi N \sin^2\theta \\
&= \pi N (a/r)^2 \quad .
\end{aligned}
$$

If we write, *ad hoc*,

$$\text{"}H_r\text{"} \quad \text{for} \quad H(x,\xi) \quad ,$$

then we have found that:

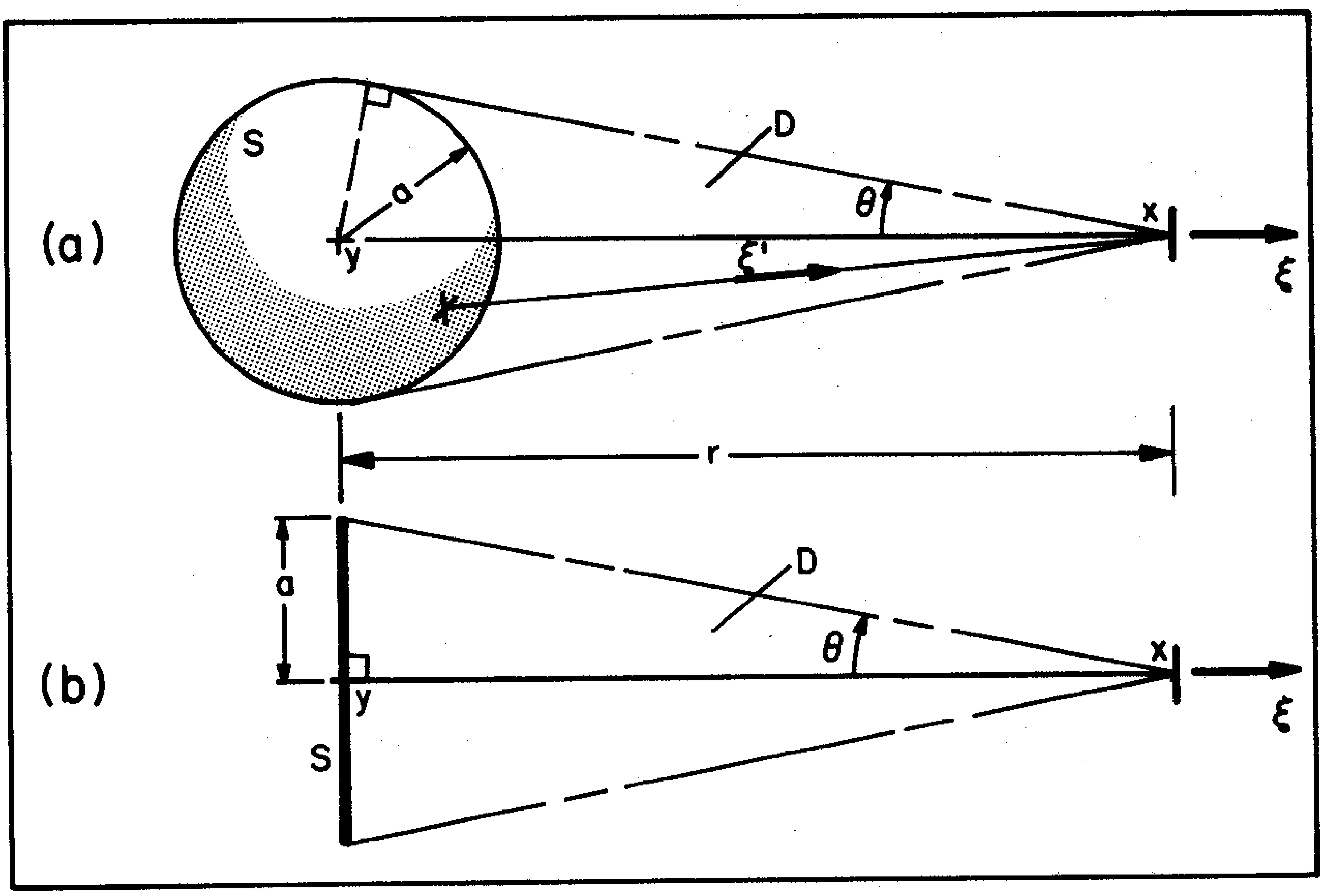

FIG. 2.31 Deriving the Irradiance Distance-Law for spheres and disks

$$\boxed{H_r = AN/r^2} \quad , \tag{8}$$

where we have written:

"A" for πa^2 ,

i.e., A is the area of a great circle of S; alternatively A is the area of projection of S on a plane perpendicular to ξ. From (14) of Sec. 2.9 applied to the present case, we may write:

$$\boxed{H_r = J/r^2} \tag{9}$$

where we have written:

"J" for AN .

It is to be particularly noted that H_r varies *precisely* as the inverse square of the distance r, where $a \leq r$. If $r = a$, then:

$$H_a = \pi N \quad . \tag{10}$$

Example 5: Irradiance Distance-Law for Circular Disks; Criterion for a Point Source

Equation (8) of Sec. 2.5 will now be used to derive the law governing the irradiance produced by a circular disk S of uniform radiance. See Fig. 2.31 (b). In that figure is depicted a circular disk of radius a and of uniform surface radiance N at each point. The disk is viewed at point x on the perpendicular through the center y of S at a distance r from the center. The set D of the lines of sight from x to S lies in a vacuum and the background radiance of S is zero. What is the radiance $H(x,\xi)$ at point x? Here ξ is the direction from y to x.

Equation (8) of Sec. 2.5 can be applied to the present situation, as in the case of Example 4. Thus, (8) of Sec. 2.5 becomes:

$$H(x,\xi) = N\int_D \xi\cdot\xi' \, d\Omega(\xi')$$

$$= N\int_0^{2\pi}\int_0^{\theta} \cos\theta' \sin\theta' \, d\theta' \, d\phi'$$

$$= 2\pi N\int_0^{\theta} \cos\theta' \sin\theta' \, d\theta'$$

$$= \pi N \sin^2\theta$$

$$= \pi N \, a^2/(a^2+r^2) \quad .$$

If we write, *ad hoc*:

"H_r'" for $H(x,\xi)$

and further, we write:

"A" for πa^2

and:

"J" for AN ,

then we have found that:

$$H_r' = AN/(a^2+r^2) \tag{11}$$

or:

$$H_r' = J/(a^2+r^2) \tag{12}$$

From this we find first of all that H_r', unlike H_r of Example 4, does not vary precisely as the inverse square of r, where $r \geq 0$. However, in the special case of r = 0, we have:

$$H_o' = \pi N \quad . \tag{13}$$

Further, in the other extreme, i.e., when r is very much larger than a, H_r' varies very nearly as the inverse square of r.

By examining more closely the difference between H_r and H_r', we arrive at the basis for the definition of a point source given in Sec. 2.9. Suppose then we compare H_r and H_r' which are, respectively, the irradiances produced by a sphere of radius a and a circular disk of radius a both of uniform radiance N. Toward this end we form the difference:

$$H_r - H_r' = J\left[\frac{1}{r^2} - \frac{1}{(a^2+r^2)}\right]$$

and then form the relative difference:

$$(H_r - H_r')/H_r' = \frac{a^2+r^2}{r^2} - 1 = a^2/r^2$$

This relative-difference expression is the basis for the following statements: *The relative difference between the irradiance* H_r *and* H_r' *is less than 1% whenever* $r > 10a$. *More generally: the irradiance produced by a finite object of uniform radiance decreases as the inverse square of the distance from that object, within an error of 1 percent, whenever the distance from the object is more than 10 times greater than the object's largest transverse linear dimension.* This alternate statement follows readily from the preceding analysis. Some further study is made in Example 6 of related questions. Observe that the associated solid angle of the circular cone of half angle 1/10 radian is very nearly $\pi(1/10)^2 = \pi/100 \approx 1/30$ steradian, in which lies the origin of the solid angle number used in the point source criterion of Sec. 2.9.

Example 6: Irradiance Distance-Law for General Surfaces

We devote this example to the elucidation of the common denominator of Examples 4 and 5; the net result being the formula for the irradiance distance-law for a general surface S of uniform radiance N viewed, as in Fig. 2.32, from an external vantage point x along a set of paths defined by a collection D of directions, each path of which lies in a vacuum.

The derivation of the required $H(x,\xi)$ begins, as in Examples 4 and 5, with (8) of Sec. 2.5, but now proceeds as follows:

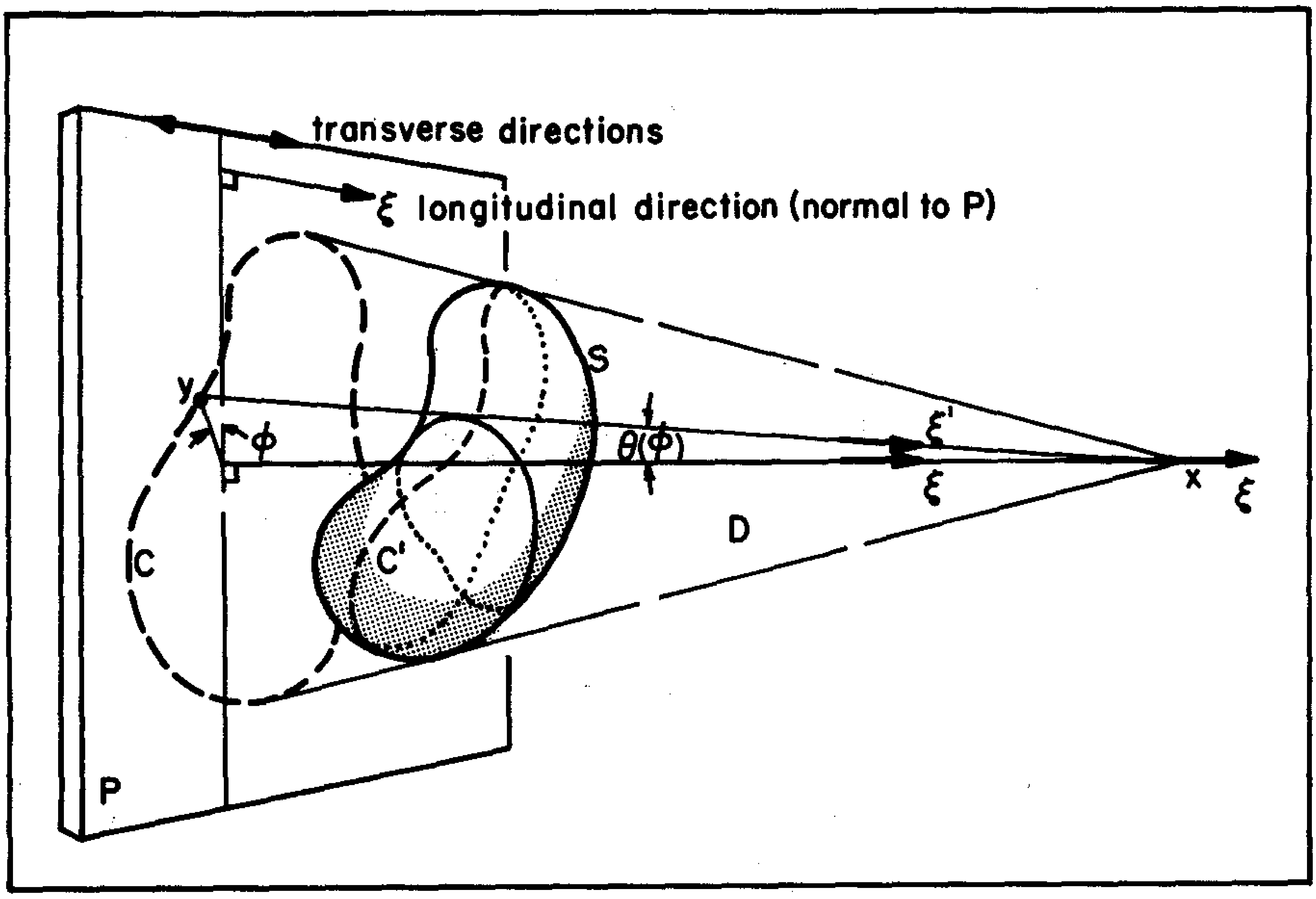

FIG. 2.32 Deriving the Irradiance Distance-Law for general surfaces

$$H(x,\xi) = N\int_D \xi\cdot\xi' \, d\Omega(\xi')$$

$$= N\int_0^{2\pi}\int_0^{\theta(\phi)} \cos\theta' \sin\theta' \, d\theta' \, d\phi$$

$$= \frac{N}{2}\int_0^{2\pi}\int_0^{\theta(\phi)} d(\sin\theta') \, d\phi$$

$$= \frac{N}{2}\int_0^{2\pi} \sin^2\theta(\phi) \, d\phi$$

Let us write, *ad hoc*:

"H" for $H(x,\xi)$.

With this, we have attained the required result:

$$H = \frac{N}{2}\int_0^{2\pi} \sin^2\theta(\phi) \, d\phi \quad . \tag{14}$$

This formula for H reduces to the expressions for H_T and H_T' when the function $\theta(\cdot)$ is suitably prescribed for all ϕ from 0 to 2π. In particular θ is a constant function in the preceding two cases. More importantly, the reader should observe the remarkable fact that the irradiance H *depends only on the integral over the outline* C *of* S, as may be seen by studying the central projection of S onto the background plane P (of Fig. 2.32) which is perpendicular to ξ. Hence it is literally immaterial to H what the longitudinal structure of S is as regards the computation of H at a fixed point x, as long as S has the given outline C on P, and also has uniform radiance N. Of course the shape of S is important when it is decided to let x *vary*, and indeed the distance-law for $H(x,\xi)$ depends critically on the longitudinal shape of S and in this context takes its most general form displayed in the above equation for H.

An alternate form of the distance-law for irradiance is obtained when we write:

$$\text{"}\boldsymbol{\Omega}\text{"} \quad \text{for} \quad \int_D \xi' \, d\Omega(\xi')$$

Hence:

$$\boxed{H = N\xi\cdot\boldsymbol{\Omega}} \quad . \tag{15}$$

When the size Ω of D is small--e.g., when S is a point source at x, then we have, very nearly:

$$\boldsymbol{\Omega} = \xi\Omega$$

and in this special case (15) yields:

$$H = N\Omega \ . \tag{16}$$

If A is the projected area of S then in this case we have very nearly:

$$\Omega = A/r^2 \quad ,$$

where r is the distance from x to S. In this way we return to the inverse square law for H in the limit of large r (or small A).

Still one more form for H, i.e., $H(x,\xi)$, is obtainable using the concept of vector irradiance introduced in Sec.2.8. Thus we have

$$\boxed{H = \xi\cdot\mathbf{H}} \tag{17}$$

where in the present case we have written:

$$\text{"}\mathbf{H}\text{"} \quad \text{for} \quad \int_D N\xi' \, d\Omega(\xi')$$

As a corollary we have:

$$\boxed{\mathbf{H} = N\mathbf{\Omega}} \tag{18}$$

An important and useful special case of (14) occurs when $\theta(\phi)$ is independent of ϕ. This happens when the surface S is a surface of revolution about the direction ξ. (See Fig. 2.32). In such a case (14) becomes:

$$\boxed{H = \pi N \sin^2 \theta} \quad .$$

In particular, if S is an infinite plane, then at all distances r from S, S subtends a half angle $\theta = \pi/2$. Hence $H = \pi N$ for all r in such a case.

Example 7: Irradiance via Line Integrals

The present example is designed to let us investigate in greater depth the irradiance integral (14) of Example 6 which showed that the irradiance produced at a fixed point x by an arbitrary surface of uniform radiance depended only on the *angular* outline of S as seen at the point x. Our goal in this example will be to cast equation (14) into explicit line integral form over the curve C which defines the outline of S.

Figure 2.33 (a) is a reconstruction of Figure 2.32 with surface S omitted. What is left is the geometric essence of the irradiance calculations done in Example 6. Specifically, we have retained the central projection of S on plane P through point x. The boundary C of this projection of S on P is a closed curve characterized by means of the function $\theta(\cdot)$ which assigns to each ϕ, $0 \leq \phi \leq 2\pi$ an angle $\theta(\phi)$, which determines point y on C as shown in Fig. 2.33. We denote by "O" the foot of the perpendicular dropped on P from x. Further, "$r(\phi)$" will denote the distance from the fixed point x to the variable point y on C.

With these preliminaries established, we can write (14) in the form:

$$H = \frac{N}{2}\int_0^{2\pi} \frac{\sin \theta(\phi)}{r(\phi)} \cdot r(\phi) \sin \theta(\phi) \, d\phi \quad .$$

The integral was rewritten this way to make use of the fact that:

$$r(\phi) \sin \theta(\phi) \, d\phi$$

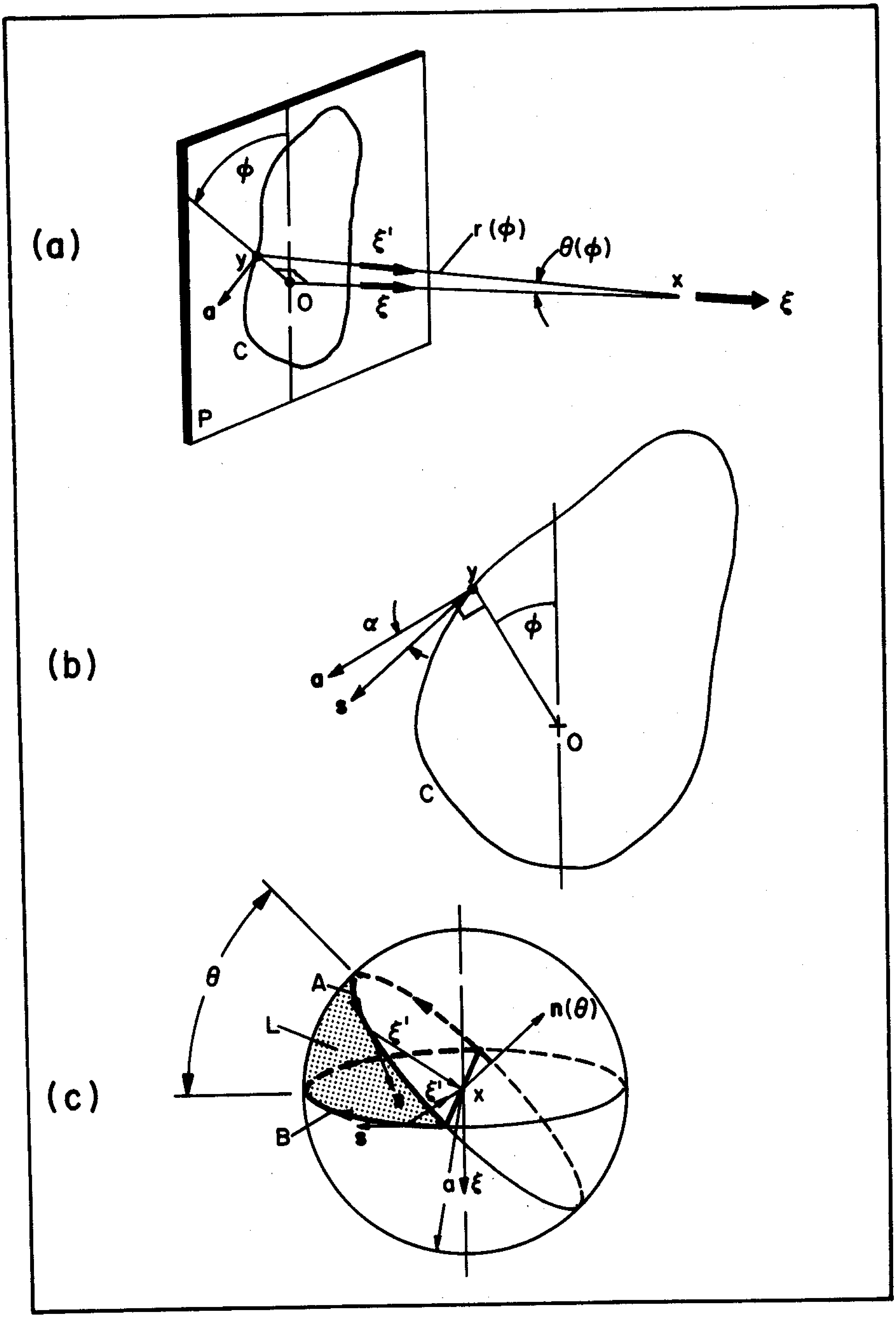

FIG. 2.33 Setting for calculating irradiances via line integrals.

is an element of length da along the direction of the unit vector **a** in P, normal to Oy, and at point y on C, as shown in Fig. 2.33 (b), which is a plan view of P. The element of length da is related to an element of arc length ds along C at y by a projection, and by definition of ds:

$$da = \cos\alpha \; ds$$

where α is the angle between **a** and the unit tangent vector **s** to C at y; hence:

$$r(\phi)\sin\theta(\phi)\,d\phi = da = \mathbf{a}\cdot\mathbf{s}\,ds \quad .$$

The preceding integral then may be written:

$$H = \frac{N}{2}\int_C \frac{\sin\theta(\phi)}{r(\phi)}\,\mathbf{a}\cdot\mathbf{s}\,ds \quad .$$

Next we observe, by means of Fig. 2.33, that:

$$\xi' \times \xi = \mathbf{a}\sin\theta(\phi) \quad ,$$

where ξ is the unit outward normal to P (outward relative to ξ', i.e., such that $\xi'\cdot\xi > 0$). Hence:

$$H = \frac{N}{2}\int_C \frac{(\xi' \times \xi)\cdot\mathbf{s}\,ds}{r(\phi)}$$

The triple box product of vectors in the integrand may be rearranged so that we obtain for C (or C'!) in Fig. 2.32:

$$\boxed{H = \frac{1}{2}\,N\xi\cdot\int_C \frac{\mathbf{s}\times\xi'\,ds}{r(\phi)}} \tag{19}$$

Comparing this with (15) we deduce that:

$$\Omega = \frac{1}{2}\int_C \frac{\mathbf{s}\times\xi'\,ds}{r(\phi)} \tag{20}$$

Equation (20) displays a line integral representation of Ω, and (19) displays the desired line integral representation of H.

As an illustration of (20), let C' be the boundary of a spherical lune L of angular opening θ, on a sphere of radius a, as shown in Fig. 2.33 (c). Thus L is now a specific instance of the general surface S of Fig. 2.32, and C may actually be taken as *any* outline of S (as, e.g., C' in Fig. 2.32). Note the present placement of point x and the direction ξ. The contribution to Ω over the upper arc A of C is clearly:

$$\frac{1}{2}\int_A \frac{\mathbf{s} \times \xi'}{r(\phi)}\, ds = \frac{\pi}{2}\, \mathbf{n}(\theta)$$

where $\mathbf{n}(\theta)$ is the unit normal to the plane containing arc A, and directed such that:

$$\mathbf{n}(\theta) = \mathbf{s} \times \xi' \quad ,$$

when A is traversed as shown in the figure. Further, $r(\phi) = a$ for all ϕ. The contribution to Ω over the arc B of C is clearly:

$$\frac{1}{2}\int_B \frac{\mathbf{s} \times \xi'}{r(\phi)}\, ds = \frac{\pi}{2}\, \xi$$

The integrals over A and B were evaluated immediately by noting that over A, $\mathbf{s} \times \xi'$ is a fixed unit vector, namely $\mathbf{n}(\theta)$; and over B, $\mathbf{s} \times \xi'$ is the unit vector ξ, the unit inward normal to the plane of arc B. The arc lengths of A and B are each $a\pi$. Hence for the present case:

$$\Omega = \frac{\pi}{2}\,(\xi + \mathbf{n}(\theta)) \quad .$$

Observe that if $\theta = 0$, then, $\mathbf{n}(\theta) = -\xi$, and $\Omega = 0$. If $\theta = \pi$, then $\mathbf{n}(\pi) = \xi$, and $\Omega = \pi\xi$. If L is of uniform radiance N, then, by (15) or (19):

$$H = N\ \xi \cdot \Omega = \frac{N\pi}{2}\,(1 + \xi \cdot \mathbf{n}(\theta)) \quad .$$

Example 8: Solid Angle Subtense of Surfaces

The integral form of the solid angle subtense $\Omega(D)$ of a set D of directions, as given in (10) of Sec. 2.5, will now be recast into a form which arises when the solid angle subtense of specific surfaces (either real or hypothetical) are under consideration. Thus, consider the surface S depicted in Fig. 2.34 (a) where S is shown viewed from an external vantage point x. Let "D(S,x)" denote the set of all directions from points of S to x. Our present goal is to derive the expression for $\Omega(D(S,x))$ (or "$\Omega(S,x)$" for short) in the form of a surface integral over S.

We begin by letting "D" in (10) of Sec. 2.5 be replaced by "D(S,x)". The result is:

$$\Omega(S,x) = \int_{D(S,x)} \sin\theta\, d\theta\, d\phi$$

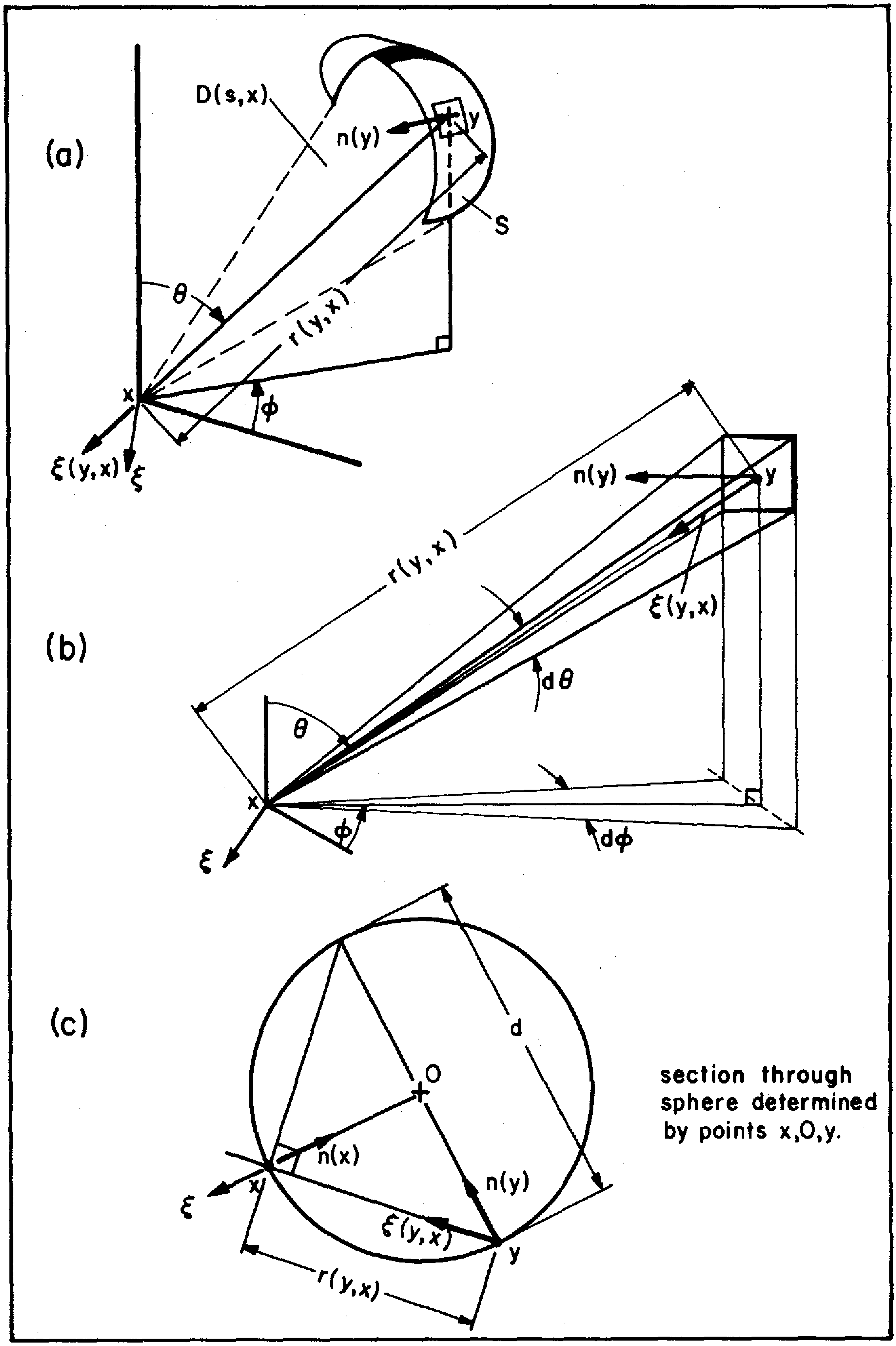

FIG. 2.34 Calculating the solid angle subtense of "tangible" surfaces.

The conventions for measuring θ and ϕ are summarized in Fig. 2.34 (a). In particular the details of the integration over a part of S about a point y on S are depicted in part (b) of Fig. 2.34. Points y and x determine a direction $\xi(y,x)$, as shown. It may be seen from part (b) of Fig. 2.34, that the relation between a small patch of S of area A(y) about y is related to its projection's area A' on a plane perpendicular to $\xi(y,x)$ by the formula:

$$A'(y) = A(y)\mathbf{n}(y)\cdot\xi(y,x) \quad (\geq 0, \text{ by choice of S})$$

where $\mathbf{n}(y)$ is the unit outward normal to S at y. Hence the solid angle subtense of the patch of S about y is:

$$\frac{A'(y)}{r^2(y,x)} = \frac{\mathbf{n}(y)\cdot\xi(y,x)A(y)}{r^2(y,x)} \quad .$$

The entire solid angle subtense of S at x is obtained by adding up all these solid angle subtenses of the component patches comprising S:

$$\boxed{\Omega(S,x) = \int_S \frac{\mathbf{n}(y)\cdot\xi(y,x)}{r^2(y,x)}\, dA(y)} \qquad (21)$$

It is of interest to observe that the set function $\Omega(\cdot,x)$ is non-negative valued, S-additive and S-continuous (compare these properties with those of the radiometric concepts in Sec. 2.3). Thus for every x and pair S_1, S_2 surfaces with disjoint sets $D(x,S_1)$ and $D(S_2,x)$ we have:

$$\Omega(S_1,x) + \Omega(S_2,x) = \Omega(S_1 \cup S_2,x)$$

which is the S-*additivity property*; further:

$$\text{If } A(S) = 0, \quad \text{then } \Omega(S,x) = 0 \quad .$$

In other words, the latter statement, the S-*continuity property* for $\Omega(\cdot,x)$, asserts that $\Omega(S,x) > 0$ only if $A(S) > 0$.* It follows from these additivity and continuity properties of Ω and the calculus that the ratio $\Omega(S,x)/A(S)$ has a limit as $S \to \{y\}$, where y is some point of S. Indeed:

$$\lim_{S\to\{y\}} \Omega(S,x)/A(S) = \frac{\mathbf{n}(y)\cdot\xi(y,x)}{r^2(y,x)} \quad .$$

*The converse clearly does not hold; thus, give a counterexample for: If $\Omega(S,x) = 0$, then $A(S) = 0$.

Writing "$d\Omega(y,x)/dA(y)$" for the limit operation above, we can then state that:

$$\frac{d\Omega(y,x)}{dA(y)} = \frac{n(y)\cdot\xi(y,x)}{r^2(y,x)} \tag{22}$$

Equation (22) yields, for a given point x, the value of the general area derivative of the solid angle function $\Omega(\cdot,x)$ at point y of an arbitrary surface, where $n(y)$ is the unit outward normal to the surface at y, $r(y,x)$ is the distance from x to y and $\xi(y,x)$ is the unit vector from y to x and such that the dot product of $\xi(y,x)$ and $n(y)$ is non-negative (this fixes the sense of "outward" for $n(y)$). As an interesting exercise the reader should show that if x *and* y in (22) are on a spherical surface S of diameter d, (see Fig. 2.34 (c)) then:

$$\frac{d\Omega(y,x)}{dA(y)} = \frac{1}{[n(y)\cdot\xi(y,x)]d^2}$$

The representation of $\Omega(S,x)$ in (21) is of particular value when the surface S is relatively concrete and has a specific analytic description, (parts of spheres, walls, and relatively tangible surfaces in general), whereas (10) of Sec. 2.5 is of greatest use when no surface S is specifiable and when instead a set D of directions *per se* is to be assigned a solid angle value. We close this example with the observation that all of Euclid's Optics [36] can be placed on a solid modern mathematical foundation using (21) and its various logical corollaries. (The translation of the first theorem in Euclid's Optics is given as a motto at the beginning of Volume I of this work. The theorem thus has several levels of meaning.)

Example 9: Irradiance via Surface Integrals

We return now to the integral for irradiance given in (8) of Sec. 2.5 and cast it into that form which is most useful when one must take into specific account the surface radiance of some surface S producing an irradiance $H(x,\xi)$ at some point x outside of S. The geometric setting for the present example is depicted by Fig. 2.34 (a), where at each point y of the surface S, there is given a surface radiance distribution $N(y,\cdot)$. We assume that all directions in $D(S,x)$ lie in a vacuum, that $D(S,x)$ is less than a hemisphere, and that the irradiance contributions to $H(x,\xi)$ come only from S so that $N(x,\xi')$ in (8) of Sec. 2.5 is zero for ξ' outside of $D(S,x)$. Hence (8) of Sec. 2.5 may be written:

$$H(x,\xi) = \int_{D(S,x)} N(x,\xi')\,\xi\cdot\xi'\; d\Omega(\xi')$$

In the present study the dummy variable "ξ'" is chosen to be the name of the variable direction $\xi(y,x)$ used in Example 8.

Then, in view of the radiance invariance law (2) of Sec. 2.6:

$$N(y,\xi') = N(x,\xi') \quad ,$$

it is clear that $H(x,\xi)$ is also represented by the integral:

$$H(x,\xi) = \int_S N(y,\xi')\xi\cdot\xi' \, d\Omega(y,x) \quad .$$

This may be written in the form:

$$H(x,\xi) = \int_S N(y,\xi')\xi\cdot\xi' \frac{d\Omega(y,x)}{dA(y)} dA(y)$$

which, by (22) is reducible to:

$$\boxed{H(x,\xi) = \int_S N(y,\xi(y,x)) \frac{\xi\cdot\xi(y,x)\mathbf{n}(y)\cdot\xi(y,x)}{r^2(y,x)} dA(y)} \qquad (23)$$

Equation (23) is the desired surface integral representation of $H(x,\xi)$. Suppose we write:

$$\text{"}\mathbf{H}(S,x)\text{"} \quad \text{for} \quad \int_S N(y,\xi(y,x)) \frac{\xi(y,x)\mathbf{n}(y)\cdot\xi(y,x)}{r^2(y,x)} dA(y) \quad .$$

This is the surface radiance counterpart to the field radiance definition of $\mathbf{H}(x)$ in (2) of Sec. 2.8. Then (23) becomes:

$$H(x,\xi) = \xi\cdot\mathbf{H}(S,x) \quad . \qquad (24)$$

Equation (24) suggests that the condition imposed at the outset, namely that $D(S,x)$ be less than a hemisphere, can evidently be relaxed. In that event (24) is generalizable to:

$$\overline{H}(x,\xi) = \xi\cdot\mathbf{H}(S,x) \quad , \qquad (25)$$

the proof of which is left to the reader.

If we assume that the point x is *inside* a closed surface S, then (23) still holds but with $\mathbf{n}(y)$ now being interpreted, if desired, as an *inward* unit normal from y to x. In that case, $\mathbf{H}(S,x)$ of (24) formally reduces to $\mathbf{H}(x)$ in (2) of Sec. 2.8. These observations suggest that the true field radiance counterpart to (25) is:

$$\overline{H}(x,\xi) = \xi\cdot\mathbf{H}(x,D) \qquad (26)$$

where we have written:

$$\text{"}\mathbf{H}(x,D)\text{"} \quad \text{for} \quad \int_D \xi' N(x,\xi') \, d\Omega(\xi') \tag{27}$$

The connection between $\mathbf{H}(x,D)$ and $\mathbf{H}(x)$ of Sec. 2.8 is clearly that:

$$\mathbf{H}(x) = \mathbf{H}(x,\Xi) \quad . \tag{28}$$

An interesting special case of (23) arises when S is part of the inside of a spherical surface. In Fig. 2.34 (a) imagine x and y to be on the same spherical surface of diameter d, and now ξ in (23) is to be the unit outward normal to S at x, i.e., $\xi = -\mathbf{n}(x)$. Under such conditions (see Fig. 2.34 (c)) it follows that:

$$\xi \cdot \xi(y,x) = \mathbf{n}(y) \cdot \xi(y,x)$$

and

$$r(y,x) = d\mathbf{n}(y) \cdot \xi(y,x) \quad .$$

Hence if $N(y,\xi(y,x)) = N$, over S, (23) yields:

$$H(x,\xi) = \frac{NA(S)}{d^2}$$

for every x on the sphere, and arbitrary subset S of the sphere.

Example 10: Radiant Flux Calculations

The irradiance integral (23) may be applied to the following radiometric setting, depicted in Fig. 2.35, which arises frequently in practice. A surface Y has a prescribed surface radiance $N(y,\cdot)$ at each point y. Surface X, which is disjoint from Y, receives an amount $P(Y,X)$ of radiant flux from Y. It is required to express the amount $P(Y,X)$ in terms of $N(y,\cdot)$ and the areas of X and Y, assuming the space between X and Y is a vacuum. Now from (23) we have for each x and ξ an expression for the irradiance $H(x,\xi)$, so that we can immediately compute $P(Y,X)$ in terms of $H(x,n(x))$, using (6) of Sec. 2.4 (in which D is now $\Xi(\mathbf{n}(x))$):

$$P(Y,X) = \int_X H\big(x,\mathbf{n}(x)\big) \, dA(x) \quad ,$$

where $\mathbf{n}(x)$ is the unit inward normal to X at x. Hence

$$P(Y,X) = \int_X \int_Y N\big(y,\xi(y,x)\big) \, \frac{\mathbf{n}(x) \cdot \xi(y,x)\, \xi(y,x) \cdot \mathbf{n}(y) \; dA(y) \; dA(x)}{r^2(y,x)}$$

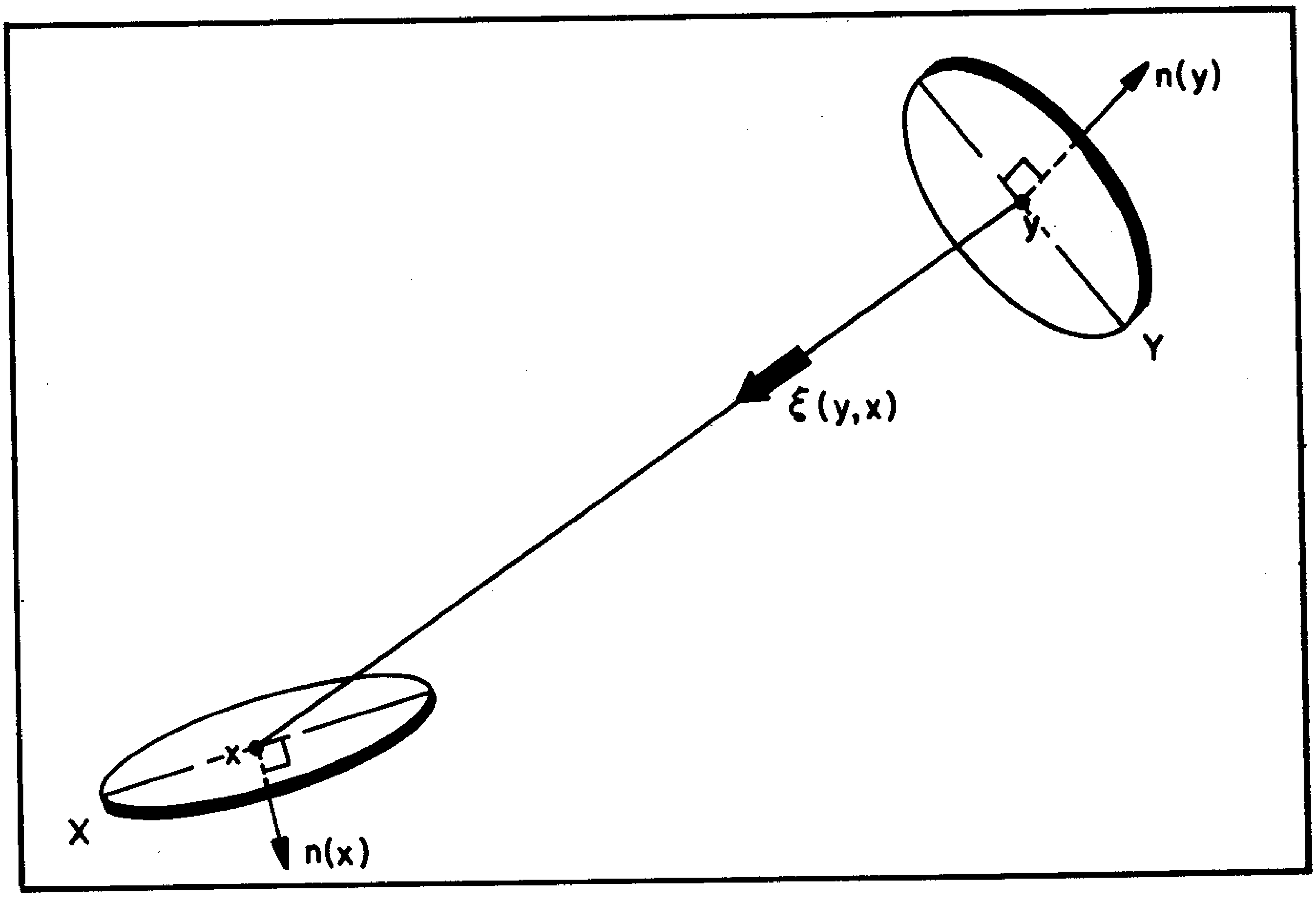

FIG. 2.35 A radiant flux calculation for two disjoint surfaces.

If we write

$$\text{"}K(y,x)\text{"} \quad \text{for} \quad \frac{\mathbf{n}(x)\cdot\xi(y,x)\,\xi(y,x)\cdot\mathbf{n}(y)}{r^2(y,x)}$$

Then, more succinctly,

$$P(Y,X) = \int_X \int_Y N\big(y,\xi(y,x)\big)K(y,x)\ dA(y)\ dA(x) \tag{29}$$

Observe that $K(\cdot,\cdot)$ is a symmetric function, i.e., for every x and y,

$$K(x,y) = K(y,x) \quad .$$

If the areas $A(X)$ and $A(Y)$ of X and Y are small--say when each is a point source with respect to any point on the other--then (29) yields the useful approximate relation:

$$P(Y,X) = N(Y,X)\ K(y,x)\ A(Y)\ A(X) \tag{30}$$

where x and y are some fixed points of X and Y, respectively and "$N(Y,X)$" denotes the surface radiance of Y in the direction

$\xi(y,x)$ of X. Writing, *ad hoc*:

$$H(Y,X) \quad \text{for} \quad P(Y,X)/A(X)$$

and

$$\text{"}J(Y,X)\text{"} \quad \text{for} \quad N(Y,X)A(Y) \quad ,$$

Equation (30) becomes:

$$H(Y,X) = J(Y,X)\; K(y,x) \quad , \tag{31}$$

which is a highly compact formulation of several well-known radiometric laws, with built-in cosine laws for both irradiance and radiant intensity, and furthermore, with built-in inverse square law for irradiance and direct square law for radiant intensity.

Example 11: Intensity Area-Law for General Surfaces

This example serves to illustrate some further facets in the duality between irradiance and radiant intensity developed in various earlier sections throughout this chapter. In particular we now direct attention to the intensity counterparts of the relations (15)-(18) in Example 6 of this section.

Thus, starting with (22) of Sec. 2.9 as a conceptual base, let us write:

$$\text{"}\mathbf{A}\text{"} \quad \text{for} \quad \int_S \xi'(x)\; dA(x) \quad .$$

Then if a surface S has a constant uniform surface radiance N over the part $S(\xi)$ defined by a direction ξ (Sec. 2.9) then

$$\boxed{J = N\xi\cdot\mathbf{A}} \tag{32}$$

which is the exact intensity-counterpart to (15), and where we have written:

$$\text{"}J\text{"} \quad \text{for} \quad J(S,\xi) \quad .$$

When the shape of S is nearly planar, e.g., when $\xi'(x)$ varies within a solid angle 1/30 steradians over S, then we have, very nearly:

$$\mathbf{A} = \xi A$$

where A is the area of S. In this special case (32) yields the present counterpart to (16):

$$J = NA \quad . \tag{33}$$

If S is a point source with respect to a point x, a distance r from S, then the apparent area of another *point* source S' similar in shape to S is related to that of S by the equation:

$$A(S') = Cl^2$$

where C is a constant and l is some given linear dimension of S. In this way we return to the direct square law for J in the limit of large r (or small A). That is, using this estimate of A(S') in (33), we obtain the present counterpart to the inverse square estimate of irradiance; and as it stands by itself, the preceding equation is the dual to the relation $\Omega = A/r^2$ used to estimate solid angle subtenses of point sources.

Still one more form for J, i.e., J(S,ξ), is obtainable using the concept of vector area introduced in Sec. 2.9. Thus we have:

$$\boxed{J = \xi \cdot \mathbf{J}} \tag{34}$$

which is the dual to (17), and where in the present case we have written:

$$\text{"}\mathbf{J}\text{"} \quad \text{for} \quad N\int_S \xi'(x)\ dA(x) \quad .$$

As a corollary we have:

$$\boxed{\mathbf{J} = N\mathbf{A}} \quad , \tag{35}$$

which is the dual to (18).

Example 12: On the Possibility of Inverse nth Power Irradiance Laws

The cumulative evidence of the preceding examples, beginning with Example 4, shows the predominant role played in radiometry by the inverse square law for irradiance. The law is evident in various guises in formulas (8) and (9) for spherical surfaces, in the point-source criterion of Example 5, in the discussion of Example 6, in (21), (22), (29), and finally, its dual (the direct square law for radiant intensity) is evident in the discussions of Example 11. All of this evidence appears to lead inexorably to the generalization that the distance fall-off of irradiance produced by flux from all real surfaces of uniform radiance must *eventually* (i.e., for large enough distance r) assume the inverse-square behavior with r. This guess is essentially correct. However, the result of Example 5 shows that for intermediate distances r, the irradiance decrease with r need not be exactly an inverse square type of decrease. A question of some interest now arises as to necessary conditions that may govern the *rate* of such decrease. For example, can a surface S be found such that the irradiance H_r over some interval of distances r from the surface falls off exactly as C/r^3, (where C is a constant)? Or, perhaps S can be found such that H_r behaves exactly as C/r over some interval of r. In general, can a surface S be found such that over some interval of r, H_r is of the form C/r^n, where n is any number? In this example we devote some attention to these questions as they are of intrinsic interest and are of the kind which aid in forming a good intuition about the laws of geometrical radiometry.

Before attempting a systematic search for surfaces of the kind discussed above, let us consider some special cases which may point the way to an appropriate methodical approach. Figure 2.36 (a) depicts a sphere S of radius a and uniform radiance N. The results of Example 4 let us conclude that the irradiance H_r at distance r from the center of S is given by:

$$H_r = N(a^2/r^2)$$

Fig. 2.36 (c) depicts an infinite plane S of uniform radiance N. The results of Example 6 indicate that:

$$H_r = \pi N$$

for every r. Further, Fig. 2.36 (b) depicts an infinite conical surface S of half angle θ_o. Once again the results of Example 6 indicate that:

$$H_r = \pi N \sin^2 \theta_o$$

for every r, i.e., H_r in this case is independent of r but less than πN by the fixed factor $\sin^2 \theta_o$. Finally, Fig. 2.36 (d) depicts a general bounded closed smooth surface S of uniform radiance N which encloses a positive volume of space. Since S encloses a positive volume of space, there is a point x in S about which a sphere S_1 can be drawn such that S_1 lies wholly in S. Since S is bounded, there is a sphere S_2 with center x, such that S lies wholly in S_2. The spheres need not be tangent to S. It follows that H_r, the irradiance produced by S at any point outside S_2 a distance r from x, must obey the following equalities:

$$N(a_1^2/r^2) \le H_r \le N(a_2^2/r^2) \quad ,$$

where a_1 and a_2 are the radii of S_1 and S_2 respectively. This set of inequalities leads us to the following assertion: if S is *any bounded surface enclosing positive volume and with uniform surface radiance* N, *then associated with* S *is an irradiance function* H_r *whose graph is bounded by two inverse square curves.* Thus for sufficiently large r, H_r is arbitrarily closely described by an inverse square relation in r.

In view of the evidence just reviewed, the first main observation toward resolving the question before us may be made: *If* S *is a surface with uniform radiance* N *and the associated* H_r *is of the form* C/r^n *with* $n \neq 2$ *for every* $r \ge a$ *, where* a *is some nonnegative number, then* S *is either (a) not*

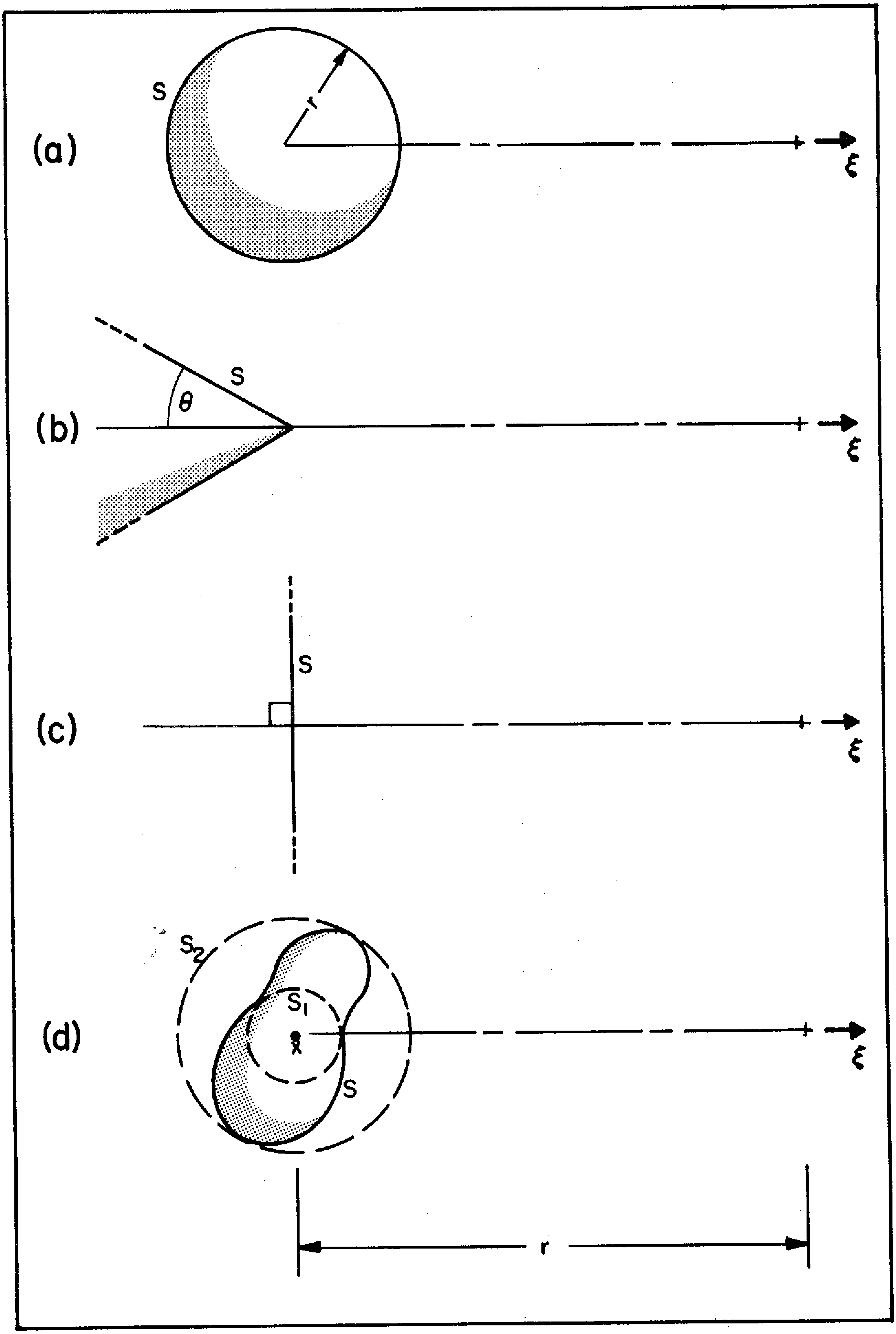

FIG. 2.36 Attempting to generalize the inverse square law for irradiance.

bounded or (b) does not enclose a positive volume.

Let us now attempt to find the actual shape of a surface S with the property that its associated irradiance varies precisely as C/r^n. Two conditions can be fixed at the outset in order to keep our initial search within reasonable confines. First we assume that $n \geq 0$. Secondly, we search only for surfaces S which, like those in (a)-(c) of Fig. 2.36, are surfaces of revolution about the direction ξ. It follows from (14) that the associated H_r is given by:

$$H_r = \pi N \sin^2 \theta(r) \quad ,$$

where $\theta(r)$ is the angle that the tangent to S from the point of observation, y, makes with the direction $-\xi$. (See Fig. 2.37.) Thus by specifying y and knowing S, we can in principle compute $\theta(r)$. Now use the hypothesized property of S to set up the following two equations:

$$\frac{C}{r^n} = H_r = \pi N \sin^2 \theta(r) \quad .$$

We can evaluate the constant C by observing that: if S is a smooth surface and $r = a$, then $\theta(r) = \pi/2$. This follows intuitively, e.g., from the observation that the surface has the appearance of an infinite plane for an observer at $r = a$. Hence for smooth surfaces:

$$\frac{C}{a^n} = \pi N$$

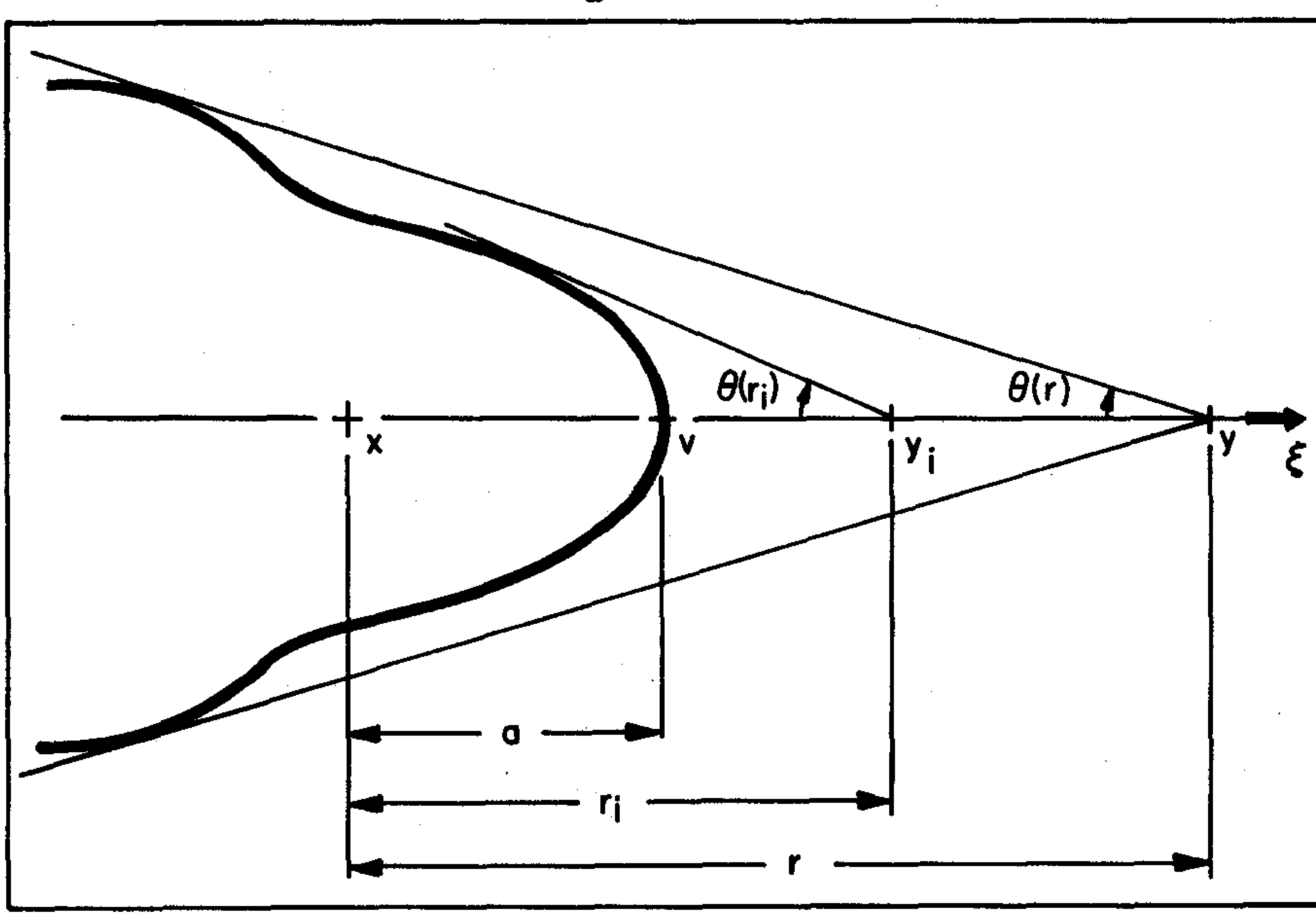

FIG. 2.37 Finding the shape of the luminous surface which has an inverse nth power irradiance law.

so that:

$$\sin\theta(r) = \left(\frac{a}{r}\right)^{n/2} \tag{36}$$

is the necessary connection between r and $\theta(r)$, where a is some fixed length associated with S. In the case of a spherical surface S and for the case of $n = 2$, a is simply the radius of S. In general if the surface S divides all of space into two separate parts (as, e.g., a plane or a paraboloid of revolution) then we agree that a is the distance from the *vertex* v of S to some *inside* point x, the *center* of S, where y is, by agreement, *outside* of S.

On the basis of relation (36) a graphical construction procedure can be evolved for the requisite surface S. First choose n, with $n > 0$ and choose a, with $a > 0$. Then select a set of distances $r_1, r_2, \ldots, r_k$, such that $a \le r_1 < r_2 < \ldots < r_k$. Equation (36) may now be used to compute the associated angles $\theta(r_1), \theta(r_2), \ldots, \theta(r_k)$. These angles are used in the following manner. At each point y_i which lies a distance r_i from the center x along the axis of revolution of S, draw two straight lines making angles $\pm\theta(r_i)$ with the direction $-\xi$, (see Fig. 2.37). If the r_i have been spaced sufficiently closely together, then one may visually detect the *envelope* of the lines just drawn, i.e., the *curve* which is tangent to each straight line of the family just constructed. This envelope is the cross section of the desired surface S; i.e., by spinning this envelope around the direction ξ, the requisite S is formed.

Some experimentation with the preceding construction procedure yields information about how the surfaces S vary in shape as a function of the power n. Thus let the parameter a be fixed. Then for every n in the range $0 < n < 2$, we find that the associated surface S(n) is unbounded. The closer n is to 0, the more of a conical shape is exhibited by S(n) about its vertex. The limiting curve S(0) is a degenerate infinite cone $\theta_o = \pi/2$, of the kind depicted in (b) of Fig. 2.36. The closer n is to 2, the more spherical is the shape of S(n) in the neighborhood of the vertex. The limiting curve S(2) is a sphere of radius a. The constructions of the surfaces S(n) for $n > 2$ at first present rather puzzling anomalies. By choosing the range of the values $r_1, \ldots, r_k$ sufficiently small and having the r_i closely packed together, one can construct the surfaces S(3), S(4), ..., within small regions around their vertices. In each case where $n > 2$, there is a critical distance r_o from the center x beyond which the envelope construction degenerates. The larger n is, the smaller is the corresponding critical distance r_o.

These graphical experiments in constructing the surface S(n) for which the inverse nth power law for irradiance holds, especially in the case of $n > 2$, indicate the need for a relatively precise analytical approach to the problem of determining S(n). We shall now briefly direct some attention to such an approach.

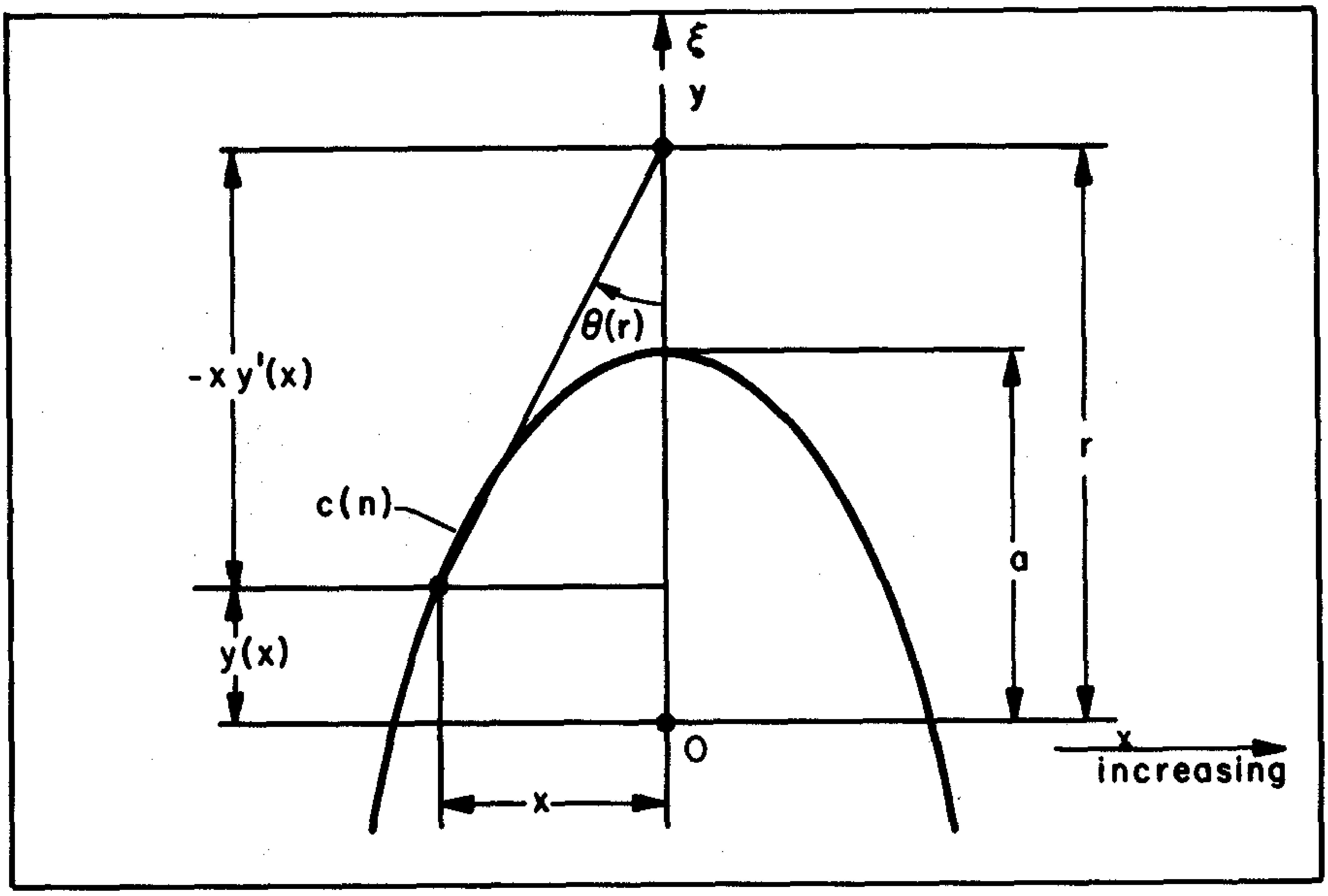

FIG. 2.38 Imbedding Fig. 2.37 in a cartesian frame.

Figure 2.38 shows an xy coordinate frame in which the cross section of surface S(n) is depicted by curve C(n). The irradiance meter is imagined to be at a point on the y-axis a distance r from the origin 0 of the frame. The unit inward normal ξ to the collecting surface of the meter is directed along the positive direction of the y-axis. The origin of the frame serves as the *center* of C(n), and the *vertex* of C(n) is a point on the y-axis a distance a from the origin. The curve C(n) is represented by some function y(·). Our primary goal is to obtain the differential equation for the function y(·) of the curve C(n). The starting point is equation (36) in the form:

$$\sin^2 \theta(r) = \left(\frac{a}{r}\right)^n$$

which, as we have seen, combines the inverse nth power requirement on the irradiance H_r with the general formula for H_r. We now systematically replace r, and θ(r), using the description of C(n) by y(·). Let us denote the derivative of y(x) with respect to x by "y'(x)". First of all r is clearly the algebraic sum of two terms: y(x) and -xy'(x), i.e., r = y(x)-xy'(x), as a glance at Fig. 2.38 would show. Secondly, it is also clear from the figure that:

$$\tan \theta(r) = x/xy'(x)$$

$$= 1/y'(x)$$

From this:

$$\sin\theta(r) = \frac{1}{\sqrt{1 + (y'(r))^2}} \quad .$$

Hence:

$$(y'(x))^2 + 1 = [y(x) - xy'(x)]^n \,(1/a^n) \quad . \qquad (37)$$

We have essentially reached our goal. Equation (37) is the differential equation for C(n). With an eye toward expediting the solution of (37) we shall rearrange it into the form:

$$\boxed{y(x) = xy'(x) + a\left[1 + (y'(x))^2\right]^{1/n}} \qquad (38)$$

Equation (38), as it stands, has the Gestalt of a Clairaut equation, an equation which is readily solvable in parametric form:

$$x(t) = -\frac{2at}{n}\left[1+t^2\right]^{\frac{1}{n}-1} \qquad (39)$$

$$y(t) = -\frac{2at^2}{n}\left[1+t^2\right]^{\frac{1}{n}-1} + a\left[1+t^2\right]^{\frac{1}{n}} \qquad (40)$$

This equation also has a singular solution of the form

$$\boxed{y(x) = t_o x + a\left[1+t_o{}^2\right]^{1/n}} \qquad (40a)$$

which represents straight lines of slope t_o. These singular solutions evidently can yield the degenerate conical case $\theta_o = \pi/2$ depicted in (c) of Fig. 2.36.

Setting n = 2 in (39) and (40), and eliminating the parameter t, we obtain:

$$x^2(t) + y^2(t) = a^2 \quad .$$

Hence C(2) is a circle with center at the origin (0,0), and of radius a, as expected. Setting n = 1 in (39) and (40), and eliminating the explicit appearance of the parameter t, we obtain

$$y(t) = -x^2(t)/4a + a$$

In this case S(1) is a paraboloid of revolution with focus at the origin (0,0), axis of symmetry along the y-axis, vertex at (0,a), and intercepting the x-axis at $x = \pm 2a$. The surface S(1) is typical, as far as size (unboundedness) and general orientation is concerned, of all S(n) with $0 < n < 2$.

For a curve-tracing analysis of a general S(n) we find from (39) and (40) that:

$$y(t)/x(t) = \left(1-\frac{n}{2}\right) t-\frac{n}{2t}$$

which is helpful in gauging the location of points on the curve. For example, if n = 2, then:

$$y(t)/x(t) = -\frac{1}{t}$$

for all t, $0 < t < \infty$. This shows that, as $t \to \infty$, a quadrant of the circle S(2) is traced out and $y(t)/x(t) \to 0$. This tracing is depicted in (b) of Fig. 2.39. Further, if n = 1, then:

$$y(t)/x(t) = \frac{t}{2} - \frac{1}{2t} \quad .$$

In this case, as t varies in the range $0 < t < \infty$, one branch of S(1) is traced out, and $y(t)/x(t) \to +\infty$. The tracing of this branch of the parabola S(1) is shown in (a) of Fig. 2.39. Finally, for the general case $n > 2$, we see that $\lim_{t\to\infty} y(t)/x(t) = \to -\infty$, indicating that y(t) becomes much larger than x(t) as $t \to \infty$. This in itself is not too informative, but when coupled with the observation that for large t,

$$x(t) \quad \text{behaves like} \quad -\frac{2a}{n}\, t^{\frac{2-n}{n}}$$

and

$$y(t) \quad \text{behaves like} \quad a\left(1-\frac{2}{n}\right) t^{\frac{2}{n}} \quad ,$$

we gain further insight into the behavior of the curves. From these observations we cull the following information: as $t \to \infty$,

$$\begin{aligned} \text{for} \quad 0 < n < 2:& \quad x(t) \to -\infty, \quad y(t) \to -\infty \\ n = 2:& \quad x(t) \to -a, \quad y(t) \to 0 \\ n > 2:& \quad x(t) \to 0, \quad y(t) \to +\infty \quad . \end{aligned}$$

The behavior for the case $n > 2$ continues to present puzzling features. Thus, when $n > 2$, $x(t) \to 0$ for large t, indicating that $|x(t)|$ attains a maximum for some t. Examining (39) for this possibility, we see that for C(n),

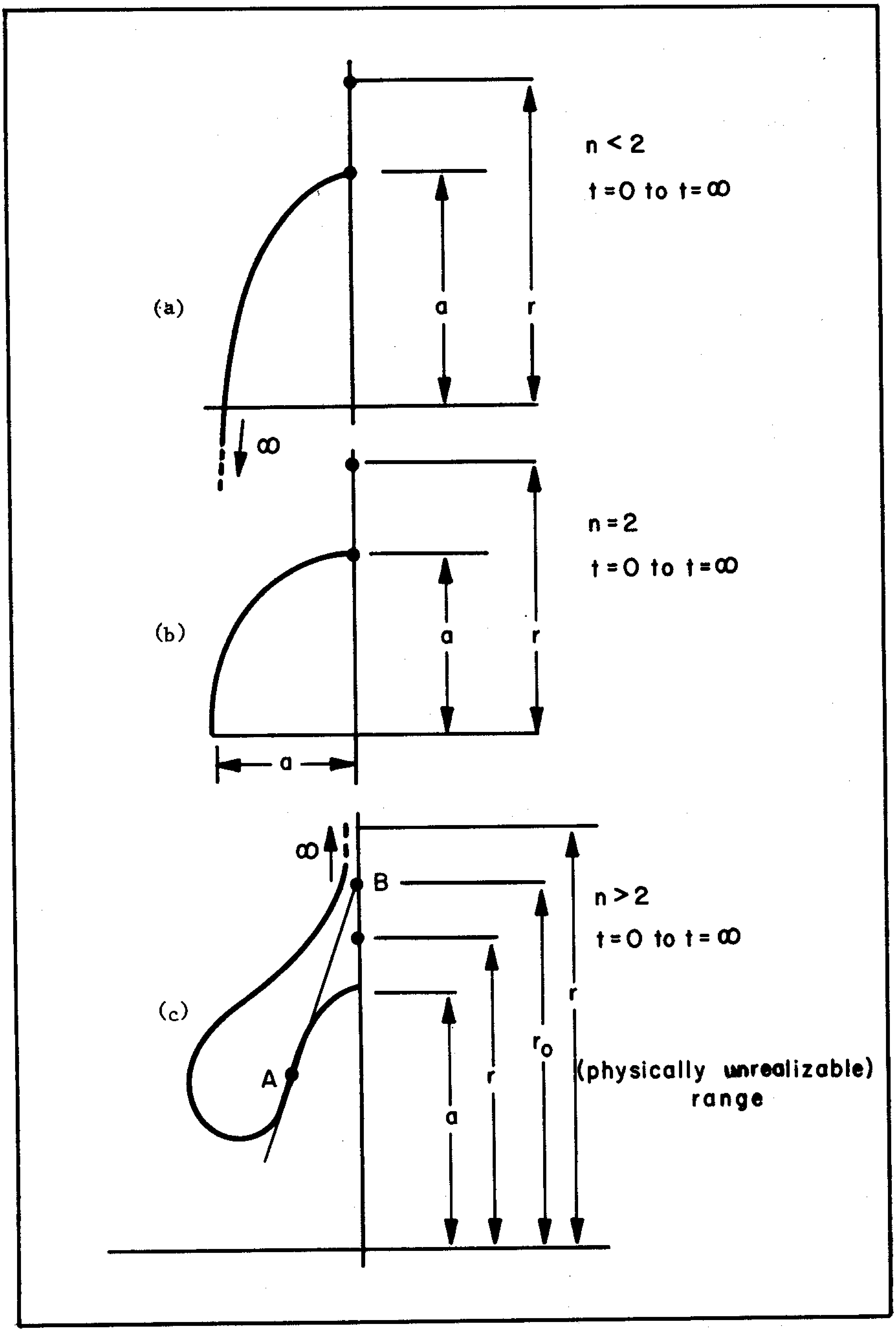

FIG. 2.39 Some cross sections of surfaces which produce irradiance fall-off like $1/r^n$ (see text).

$$\max |x(t)| \quad \text{occurs at} \quad t = \sqrt{\frac{n}{n-2}}$$

From this relation, it is at once clear that we obtain observable maxima for $n > 2$, and no observable (i.e., real) maxima for $0 < n < 2$. For example, we expect for $n = 3$, a maximum value of $|x(t)|$ at $t = \sqrt{3/(3-2)} = \sqrt{3}$. However, we expect no real maximum for $n = 1$, since $t = \sqrt{1/(1-2)} = \sqrt{-1}$. Finally, we inquire if, for the case $n > 2$, S(n) ever intersects the x-axis as do its lower-n counterparts. Thus in (40) we set $y(t) = 0$, and find that:

$$y(t) = 0 \quad \text{when} \quad t = \sqrt{\frac{n}{2-n}} \quad .$$

Hence if $n > 2$, then $y(t) \neq 0$ for every real t. This means that C(n) does not meet the x-axis for $n > 2$.

Summarizing the behavior of C(n) for $n > 2$: as t varies from 0 to $+\infty$, a branch of C(n) is traced out which is of the general form shown in (c) of Fig. 2.39. This figure explains the source of our difficulties in the geometric constructions based on equation (36). The construction is able to generate a branch of S(n) from the vertex to the first point of inflection at point A. Beyond A, the branch of C(n) and S(n) itself has no conventional physical interpretation. Some interesting unconventional interpretations can be made; however, we leave it to the reader's initiative to interpret the meaning of C(n) beyond point A. The tangent to C(n) at A meets the y-axis at a point B, which determines the critical range r_0 for which the physically realizable inverse nth power law holds. Part (c) of Fig. 2.39 is drawn from computed data for the case $n = 3$. The associated set of parametric coordinates are given in the table below, which was computed by Mrs. Judith Marshall.

Table 1

Computed Values for Part (c) of Figure 2.39
(The case $n = 3$)

t	x(t)	y(t)	t	x(t)	y(t)
0	-0	1	2.0	-0.456	0.797
0.1	-0.066	0.997	2.5	-0.444	0.826
0.2	-0.130	0.987	3.0	-0.430	0.802
0.3	-0.194	0.971	3.5	-0.417	0.906
0.4	-0.242	0.954	4.0	-0.403	0.958
0.5	-0.287	0.933	10.	-0.307	1.580
1.0	-0.420	0.840	20.	-0.246	2.43
1.5	-0.455	0.798	30.	-0.213	3.25
1.6	-0.458	0.793	40.	-0.194	3.91
1.7	-0.460	0.790	50.	-0.180	4.60

Several general concluding comments can now be made on the problem of the inverse nth power law for irradiance. First, there is the constantly recurring use of or reference to the integer 2 throughout the most general of the preceding discussions. Observe how critically 2 enters into the following tabular classification of our main results:

Table 2

n	The surface S(n) inducing $H_r = C/r^n$	Radius of curvature at vertex	Range of validity
0	Curve with half-angle $\theta_0 = \pi/2$ (Fig 2.36 (c))	0 (= 0/2a)	$r \geq a$
1	Paraboloid of revolution (Fig. 2.38)	1/2a	$r \geq a$
2	Sphere (Fig. 2.38)	1/a (= 2/2a)	$r \geq a$
3	3rd order luxoid (Fig. 2.39 (c))	3/2a	$r_0 \geq r \geq a$
n	nth order luxoid (Fig. 2.39 (c))	n/2a	$r_0 \geq r \geq a$

In this classification we encounter classical euclidean surfaces for all n, $0 < n \leq 2$, but a definite break occurs at $n = 2$, as has been repeatedly evident in the curve-tracing discussion above. All this is apparently closely related to the fact that we live in a three dimensional world, or at any rate, the radiometric laws above are represented most naturally in euclidean frames of dimension 3. The three dimensional geometric frame has been used implicitly throughout our discussions. We are thus led to conjecture that radiometry in a two dimensional world would have a ubiquitous inverse first power "irradiance" law and radiometry on a line would have its inverse zero power irradiance law. It is interesting to speculate on the theory and utility of k-dimensional geometric radiometry in which very likely the "irradiance" in such a geometry will obey an inverse k-1 power law, and to contemplate the potentially rich multiplicity of irradiances, scalar irradiances, and radiant intensities and their manifold interconnections latent in such geometries. Here the dualities brought out between irradiance and radiant intensity in the preceding examples are likely to attain their deepest and broadest meanings. These observations are commensurate with the conclusions, already brought out in the studies in Sec. 99 of Ref. [251], that radiometry and radiative transfer are meaningfully formulable in arbitrarily general spaces. These interesting matters are left to the reader for further consideration.

Example 13: Irradiance from Elliptical Radiance Distributions

We shall now illustrate the use of equation (17) of Sec. 2.5 by computing the irradiance distribution associated with an important type of theoretical radiance distribution, namely the elliptical radiance distribution. The elliptical radiance distribution arises in the study of light patterns at great depths in oceans, lakes, and other natural hydrosols. It is a convenient theoretical standard against which to compare the light patterns that actually subsist in nature. The physical basis for the elliptical radiance distribution will be considered in Chapter 4. For the present we shall be concerned only with its geometric properties. In particular we shall investigate its structure with respect to direction, to variation in eccentricity, and also compute its associated vector and scalar irradiances.

We begin by setting the geometric stage of the computation. Figure 2.40 (a) depicts a terrestrial coordinate frame (Sec. 2.4) and the plane of a radiant flux collector oriented as usual by its unit inward normal ξ. Let the associated direction angles of ξ be θ and ϕ. Thus if $\theta = \pi$ and $\phi = 0$, say, then $\xi = -\mathbf{k}$ and the collector receives flux from the upper hemisphere, i.e., receives flux flowing in the directions of Ξ_-. In general, when ξ is the unit inward normal to the collecting surface, the incident radiant flux is along the directions of the hemisphere $\Xi(\xi)$, as defined in Sec. 2.4.

Next we define an *elliptical radiance distribution* at x, of eccentricity ε, $0 \le \varepsilon \le 1$, and magnitude N, to be a radiance distribution of the form:

$$N(x,\xi') = N/(1+\varepsilon\xi'\cdot\mathbf{k}) \tag{41}$$

An alternate mode of representation of $N(x,\xi')$ is by means of polar and azimuthal angles. Thus (41) may be written

$$N(x,\theta',\phi') = N/(1+\varepsilon \cos \theta') \quad . \tag{42}$$

The upper diagrams of Fig. 2.41 show four plots of elliptical radiance distributions $N(x,\cdot)$ of eccentricity ε = .25,.50,.75, .95. For small values of ε near 0 the associated distribution is predominantly spherical. For values of ε near 1, the associated distribution is long and narrow. When $\theta = 0$, the flow is upward and smallest; when $\theta = \pi$, the flow is downward and greatest, thus simulating, at least qualitatively, the real flows in nature. (We are using surface rather than field radiance.) The "size" of the distribution is governed by N, being the radiance in the horizontal directions, i.e., ξ's with angles of the form $(\pi/2,\phi)$. The ratio of downward (zenith) to horizontal radiances in the present model is given by:

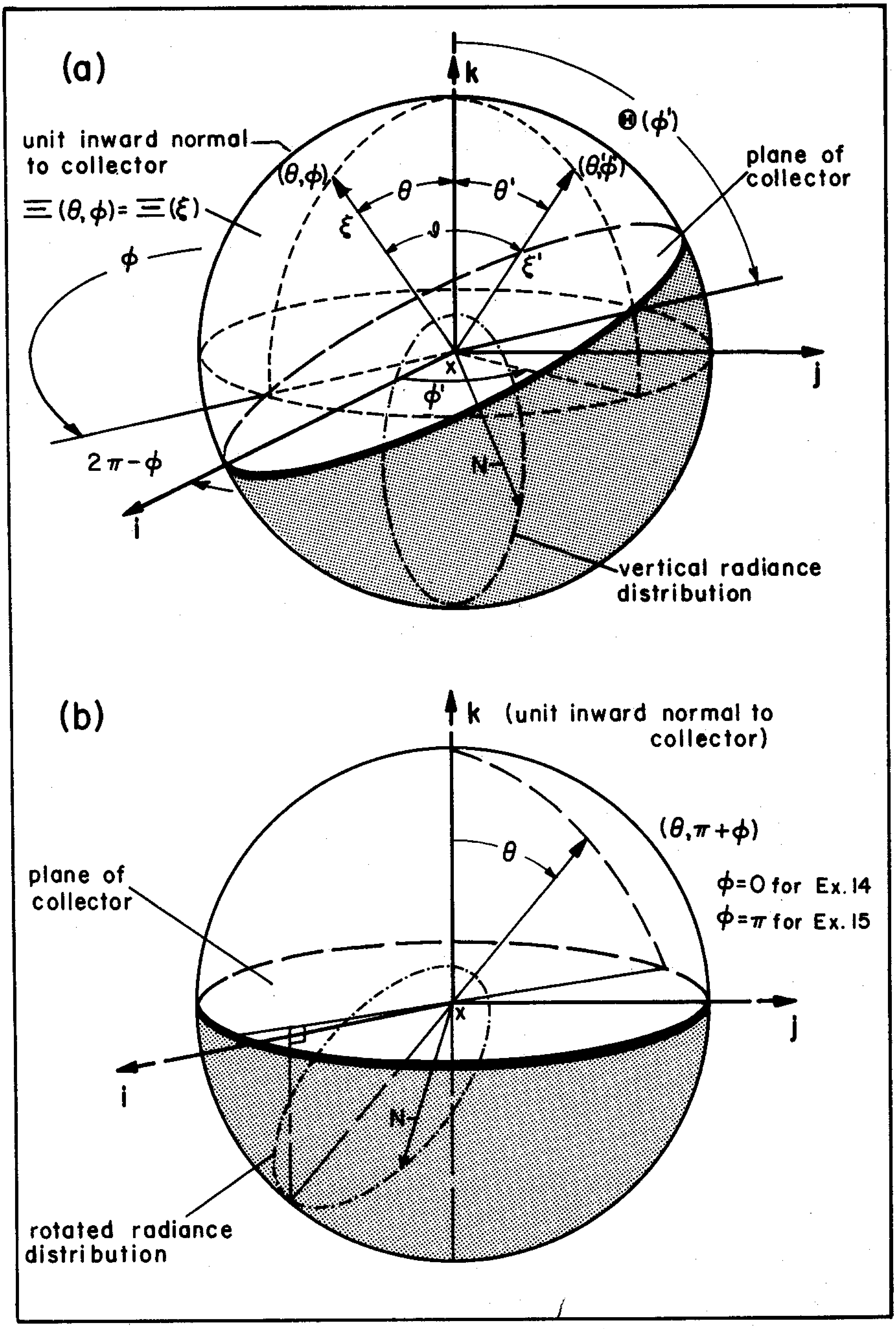

FIG. 2.40 Some calculation details for irradiance from elliptical radiance distributions.

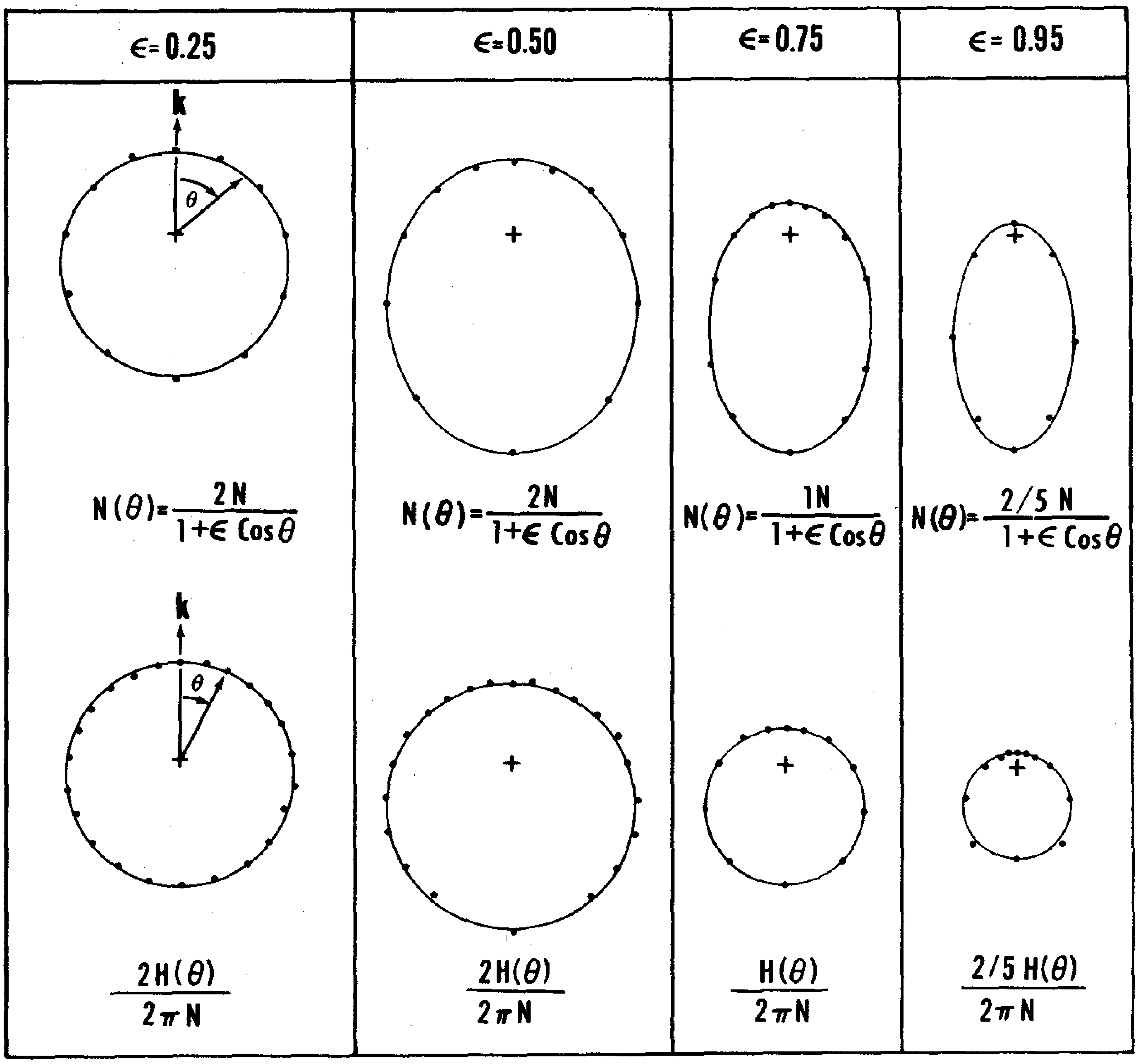

FIG. 2.41 Some representative irradiance distributions H(θ) associated with elliptical radiance distributions. The points are calculated from (48). The solid curves are circles, showing a possible simplification of (48) for engineering calculation purposes.

$$N(x,\pi,\phi')/N(x,\pi/2,\phi') = 1/(1-\varepsilon)$$

Thus, the nearer ε is to 1, the greater is this ratio. The ratio of zenith (downward) to nadir (upward) radiance is:

$$N(x,\pi,\phi')/N(x,0,\phi') = \frac{1+\varepsilon}{1-\varepsilon}$$

Turning now to the computation of $H(x,\theta,\phi)$, we start with (17) of Sec. 2.5:

$$H(x,\theta,\phi) = \int_{\Xi(\theta,\phi)} N(x,\theta',\phi') \cos z\ell \sin\theta' \, d\theta' \, d\phi' \qquad (43)$$

where $\Xi(\theta,\phi)$ is the hemisphere $\Xi(\xi)$, as shown in part (a) of Fig. 2.40. This range of integration in (43) may be given explicitly:

$$H(x,\theta,\phi) = \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\Theta(\phi')} N(x,\theta',\phi') \cos\vartheta \sin\theta' \, d\theta' \, d\phi' \qquad (44)$$

where $\Theta(\phi')$ is the angle between **k** and ξ', i.e., the variable direction of integration in the collector's plane which has azimuth ϕ'. $\Theta(\phi')$ may be determined from the functional relation:

$$\Theta(\phi') = \text{arc cot } \{-\tan\theta \cos(\phi-\phi')\} \quad . \qquad (45)$$

Thus, e.g., if $\theta = 0$, $\Theta(\phi') = \text{arc cot } \{0\} = \pi/2$ for every ϕ', $0 \leq \phi \leq 2\pi$. Eq. (43) can be put into a more convenient form by using the fact that, as far as the quantity $H(x,\theta,\phi)$ is concerned, it is immaterial whether, on the one hand, the collector is tipped in the frame of reference of the radiance distribution as in (a) of Fig. 2.40; or on the other hand, the collector is held still and the radiance distribution is appropriately tipped in the frame of the collector as in (b) of Fig. 2.40. The computational merit of the arrangement in (b) is superior to that in part (a) of Fig. 2.40, and we shall adopt it in the present illustration. The salient change resulting in this switch of points of view is in the functional form of $N(x,\theta',\phi')$. Indeed, a glance at (a) and (b) of Fig. 2.40 shows that the "vertical" radiance distribution in part (a) has undergone a rigid rotation to the "tipped" form in part (b), and rotated about the vertical axis so that **k** goes into the unit vector whose angles are $(\theta,\pi+\phi)$.

The details of the transformation of $N(x,\theta',\phi')$ into its new form $N'(x,\theta',\phi')$ constitute a simple exercise in analytic geometry and are left for the reader to formulate (recall (18) of Sec. 2.5). The resultant form is:

$$N'(x,\theta',\phi') = \frac{N}{1 + \varepsilon(-\sin\theta \sin\theta' \cos\phi' + \cos\theta \cos\theta')} \qquad (46)$$

in which we have set $\phi = 0$ since the desired irradiance $H(x,\theta,\phi)$ is independent of ϕ for the present radiance distribution, which is assumed symmetrical about the vertical. We can partially check (46) by letting $\theta' = \theta$ and $\phi' = \pi$. The resultant radiance is:

$$N'(x,\theta,\pi) = N/(1+\varepsilon)$$

which is precisely the magnitude of $N(x,0,\phi)$, as was to be expected. Using (46), it is clear from (b) of Fig. 2.40 that the desired irradiance $H(x,\theta,\phi)$ is given by:

$$H(x,\theta,\phi) = \int_{\phi=0}^{2\pi}\int_{\theta'=0}^{\pi/2} N'(x,\theta',\phi') \cos\theta' \sin\theta' \, d\theta' \, d\phi' \qquad (47)$$

The integration details of (47) are straightforward and are therefore omitted. Writing, *ad hoc*:

"H(θ)" for H(x,θ,φ)

we then evaluate (47) to obtain:

$$\boxed{H(\theta) = \frac{2\pi N}{\varepsilon^2}\left\{1+\varepsilon\cos\theta-\sqrt{1-\varepsilon^2\sin^2\theta} - (\cos\theta)\ln\left[\frac{(1+\varepsilon)(1+\cos\theta)}{\sqrt{1-\varepsilon^2\sin^2\theta}+\cos\theta}\right]\right\}} \qquad (48)$$

which is the desired functional form of the irradiance distribution under an elliptical radiance distribution of eccentricity ε and magnitude N. The reader should now show how to use (48), without the need of further computation, for the case where the axis of the elliptical distribution is originally tilted at θ_o from the vertical, and the angle between this axis and the unit inward normal to the collecting surface is ϑ_o.

Let us study some of the properties of H(θ). First of all, by setting $\theta = \pi$, we obtain the downward irradiance induced by the radiance distribution in (41) on the collecting surface:

$$H(\pi) = -\frac{2\pi N}{\varepsilon^2}\left[\varepsilon + \ln(1-\varepsilon)\right] \quad . \qquad (49)$$

The upward irradiance is obtained by setting $\theta = 0$:

$$H(0) = \frac{2\pi N}{\varepsilon^2}\left[\varepsilon - \ln(1+\varepsilon)\right] \quad . \qquad (50)$$

The net downward irradiance is therefore:

$$\overline{H}(\pi) = H(\pi) - H(0)$$

$$= \frac{2\pi N}{\varepsilon^2}\left[-2\varepsilon + \ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\right] \qquad (51)$$

which is positive for all ε, $0 < \varepsilon < 1$. $\overline{H}(\pi)$ is the magnitude of the vector irradiance **H**(x) associated with (41). The direction of **H**(x) is evidently -**k**. It may be verified directly from (48) that:

$$\overline{H}(\theta) = \overline{H}(\pi) \cos \psi \qquad (52)$$

where we have written

$$"\overline{H}(\theta)" \quad \text{for} \quad H(\theta) - H(\psi)$$

and

$$"\psi" \quad \text{for} \quad \pi - \theta$$

Equation (52) is a specific example of (16) of Sec. 2.8.

Observe next that $H(\theta) \to \pi N$ as $\varepsilon \to 0$. This is to be expected since the elliptical radiance distribution becomes spherical as $\varepsilon \to 0$, and as we now know, a spherical radiance distribution of magnitude N, induces an irradiance πN. This fact about the limit of $H(\theta)$ as $\varepsilon \to 0$ may be seen relatively readily for a special case by letting $\varepsilon \to 0$ in (49). Indeed, expanding $\ln(1-\varepsilon)$ in a power series in ε, we have, for very small ε, as an approximation:

$$H(\pi) = 2\pi N \left(\frac{1}{2} + \frac{\varepsilon}{3} + \frac{\varepsilon^2}{4} \right) \quad ,$$

$$H(0) = 2\pi N \left(\frac{1}{2} - \frac{\varepsilon}{3} + \frac{\varepsilon^2}{4} \right) \quad ,$$

whence:

$$\overline{H}(\pi) = 4\pi\varepsilon N/3$$

The scalar irradiance induced by the elliptical radiance distribution (41) is also of interest. Using the representation (41) in (3) of Sec. 2.7 we have:

$$\boxed{h(\varepsilon) = \frac{2\pi N}{\varepsilon} \ln \left(\frac{1+\varepsilon}{1-\varepsilon} \right)} \qquad (53)$$

where we have written, *ad hoc*:

$$"h(\varepsilon)" \quad \text{for} \quad h(x,t) \quad ,$$

Note that:

$$\lim_{\varepsilon \to 0} h(\varepsilon) = 4\pi N \quad .$$

For small ε, we have, very nearly:

$$h(\varepsilon) = 4\pi N \left[1 + \frac{\varepsilon^2}{3} \right] \quad .$$

Comparing (53) and (51) we see that:

$$\bar{H}(\pi) = \frac{1}{\varepsilon}[h(\varepsilon)-h(0)] \quad . \tag{54}$$

Thus, the magnitude of the vector irradiance associated with an elliptical radiance distribution of eccentricity ε and magnitude N is $1/\varepsilon$ times the difference between the scalar irradiance $h(\varepsilon)$ associated with the distribution and the scalar irradiance associated with a *uniform* radiance distribution of magnitude N.

Finally, we consider the hemispherical scalar irradiances associated with (41) (see (7) of Sec. 2.7), writing *ad hoc*:

$$"h(\varepsilon,-)" \quad \text{for} \quad h(x,-\mathbf{k},t)$$

$$"h(\varepsilon,+)" \quad \text{for} \quad h(x,\mathbf{k},t)$$

we have for an elliptical radiance distribution:

$$h(\varepsilon,-) = -\frac{2\pi N}{\varepsilon}\ln(1-\varepsilon) \tag{55}$$

$$h(\varepsilon,+) = \frac{2\pi N}{\varepsilon}\ln(1+\varepsilon) \tag{56}$$

Adding these two and comparing the sum with (53) we have:

$$h(\varepsilon,+) + h(\varepsilon,-) = h(\varepsilon) \quad ,$$

which illustrates (9) of Sec. 2.7. In the two-flow theory of light fields, to be studied in Chapters 8 and 9, the following ratios are of interest in that theory (see also (30) of Sec. 10.7):

$$h(\varepsilon,-)/H(\pi) = \varepsilon\ln(1-\varepsilon)/(\varepsilon+\ln(1-\varepsilon)) \tag{57}$$

$$h(\varepsilon,+)/H(0) = \varepsilon\ln(1+\varepsilon)/(\varepsilon-\ln(1+\varepsilon)) \tag{58}$$

These ratios constitute convenient measures of the "collimatedness" of the elliptical radiance distribution. Thus for the case $\varepsilon = 0$, the distribution is spherical and the very antithesis of collimatedness. In this case:

$$\lim_{\varepsilon\to 0} h(\varepsilon,-)/H(\pi) = 2 \quad .$$

A similar limit, namely 2, holds for $h(\varepsilon,+)/H(0)$. In the other extreme, i.e., when ε is near 1, the elliptical distribution of downward flux is relatively collimated. In this case:

$$\lim_{\varepsilon\to 1} h(\varepsilon,-)/H(\pi) = 1 \quad .$$

On the other hand, and somewhat unexpectedly, the upward radiance approaches a certain shape for which:

$$\lim_{\varepsilon \to 1} h(\varepsilon,+)/H(0) = \ln 2/(1-\ln 2) \quad .$$

Equation (48) is plotted for the four values of ε given in Fig. 2.41. The plots of $2H(\theta)/2\pi N$ are shown in the lower line of that figure.

Example 14: Irradiance from Polynomial Radiance Distributions

The present example is assigned the task of developing a generalization of the elliptical radiance distribution considered in Example 13, and of developing a formula for the associated irradiance distribution. The main lesson of this example is one not of importance to radiometry *per se*. Rather, it is designed to bring to the reader's attention the fact that many of the techniques of classical polynomial and power series theory are available to help obtain analytical representations of the radiance distributions measured in nature and on which, in turn, one can base practical methods of computing the associated irradiance distributions.

Suppose then, that an empirical radiance distribution $N(x,\cdot)$ can be represented for each θ and ϕ by the following polynomial in cos θ:

$$N(x,\theta,\phi) = \sum_{j=0}^{n} a_j P_j(\cos\theta) \tag{59}$$

where $P_j(\cos\theta)$ is the Legendre polynomial (in cos θ) of the first kind and of integral order j. The number n may be finite or infinite, as required. Here we are assuming that $N(x,\cdot)$ is a radiance distribution symmetrical about the vertical but of a form which has a quite general θ-dependence. As in the case of the elliptical radiance distribution in Example 13, we can use the fact that the irradiance produced by $N(x,\theta,\phi)$ in (59) depends only on the angle ϑ between its axis of symmetry and the inward normal to the collecting surface. Therefore we can use the results of this example, without further effort, to compute irradiance on any collecting surface when the angle ϑ between the axis of the symmetrical distribution and the unit inward normal to the collecting surface is known. Hence the assumption of the form (59) constitutes no loss of generality in this sense.

We first observe that the coefficients a_j are readily determinable from the tabulated data $N(x,\theta,\phi)$. Indeed, using the orthogonality properties of $P_j(\cos\theta)$, we have from (59) (and cf. (3) of Sec. 6.3):

$$a_k = \frac{(2k+1)}{2}\,\frac{1}{2\pi}\int_{\Xi} N(x,\theta',\phi')P_k(\cos\theta')\sin\theta'\,d\theta'\,d\phi'$$

i.e.,

$$a_k = \frac{(2k+1)}{2} \int_0^{\pi} N(x,\theta',\phi') P_k(\cos\theta') \sin\theta' \, d\theta' . \qquad (60)$$

Hence, if N(x,•) is known, performing the operation on N(x,•) as defined in (60), yields a_k for every k = 1,...,n. By writing:

"μ" for $\cos\theta'$

and

"$N(\mu)$" for $N(x,\theta',\phi')$,

Equation (60) takes the relatively compact form:

$$a_k = \frac{(2k+1)}{2} \int_{-1}^{+1} N(\mu) P_k(\mu) \, d\mu \qquad (61)$$

Equation (61) can be evaluated by any of several avaliable numerical quadratures, given the radiance data $N(\mu)$. Having obtained the a_k in this way, we now can go on to consider the computation details of H(x,•) associated with N(x,•). With (17) of Sec. 2.5 as a starting point and using (59) we can write:

$$H(x,\theta,\phi) = \iint_{\Xi(\theta,\phi)} \left(\sum_{j=0}^{n} a_j P_j (\cos\theta') \right) \cos \vartheta \, \sin\theta' \, d\theta' \, d\phi' \qquad (62)$$

where ϑ is the angle between the directions (θ',ϕ') and (θ,ϕ). The preceding integral can be transformed into an alternate form by adopting the technique used in Example 13. (See Fig. 12.40 (b).) Thus Equation (62) can be written:

$$H(x,\theta,\phi) = \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi/2} \left(\sum_{j=0}^{n} a_j P_j (\cos \vartheta') \right) \cos\theta' \sin\theta' \, d\theta' \, d\phi' \qquad (63)$$

and which may be viewed as the present counterpart of (47), wherein:

$$\cos\vartheta' = \sin\theta \sin\theta' \cos\phi' + \cos\theta\cos\theta' \quad .$$

Equation (63) now stands in a form which is readily evaluable. Toward this end, observe that the sum of Legendre polynomial terms can be written in the form:

$$\sum_{j=0}^{n} a_j P_j(\cos \vartheta') = \sum_{j=0}^{n} b_j \cos^j \vartheta' \qquad (64)$$

where the numbers b_j, $j = 0, \ldots, n$, are obtained by expanding each $P_j(\cos \vartheta')$ in the left side of (64) in powers of $\cos \vartheta'$ and collecting together like powers of $\cos \vartheta'$. Each "b_j" therefore denotes the coefficient of $\cos^j \vartheta'$ so obtained. Hence from knowledge of the a_j, the numbers b_j are readily computed. Tables of Legendre Polynomials are available for the determinations of the b_j.

Next write, *ad hoc*:

"x" for $\sin \theta \sin \theta'$

"y" for $\cos \theta \cos \theta'$.

Then for every j, 1,..., n

$$(\cos \vartheta')^j = (x \cos \phi' + y)^j$$

$$= \sum_{i=0}^{j} {}^jC_i \, x^{j-i} y^i \cos^{j-i} \phi' \quad ,$$

where "${}^jC_i$" as usual denotes the combinatorial coefficient of the ith term in the jth power of a binomial. Using this expansion in (63), with the help of (64), we can rearrange (63) to read:

$$H(x,\theta,\phi) =$$

$$= \sum_{j=0}^{n} b_j \int_{\phi'=0}^{2\pi} \int_{\theta'=0}^{\pi/2} \sum_{i=0}^{j} {}^jC_i x^{j-i} y^i \cos^{j-i}\phi' \cos \theta' \sin \theta' \, d\theta' \, d\phi'$$

$$= \sum_{j=0}^{n} b_j \sum_{i=0}^{j} {}^iC_j \int_{\theta'=0}^{\pi/2} x^{j-i} y^i \cos \theta' \sin \theta' \left[\int_{\phi'=0}^{2\pi} \cos^{j-i}\phi' \, d\phi' \right] d\theta' \; . \qquad (65)$$

Observe that:

$$\int_{\phi'=0}^{2\pi} \cos^{j-i}\phi' \, d\phi' = 2\sqrt{\pi}\, \frac{\Gamma\left(\frac{j-i+1}{2}\right)}{\Gamma\left(\frac{j-i+2}{2}\right)} \Delta_{ij} \quad ,$$

where

$$\Delta_{ij} = \begin{cases} 0 \text{ if } j-i \text{ is odd} \\ 1 \text{ if } j-i \text{ is even} \end{cases} \quad .$$

Let "I_{ij}" momentarily denote the value of this integral of $\cos^{j-i}\phi'$. Then (65) becomes:

$$H(x,\theta,\phi) =$$

$$= \sum_{j=0}^{n} b_j \sum_{i=0}^{j} {}^{i}C_j I_{ij} \int_0^{\pi/2} \sin^{j-i}\theta \sin^{j-i+1}\theta' \cos^{i}\theta \cos^{i+1}\theta' \, d\theta'$$

$$= \sum_{j=0}^{n} b_j \sum_{i=0}^{j} {}^{i}C_j I_{ij} \sin^{j-i}\theta \cos^{i}\theta \int_0^{\pi/2} \sin^{j-i+1}\theta' \cos^{i+1}\theta' \, d\theta' \qquad (66)$$

Observe that:

$$\int_0^{\pi/2} \sin^{j-i+1}\theta' \cos^{i+1}\theta' \, d\theta' = \frac{1}{2} \frac{\Gamma\left(\frac{j-i+2}{2}\right) \Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(\frac{j+4}{2}\right)} \quad ,$$

where "$\Gamma(z)$" once again denotes the value of the gamma function Γ at z. Let us write "J_{ij}" for the product of I_{ij} and the latter integral. Hence:

$$J_{ij} = \sqrt{\pi} \cdot \frac{\Gamma\left(\frac{j-i+1}{2}\right) \Gamma\left(\frac{i+2}{2}\right)}{\Gamma\left(\frac{j+4}{2}\right)} \Delta_{ij} \quad .$$

If we write:

$$\text{"}C_j(\theta)\text{"} \quad \text{for} \quad \sum_{i=0}^{j} {}^{i}C_j \, J_{ij} \sin^{j-i}\theta \cos^{i}\theta$$

and

$$\text{"}H(\theta)\text{"} \quad \text{for} \quad H(x,\theta,\phi) \quad ,$$

then (66) becomes:

$$H(\theta) = \sum_{j=0}^{n} b_j C_j(\theta) \quad . \tag{67}$$

This is the desired formula for the irradiance distribution associated with the radiance distribution in (59). Observe that the numbers ${}^iC_j\, J_{ij}$ are evaluable once for all, so that to use (67) with particular radiance distribution $N(x,\cdot)$ it is required only to evaluate the a_k by means of (61) to the desired degree of accuracy, and to obtain the b_k using (64). It is left as an exercise for the reader to evaluate ${}^iC_jJ_{ij}$ and to obtain explicit formulas for the b_k in terms of the a_k for the first few values of $k = 1, 2, \ldots, n$, and to make a list of them so that the use of (67) is reduced to simply finding the a_k for each new application.

The reader may verify that the scalar irradiance h associated with a radiance distribution $N(x,\cdot)$ of the form in (59) is given by:

$$h = 2\pi \sum_{j=0}^{n} b_j \left(1 - (-1)^{j+1}\right)\Big/ j + 1 \quad . \tag{68}$$

We close this example by observing two special cases of the polynomial distributions. First we note that the set of polynomial radiance distributions discussed above contains as a special case the elliptical radiance distribution of Example 13. To see this, in (64) choose the a_j subject to the condition that:

$$b_j = (-\varepsilon)^j N$$

for every integer $j \geq 0$, where $0 \leq \varepsilon < 1$, and where N is a nonnegative number. Then:

$$N(x,\theta,\phi) = N \sum_{j=0}^{\infty} (-\varepsilon \cos\theta)^i$$

$$= N/(1 + \varepsilon \cos\theta) \quad ,$$

which is the form of (42). Secondly, an interesting radiance distribution associated with heavily overcast skies is a special case of (59). This is the "Moon-Spencer Sky" representation of radiance distributions and takes the following form. For every θ, $\pi/2 \le \theta \le \pi$:

$$N(x,\theta,\phi) = N(x,\pi/2,\phi)(1-2 \cos \theta) \ .$$

The empirical details of determining this distribution may be found in Ref. [186].

Example 15: On the Formal Equivalence of Radiance and Irradiance Distributions

The present sequence of illustrations of the radiometric concepts is concluded with a discussion of the theoretical possibility of reversing the usual path between radiance and irradiance distributions. *We shall show that, given an irradiance distribution* $H(x,\cdot)$ *it is possible, in principle, to deduce the associated radiance distribution* $N(x,\cdot)$. This course of action is the reverse of that taken in the various Examples above, and in the discussion of Sec. 2.5. The theoretical and experimental significance of this reversal of the usual computation procedure was touched on briefly in Sec. 2.5 wherein also a practical scheme for obtaining $N(x,\cdot)$ from $H(x,\cdot)$ was suggested. The main purpose of this example is to show that this reverse path is possible not only on a numerical level, but also on an exact function-theoretic level. This is tantamount to showing that (8) of Sec. 2.5, when viewed as an integral equation with unknown $N(x,\cdot)$, has a unique solution in terms of the irradiance distribution $H(x,\cdot)$. We shall discuss this point of view in detail, as it affords an opportunity to illustrate how the use of advanced vector space concepts can facilitate the solutions of certain radiometric problems.

We can phrase the present problem in precise terms as follows: *Given:* the irradiance distribution $H(x,\cdot)$ at a point x in an optical medium. *Required:* the associated radiance distribution $N(x,\cdot)$. Now, for every direction ξ and point x we have, by (8) of Sec. 2.5:

$$H(x,\xi) = \int_{\Xi(\xi)} N(x,\xi')\xi\cdot\xi' \, d\Omega(\xi') \ . \qquad (69)$$

Let us write:

$$\text{"}C(x)\text{"} \quad \text{for} \quad \frac{1}{2\pi}\int_{\Xi(\xi)} [\ ]\xi\cdot\xi' \, d\Omega(\xi') \ .$$

Errata

We call $C(x)$ the *cosine operator*, for obvious reasons. Then (69) can be written as:

$$\frac{H(x,\cdot)}{2\pi} = N(x,\cdot)\; C(x) \quad , \tag{70}$$

where "$N(x,\cdot)C(x)$" means: "operate on the radiance distribution $N(x,\cdot)$ by substituting $N(x,\cdot)$ into the square bracket of the integral operator $C(x)$." For example, if at point x, $N(x,\cdot)$ is a uniform radiance distribution with magnitude N, then for every ξ:

$$\begin{aligned} N(x,\cdot)C(x) &= \frac{1}{2\pi}\int_{\Xi(\xi)} N\xi\cdot\xi'\; d\Omega(\xi') \\ &= \frac{N}{2\pi}\int_{\Xi(\xi)} \xi\cdot\xi'\; d\Omega(\xi') \\ &= \frac{N}{2} \quad . \end{aligned} \tag{71}$$

Up to this point in the present example our deliberations have been relatively elementary and were without exception motivated by physical intuition. But now when we ask: "Can we determine $N(x,\cdot)$ knowing $H(x,\cdot)$ and $C(x)$?", we leave the domain of physical intuition and are asking a purely mathematical question. Perhaps even in this general radiometric setting some reader may see a physical reason for an affirmative answer to the query. For instance, by starting with the simpler setting in (21) of Sec. 2.5 and by letting the number of equations of the type considered there increase indefinitely and by being assured at each step along such a course that $N(x,\cdot)$ is determinable from $H(x,\cdot)$, perhaps by following such a line of thought one can be convinced of the general determinability of $N(x,\cdot)$ from $H(x,\cdot)$. Indeed, it is most desirable that some assurance be generated in such a manner. But at the present moment we are confronted by a mathematical question and in view of its important relevance to applications we prefer to settle it using now the rules of mathematics.

To begin to answer the preceding question we generate a mathematical setting in which the question suggests some further action toward the present goal. The appropriate setting is obtained by considering the set $\mathcal{N}(x)$ of all radiance distributions at point x. Next we observe the interesting fact that the sum of any two such radiance distributions is again in the set $\mathcal{N}(x)$. For example, if $N(x,\cdot)$ and $N'(x,\cdot)$ are in $\mathcal{N}(x)$, then the function:

$$N(x,\cdot) + N'(x,\cdot)$$

is in $\mathcal{N}(x)$ and by definition assigns to each direction ξ at x the sum $N(x,\xi) + N'(x,\xi)$ of the two radiances $N(x,\xi)$ and

$N'(x,\xi)$. The sum of two radiances is again a radiance*. That is, if each of $N(x,\cdot)$ and $N'(x,\cdot)$ is physically admissible then a lighting arrangement could be conceived so that $N(x,\cdot) + N'(x,\cdot)$ was realizable. Observe, however, that we need not introduce the preceding observation as an additional justification for the assertion about $\mathcal{N}(x)$ containing the sum of $N(x,\cdot)$ and $N'(x,\cdot)$ whenever it contains each. That assertion is simply the result of the present definition of $\mathcal{N}(x)$, and of the general definition of radiance. Next we observe that if $N(x,\cdot)$ is in $\mathcal{N}(x)$, so is $cN(x,\cdot)$, where c is a non negative real number. The physical plausibility of this assertion is obvious. As a result of these observations we see that $\mathcal{N}(x)$ is part of the vector space $V(x)$ of *all* functions $f(x,\cdot)$ at x over the domain Ξ, and with dimensions of radiance. In fact $\mathcal{N}(x)$ forms what is referred to in mathematical terminology as a *non negative cone* of $V(x)$ which, by definition, is closed under formation of sums, and multiplication by non negative real numbers. (Take all the unit vectors in a subset Ξ_0 of Ξ and form the set of all products $c\xi$, with $c \geq 0$, and ξ in Ξ_0. Describe the geometrical appearance of this set.)

Now what is the purpose of all this collecting together of huge families $\mathcal{N}(x)$ of radiance distributions? Simply this: by collecting together the members of $\mathcal{N}(x)$ in the fashion just exhibited, the operator $C(x)$ defined above takes on the crucial role of a *linear transformation* from $V(x)$ to $V(x)$ and in this setting our original question, "Can we obtain $N(x,\cdot)$ from $H(x,\cdot)$?," takes a deeper and mathematically meaningful cast.

Before rephrasing the question in the vector space terminology it may be well to include, simply for completeness, a comment about what it means for $C(x)$ to be a linear transformation. It means this: If $f(x,\cdot)$ and $g(x,\cdot)$ are any two functions in $V(x)$ and a and b are any two real numbers, then

$$[af(x,\cdot) + bg(x,\cdot)]\, C(x) = a[f(x,\cdot)C(x)] + b[g(x,\cdot)C(x)]$$

where $f(x,\cdot)C(x)$ and $g(x,\cdot)C(x)$ are again members of $V(x)$, being images of $f(x,\cdot)$ and $g(x,\cdot)$ under $C(x)$. Observe that $C(x)$ acting on a function with dimensions of radiance yields up once again a function with dimensions of radiance.

We can now ask our question about $H(x,\cdot)$ and $N(x,\cdot)$ as follows: "Is the linear transformation $C(x)$ from $V(x)$ to $V(x)$ a one-to-one transformation when restricted to the part $\mathcal{N}(x)$ of $V(x)$?" By $C(x)$ being "one-to-one", is meant that $C(x)$ sends exactly one radiance function into each modified irradiance function of the form: $H(x,\cdot)/2\pi$ defined in (70). Then, having given an irradiance distribution $H(x,\cdot)$, we are thereby assured that there is one and only one radiance function that it comes from (i.e., is associated with). Hence, whenever

*This may be taken as intuitively obvious at this point of the exposition. Formally, it is a consequence of the interaction principle of Chapter 3.

C(x) is one-to-one we are encouraged to find that unique radiance distribution N(x,•) which, by (70), yields up H(x,•).

It turns out that the linear transformation C(x) is indeed one-to-one when restricted to $\mathcal{N}(x)$ and we find its inverse $C^{-1}(x)$ as follows. We begin by observing that:

$$\int_{\Xi} \left(H(x,\xi)/2\pi\right) \, d\Omega(\xi) = \int_{\Xi} \left(N(x,\cdot)C(x)\right) \, d\Omega(\xi)$$

$$= \frac{1}{2\pi} \int_{\Xi} \left[\int_{\Xi(\xi)} N(x,\xi')\xi'\cdot\xi \, d\Omega(\xi') \right] d\Omega(\xi)$$

$$= \frac{1}{2\pi} \int_{\Xi} N(x,\xi') \left[\int_{\Xi(\xi')} \xi'\cdot\xi \, d\Omega(\xi) \right] d\Omega(\xi')$$

$$= \frac{1}{2} \int_{\Xi} N(x,\xi') \, d\Omega(\xi')$$

$$= \frac{1}{2} h(x) \quad .$$

For our present purpose, let us write:

$$"|f(x,\cdot)|" \quad \text{for} \quad \int_{\Xi} |f(x,\xi)| \, d\Omega(\xi) \quad ,$$

where f(x,•) is in V(x). In particular, if f(x,•) is in $\mathcal{N}(x)$, then |f(x,•)| is the scalar irradiance associated with a general radiance distribution f(x,•) at point x. The main thing the preceding calculation has shown is that:

$$|N(x,\cdot)C(x)| = \frac{1}{2} |N(x,\cdot)| \quad . \qquad (72)$$

The significance of this equality for the present discussion is crucial, and we pause to make this significance clear. The significance becomes clear when it is pointed out that the scalar irradiance h(x) acts as the "length" of the vector N(x,•) in V(x). Indeed, the bars around "f(x,•)" in the definition above are there to point up the easily verified fact that |f(x,•)| is analogous to the absolute value of a number or vector; and it may be shown that all the essential properties of length that we carry with us from euclidean space hold also for the numbers |N(x,•)|. We call |N(x,•)| (i.e., h(x) in this case) the *radiometric norm* of N(x,•), to point up this similarity between N(x,•) and the usual concept of norm or length of a vector.

The significance of (72) can now be stated: the linear transformation C(x) has the property that it maps a radiance function into one which has exactly half the norm (i.e., "length") of the original radiance distribution. In short, C(x) has the *norm contracting* property with contracting factor 1/2. The mathematical consequence of this fact is immediate: we now can use the well known norm-contracting theorem of vector space theory, as stated, e.g., in Ref. [251] for the radiative transfer context, to assert that the inverse $C^{-1}(x)$ of C(x) exists, and that, indeed:

$$C^{-1}(x) = \sum_{j=0}^{\infty} \left(I - C(x)\right)^j \quad , \tag{73}$$

where I is the identity transformation, i.e.,

$$f(x,\cdot)I = f(x,\cdot)$$

for every f(x,•) in V(x). This identity transformation can be written as an integral operator. Thus if we write:

$$\text{"I"} \quad \text{for} \quad \int_{\Xi(\xi)} [\ ]\delta(\xi-\xi')\, d\Omega(\xi')$$

it may be verified that I is the identity operator on V(x) whenever δ is the Dirac delta function (on the space with Ω as measure). Then if we go on to write:

$$\text{"D(x)"} \quad \text{for} \quad \frac{1}{2\pi}\int_{\Xi(\xi)} [\ ]\left(2\pi\delta(\xi-\xi') - \xi\cdot\xi'\right) d\Omega(\xi')$$

we have the equivalent form for (73), where

$$D(x) = \left(I-C(x)\right)$$

and

$$\boxed{C^{-1}(x) = \sum_{j=0}^{\infty} D^j(x)} \tag{74}$$

and

$$\boxed{N(x,\cdot) = \frac{H(x,\cdot)}{2\pi}\, C^{-1}(x)} \tag{75}$$

where, in turn, we have defined $D^j(x)$ recursively by writing:

$$\text{"}D^{o}(x)\text{"} \quad \text{for} \quad I \quad ,$$

and for every positive integer j:

$$\text{"}D^{j}(x)\text{"} \quad \text{for} \quad D^{j-1}(x)D(x) \quad ,$$

and where, finally, "$D^{j-1}(x)D(x)$" denotes the customary integral operation on $D^{j-1}(x)$ as an integrand in D(x). Equation (74) yields the requisite inverse of C(x), and the solution of our present problem is summarized in (75).

Observe that to use the norm-contracting theorem in Ref. [251] we actually need the fact that I-C(x) is a norm-contracting operator. The reader may now easily verify that:

$$|N(x,\cdot)(I-C(x))| = \frac{1}{2}|N(x,\cdot)| \quad ,$$

so that I-C(x) is, indeed, norm-contracting with contracting factor 1/2, and the norm-contracting theorem statement yields (73) and hence (74).

Aside from the relatively advanced mathematical objects involved in (74) (namely, Dirac delta functions, and two-dimensional iterated integration) the algebraic essence of (74) is identical to that of the formula used by every high school student summing a geometric series of the form:

$$(1-x) + (1-x)^2 + (1-x)^3 + \ldots$$

whose value is clearly 1/x and where x is any number with absolute value less than 1. Now, instead of squaring (1-x), i.e., multiplying (1-x) by itself, we are required to operate with I-C(x) on itself. Thus, e.g.,

$$N(x,\cdot)(I-C(x))^2 =$$

$$= \frac{1}{2\pi}\int_{\Xi(\xi'')}\left[\frac{1}{2\pi}\int_{\Xi(\xi')} N(x,\xi)\left(2\pi\delta(\xi'-\xi)-\xi'\cdot\xi\right)d\Omega(\xi)\right]\left(2\pi\delta(\xi''-\xi')-\xi''\cdot\xi'\right)\,d\Omega(\xi')$$

To obtain the form for $(I-C(x))^2$ itself, simply remove "$N(x,\cdot)$" and "$N(x,\xi)$" where they occur in the preceding equality. Thus, as in the case of computing the "fraction" 1/x by using solely multiplication, addition and subtraction repeatedly, so too can we compute "1/C(x)", i.e., $C^{-1}(x)$ using solely integration, multiplication, addition and subtraction, repeatedly. The norm-contracting theorem states that by continuing sufficiently far, $C^{-1}(x)$ can be arbitrarily closely approximated.

The error engendered by stopping the computation of $C^{-1}(x)$ in (74) at the kth term may be readily computed. Thus, write,

$$"N^{(k)}(x,\cdot)" \quad \text{for} \quad \frac{1}{2\pi}\sum_{j=0}^{k} H(x,\cdot)D^j(x)$$

so that $N^{(k)}(x,\cdot)$ serves as the kth order approximation to the desired distribution $N(x,\cdot)$. Then the radiometric norm of the difference between $N(x,\cdot)$ and $N^{(k)}(x,\cdot)$ is:

$$|N(x,\cdot)-N^{(k)}(x,\cdot)| = \left|\sum_{j=k+1}^{\infty} \frac{H(x,\cdot)}{2\pi} D^j(x)\right|$$

$$= \sum_{j=k+1}^{\infty} \left|\frac{H(x,\cdot)}{2\pi} D^j(x)\right|$$

$$= \sum_{j=k+1}^{\infty} \frac{1}{2^j} \left|\frac{H(x,\cdot)}{2\pi}\right|$$

$$= \frac{1}{2^k} \left|\frac{H(x,\cdot)}{2\pi}\right|$$

The reader may use as specific cases in (75) the formulas for $H(x,\theta,\phi)$ in (48) and (67) of Examples 13 and 14 in order to recover the associated radiance distributions of those examples. These will afford non trivial examples of (75).

We close this discussion with some general assertions to which one is naturally led after contemplating the lesson of the present example. The assertions concern the possibility of still further equivalences between radiance and other radiometric concepts which are natural generalizations of the concept of irradiance. Recall that irradiance was defined empirically by specifying a small plane surface S onto each point of which radiant flux could be incident within the set $\Xi(\xi)$, where ξ is the unit inward normal to S. If now we replace $\Xi(\xi)$ by *any* fixed conical set $D(\xi)$ of directions of positive solid angle content specified in some way with respect to ξ, then the generalized irradiance distribution $H(x,D(\cdot))$, as defined in (4) of Sec. 2.4, is equivalent to $N(x,\cdot)$ in the same sense that $H(x,\cdot)$ and $N(x,\cdot)$ were shown to be equivalent in the present example. This is the first assertion to which we are led. Its proof is left to the reader.

The lesson of the present example can be carried still further than the point reached in the preceding paragraph. Let "$S(x,\xi)$" denote a collecting surface S which is a convex surface of revolution of fixed shape and size whose location and orientation in an optical medium X are uniquely specified

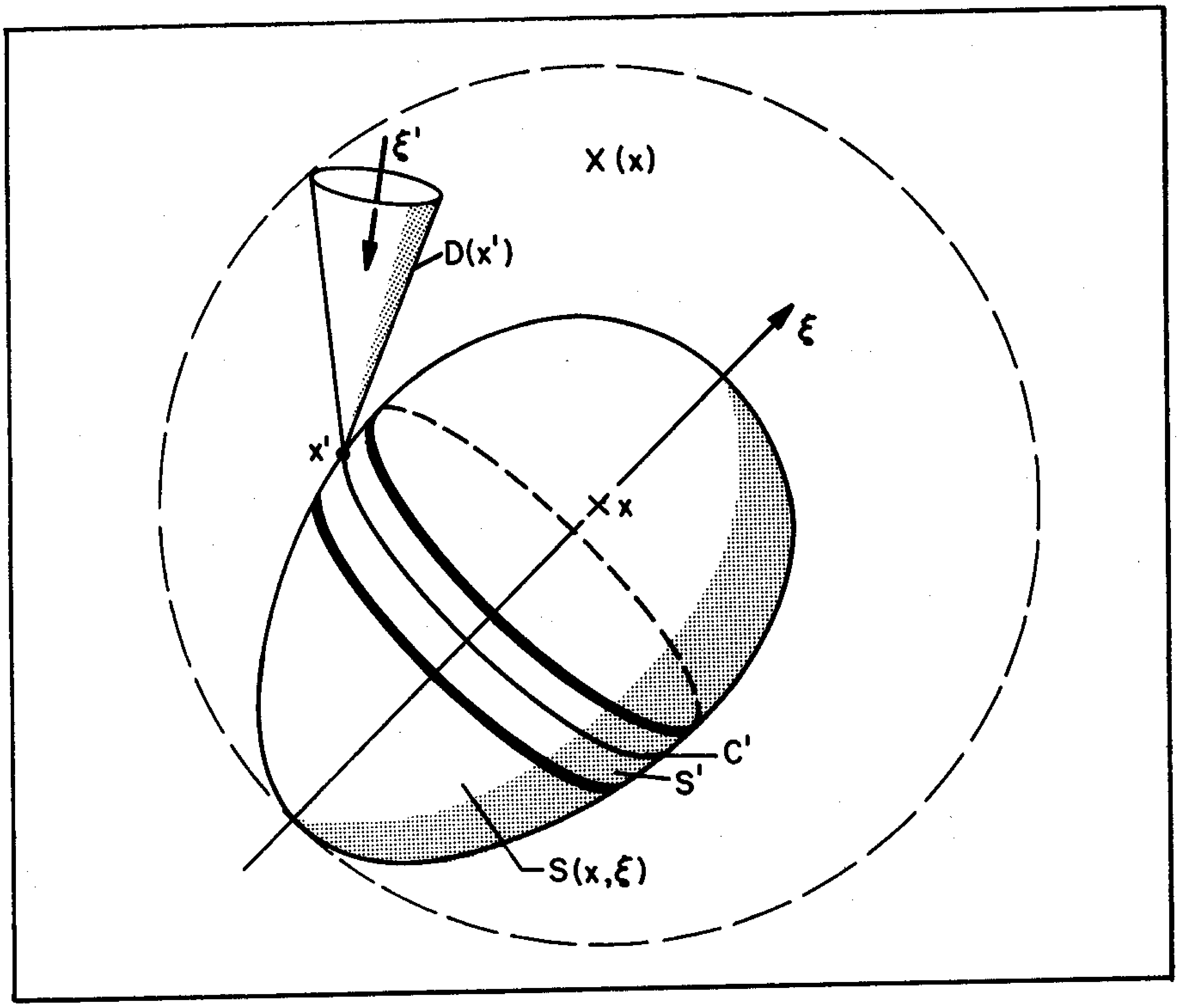

FIG. 2.42 A diagram of a radiometrically adequate collector. How many of them are there? (See text)

by locating a fixed point x on (or within) S and giving the direction ξ of the sensed axis of revolution of S. A typical surface of the type S(x,ξ) is pictured in Fig. 2.42. Let S' be a *proper band* of latitude circles on S, i.e., such that S' has positive area and such that to the points x' of each latitude circle C' of S' there is assigned a right circular cone D(x') of directions whose axis direction ξ' lies in the plane of x' and ξ and makes a given angle with ξ, and of common positive solid angle opening Ω(D(x')). We shall require that the values ξ·ξ' and Ω(D(x')) are fixed for the points x' on each latitude circle C' on S' but may vary from circle to circle on S'. Let X(x) be the spherical region swept out by S(x,ξ) as x is held fixed and ξ allowed to vary through all of Ξ. Finally, assume that a general radiance distribution of fixed structure is defined at each point within X(x). Then if "P(S(x,ξ))" denotes the radiant flux collected by S for a given x and ξ, we make the following plausible assertion with the above conditions in mind: *For every point* x *in the optical medium* X, *the radiance distribution* N(x,·) *is equivalent to the radiant flux distribution* P(S(x,·)) *in the sense that there is a one-to-one integral operator* E(S,x) *such that:*

$$P\big(S(x,\cdot)\big) = N(x,\cdot)\;E(S,x) \tag{76}$$

The preceding assertion clearly contains the irradiance assertions above as special cases. For example, let S be a plane circular surface of positive area, with unit inward normal ξ and center x. Let S' be one side of S such that $D(x') = \Xi(\xi)$ for every x' in S'. Then under the conditions of the preceding assertion, we have:

$$P\big(S(x,\xi)\big) = H(x,\xi)\ A(S) \quad ,$$

so that, according to (70) and (76):

$$E(S,x) = 2\pi C(x)\ A(S) \quad ,$$

where A(S) is the area of the plane circular surface S.

These examples do not exhaust the possibilities inherent in (70) and (76); however, they will suffice for the present to show that there is an infinite class of radiometric functions each member of which is equivalent to the radiance function in the sense of there being a one-to-one linear transformation between the vector spaces of radiance distributions and radiant flux distributions of such functions. Let us say that an arbitrary convex surface S is a *radiometrically adequate collector* in an optical medium X if its associated radiant flux distribution $P(S(x,\cdot))$ is equivalent, in the sense of the present example, to $N(x,\cdot)$ for every point x in X. We close this example with the following problem directed to interested readers: *Characterize the most general class of radiometrically adequate collectors. (In other words: give the necessary and sufficient conditions that a surface* S *be a radiometrically adequate collector.)* We have shown in the present example that plane circular surfaces, and more generally, have conjectured that surfaces of revolution such as cylinders, spheres, hemispheres, spherical caps, prolate and oblate spheroids, etc., can be radiometrically adequate collectors. It is certainly clear, at least intuitively, that the class of radiometrically adequate collectors is quite large and could, under suitable qualifications, contain surfaces not necessarily surfaces of revolution, such as the Platonic "solids", rectangular parallelepipeds, convex surfaces, and even certain non convex surfaces. However, non convex surfaces introduce self-interreflection complications which cannot be handled until the interaction principle (Chapter 3) has been studied, and therefore for the present at any rate, will be omitted from the problem stated above.

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2.12 Transition from Radiometry to Photometry

The concepts of classical photometry, to which we turn our attention in this section, are designed to give quantitative measures of the capability of radiant flux to evoke the sensation of brightness in human eyes. These measures all rest in the single concept of the *standard luminosity function* the key concept in the science of photometry. Photometry is principally concerned with the precise description of and the deductions from the relative visibility of monochromatic radiant flux as a function of wavelength and as embodied in the