The preceding assertion clearly contains the irradiance assertions above as special cases. For example, let S be a plane circular surface of positive area, with unit inward normal ξ and center x. Let S' be one side of S such that $D(x') = \Xi(\xi)$ for every x' in S'. Then under the conditions of the preceding assertion, we have:

$$P\big(S(x,\xi)\big) = H(x,\xi)\ A(S) \quad ,$$

so that, according to (70) and (76):

$$E(S,x) = 2\pi C(x)\ A(S) \quad ,$$

where A(S) is the area of the plane circular surface S.

These examples do not exhaust the possibilities inherent in (70) and (76); however, they will suffice for the present to show that there is an infinite class of radiometric functions each member of which is equivalent to the radiance function in the sense of there being a one-to-one linear transformation between the vector spaces of radiance distributions and radiant flux distributions of such functions. Let us say that an arbitrary convex surface S is a *radiometrically adequate collector* in an optical medium X if its associated radiant flux distribution $P(S(x,\cdot))$ is equivalent, in the sense of the present example, to $N(x,\cdot)$ for every point x in X. We close this example with the following problem directed to interested readers: *Characterize the most general class of radiometrically adequate collectors. (In other words: give the necessary and sufficient conditions that a surface* S *be a radiometrically adequate collector.)* We have shown in the present example that plane circular surfaces, and more generally, have conjectured that surfaces of revolution such as cylinders, spheres, hemispheres, spherical caps, prolate and oblate spheroids, etc., can be radiometrically adequate collectors. It is certainly clear, at least intuitively, that the class of radiometrically adequate collectors is quite large and could, under suitable qualifications, contain surfaces not necessarily surfaces of revolution, such as the Platonic "solids", rectangular parallelepipeds, convex surfaces, and even certain non convex surfaces. However, non convex surfaces introduce self-interreflection complications which cannot be handled until the interaction principle (Chapter 3) has been studied, and therefore for the present at any rate, will be omitted from the problem stated above.

2.12 Transition from Radiometry to Photometry

The concepts of classical photometry, to which we turn our attention in this section, are designed to give quantitative measures of the capability of radiant flux to evoke the sensation of brightness in human eyes. These measures all rest in the single concept of the *standard luminosity function* the key concept in the science of photometry. Photometry is principally concerned with the precise description of and the deductions from the relative visibility of monochromatic radiant flux as a function of wavelength and as embodied in the standard luminosity function. The depth to which we shall study photometry will be only so far that the reader may gain an insight into the principal features of the subject and a competence in working with photometric concepts, in the forms they commonly occur in the study of applied hydrologic optics. Such interesting problems as the physiological basis of color vision, which lie at the base of the subject, transcend the scope of the present discussion.

We shall initially motivate the transition to the photometric concepts by means of hypothetical experiments designed to acquaint the reader with the main empirical features of photometry. The experiments described are to be understood as didactic devices and as such omit the wealth of detail required for the implementation of their real counterparts. Once the essential idea of the transition has been explained and the transition made from the concept of radiance to that of its photometric counterpart, luminance, then we shall embark on a systematic transition to geometrical photometry and compile our results in compact tabular form suitable for convenient reference.

The Individual Luminosity Functions

Figure 2.43 depicts an observer viewing a screen in a well-lighted room. The screen is divided into two equal areas, and is devised so that on the left half a radiance of fixed amount $N(\lambda_0)$ of fixed wavelength λ_0 is constantly displayed throughout the experiment. The magnitude of $N(\lambda_0)$ is chosen comparable to daylight radiances. The right half of

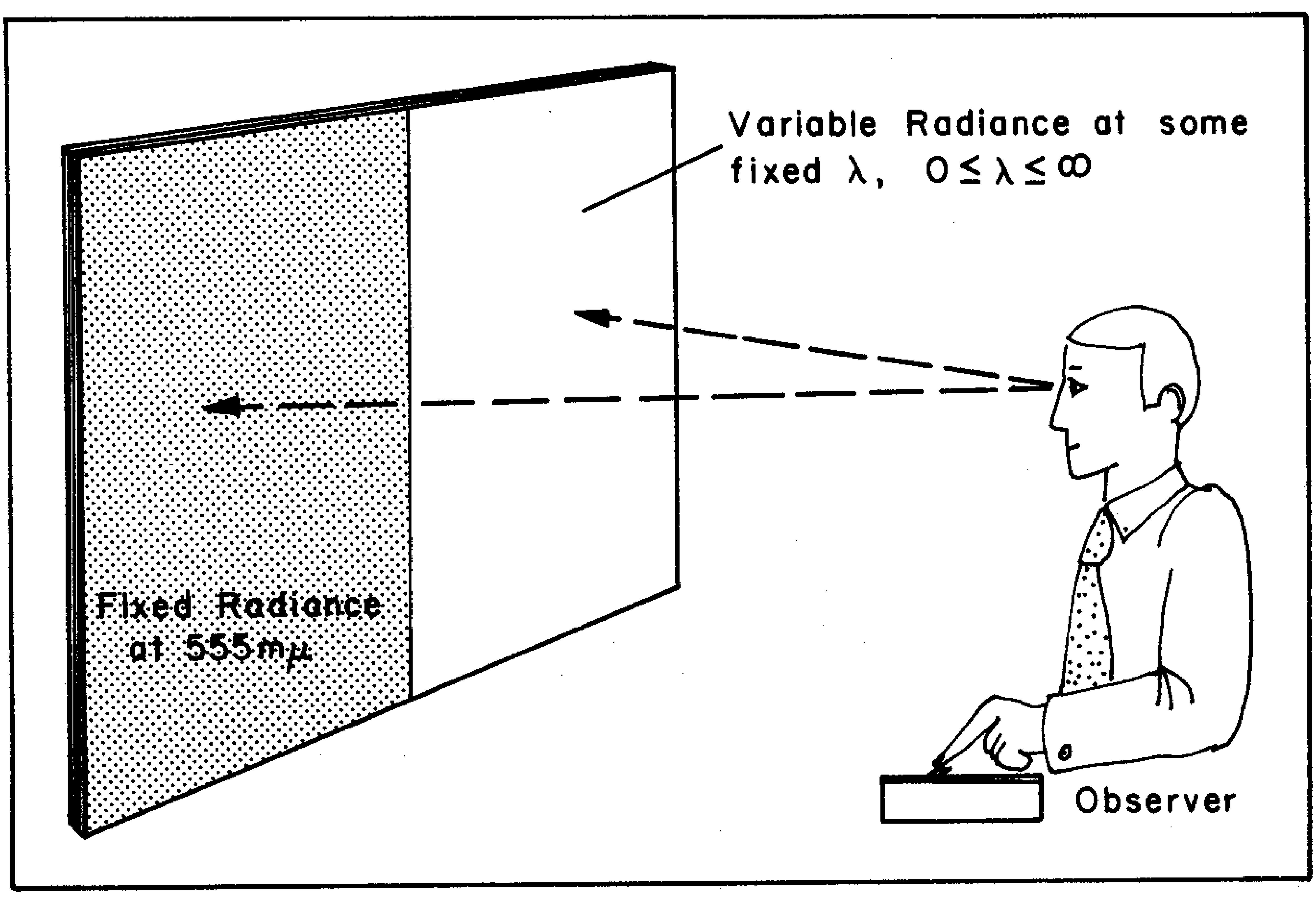

FIG. 2.43 A schematic setting for the empirical determination of the individual luminosity function.

the screen displays a radiance of variable amount $N(\lambda_j)$ of fixed wavelength λ_j in some set L of wavelengths. The observer begins the experiment with $N(\lambda_j)=0$ and, by means of a controlling device, slowly increases $N(\lambda_j)$ from 0 *until that magnitude* $N(\lambda_j)$ *is attained for which he believes the brightness sensation produced in his brain by* $N(\lambda_j)$ *is equal to that produced by* $N(\lambda_o)$. This decision process may require some preliminary trials on the observer's part. Soon, however, he makes a decision that $N(\lambda_j)$ and $N(\lambda_o)$ are of equal "brightness" and presses a button, thereby recording $N(\lambda_j)$. This procedure is now repeated for the other wavelengths over the electromagnetic spectrum.

The experiment with observer a_i just described results in a set of radiance values $N(\lambda_j)$ with λ_j in the set L. These values, it should be noted, are associated with the particular observer a_i used in the preceding experiment. To occasionally point up this fact let us write "$N(a_i,\lambda_j)$" for the radiance that matches $N(\lambda_o)$ as judged by observer a_i in some set A of observers which have performed the experiment. As a result of these experiments, to each observer a_i we may assign his particular *luminosity function* defined as follows. We write:

$$\text{"}\bar{y}(a_i,\lambda_j)\text{"} \quad \text{for} \quad N(\lambda_o)/N(a_i,\lambda_j) \tag{1}$$

for every λ_j in L, and call $y(a_i,\cdot)$ the *luminosity function for observer* a_i. The value $\bar{y}(a_i,\lambda_j)$ is called the *luminosity* of the wavelength λ_j, as judged by observer a_i.

Matters have been arranged (on the basis of earlier experiments with observer a_i, not recorded here) so that wavelength λ_o was the wavelength of maximum luminosity for observer a_i. To see what this means, recall that $N(a_i,\lambda_j)$ is chosen to be of such a magnitude as to match $N(\lambda_o)$ in its capability of evoking the sensation of brightness. Since $N(\lambda_o)$, the radiance with wavelength λ_o of maximum luminosity is fixed in magnitude, all other radiances $N(a_i,\lambda_j)$ must then be increased to give the same brightness sensation to a_i as did the radiance $N(\lambda_o)$. Hence a plot of $\bar{y}(a_i,\lambda_j)$ versus λ_j for each observer a_i in A will have a graph of the general form in Fig. 2.44. At $\lambda_j = \lambda_o$, $y(a_i,\lambda_o) = 1$. For every other λ_j, $\bar{y}(a_i,\lambda_j) < 1$. To point up the fact that λ_o varies from observer to observer, let us write, alternatively, "$\lambda_o(a_i)$" for the λ_o of observer a_i.

Once each observer in the experimental group A has been assigned a luminosity function, this information could be used to predict the subjective sensation of brightness of a given sample of monochromatic radiant flux in the following sense. Suppose that observer a_i encounters a radiance of magnitude $N(\lambda_j)$. Then by (1) we can predict that this radiance would appear to him to have the same "brightness" as a sample of radiant flux of wavelength $\lambda_o(a_i)$ and radiance:

$$N(\lambda_j)\bar{y}(a_i,\lambda_j) \quad . \tag{2}$$

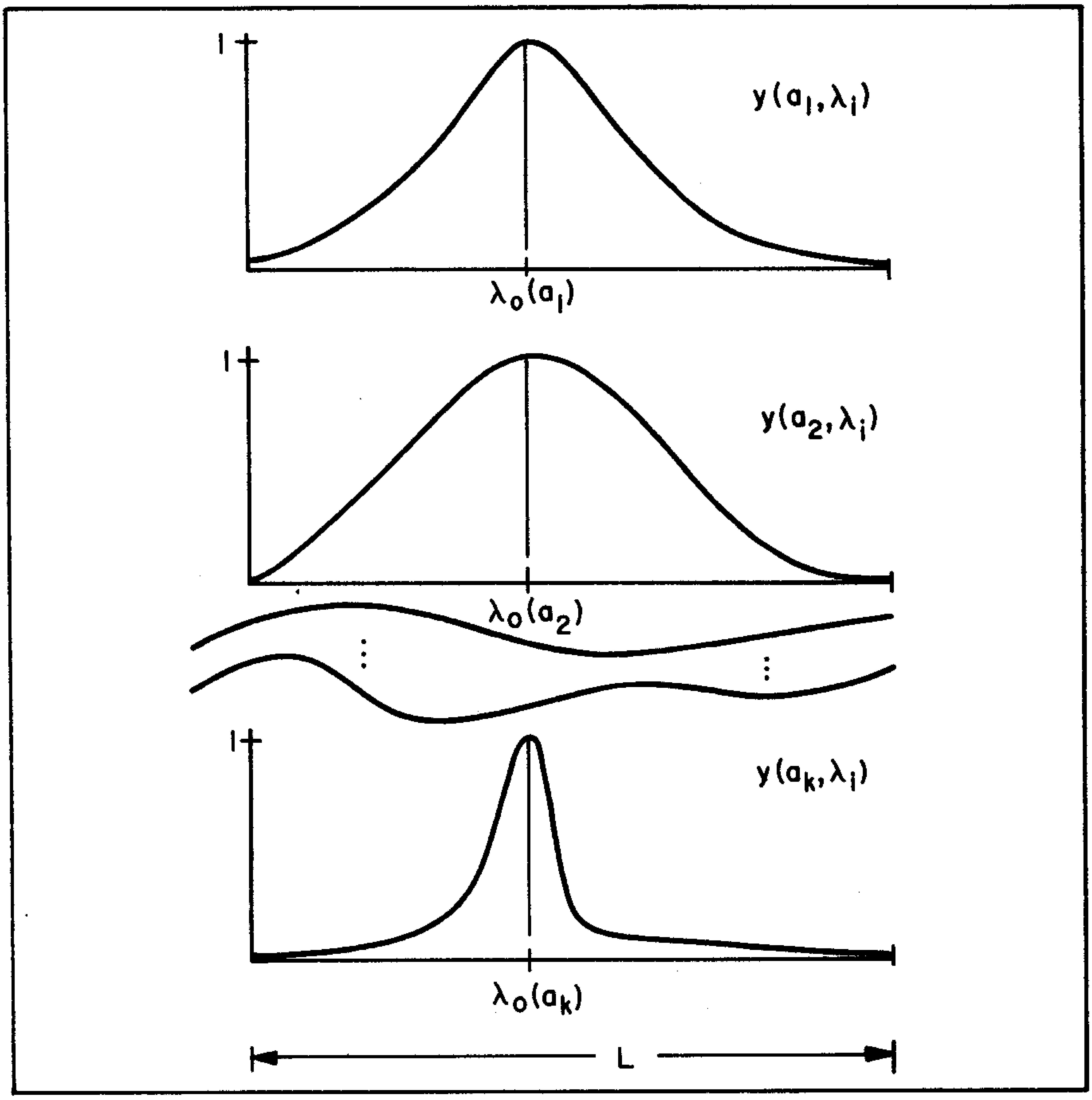

FIG. 2.44 Some individual luminosity functions (schematic only).

The term follows very simply and logically from (1). But the *interpretation of this term, as just stated, is not compelled to follow from (1) by the laws of algebra.* To make this interpretation we must first make an assumption (preferably explicitly) that the subjective sensation of brightness that can be produced by a radiance $N(\lambda_j)$ *varies linearly* with the magnitude of $N(\lambda_j)$. Thus if we were to double $N(\lambda_j)$, then the sensation would be the same as that produced by viewing radiant flux of wavelength λ_o and of double the radiance $N(\lambda_j)\bar{y}(a_i,\lambda_j)$. The reasonableness of this assumption rests critically on the stability of a_i's luminosity curve with respect to the absolute magnitude of $N(\lambda_o(a_i))$ used in the experiment, and on the general lighting level within the experimental room. Actual experimental evidence indicates that the luminosity function for a_i is dependent to a measurable degree on both $N(\lambda_o(a_i))$ and the background radiance. The description of the hypothetical experiment above was careful to note that the experiment took place in a well-lighted room. To point up this fact the resultant luminosity curves in Fig. 2.44 are called *photopic luminosity* curves. When the observers view the screen in a darkened room with dark-adapted vision, it is found that the luminosity curves shift en masse 50-60 mμ to the left with very slight overall change in shape. The resultant curves are called *scotopic luminosity* curves.

To summarize, a consistent, workable interpretation of the meaning of (2) requires an explicit assumption of the linearity of subjective brightness sensations with respect to the magnitude $N(\lambda_j)$. This assumption might be at slight variance with experimental fact over some ranges of values of $N(\lambda_j)$, but it has the virtue of leading the way to a scientific basis of photometry. A science, it will be recalled, is an organized body of knowledge sustained within the webwork of a set of generally accepted conventions. To raise photometry to the level of a science, for better or worse, requires the explicit statement of adopted conventions of the kind just discussed.

We return now to experimental subject a_i, whose photopic luminosity function has been determined, and attempt to predict a new kind of response of a_i to radiant flux. Suppose now that a_i is confronted with a radiance in the right half of the screen in Fig. 2.43 which consists of a radiance which is a superimposed mixture of two monochromatic radiances $N(\lambda_j)$, $N(\lambda_k)$ from the set L, say of distinct wavelengths λ_j and λ_k. Were he confronted with each separately, we would be able to predict the equivalent sensation producing radiance of the wavelength $\lambda_o(a_i)$ by performing twice the operation in (2): once for λ_j and then again for λ_k. In an attempt to predict the sensation producing capabilities of radiance of wavelength $\lambda_o(a_i)$ equivalent to that of the radiance mixture $N(\lambda_j)+N(\lambda_k)$ we are tempted by simple energy-addition arguments to say that:

$$N(\lambda_j)\bar{y}(a_i,\lambda_j) + N(\lambda_k)\bar{y}(a_i,\lambda_k) \qquad (3)$$

is the requisite radiance. However, there appears to be no experimental evidence to substantiate this attempt, although practical calculations based on (3), and physiological eye-mechanisms tend to lend some support of (3). In the absence of such experimental evidence and in the presence of a desire to progress to a scientific discipline, we must make an explicit assumption to the effect that: *the radiance of wavelength* $\lambda_o(a_i)$, *capable of producing the same sensation of brightness as a mixture of two radiances of wavelengths* λ_j *and* λ_k, *is given by (3) above.* Clearly this is a generalization of the linearity assumption above, the earlier form being obtained by setting $\lambda_j = \lambda_k$.

Once the preceding assumption--(or *definition* of equivalent radiance of wavelength λ_o, as it should preferably be called)--is made, the path toward a sound basis for the science of photometry is cleared of one further obstacle. Indeed, it is but a formal step from (3) to the following general definition for the relative luminance distribution associated with a radiance distribution at a point x in an optical medium: Let $N(x,\cdot,t,\lambda)$ be the radiance distribution at x at time t for wavelength λ. Then the associated *relative luminance distribution* with respect to observer a_i is the function:

$$\int_0^\infty N(x,\cdot,t,\lambda)\bar{y}(a_i,\lambda)\ d\lambda \qquad (4)$$

which assigns to each ξ at x at time t, the *relative luminance*, with respect to a_i, of the *integrated radiance* distribution $\int_0^\infty N(x,\cdot,t,\lambda)\ d\lambda$. We shall denote the latter by "$N(x,\cdot,t,\Lambda)$".

A minor technical point should be noted here before going further, a point which concerns the integration of radiance with respect to *wavelength* λ rather than *frequency* ν. It will be recalled that the basis for integrating radiance over the spectrum of frequencies was established in Sec. 2.3, and that the possibility of such an operation is guaranteed by the additivity and continuity properties of Φ with respect to frequency (cf., (1) and (2) in Sec. 2.2). By noting that $\nu\lambda = v$ implies $d\nu = -(v/\lambda^2)\ d\lambda$, each integration with respect to ν can be cast into an integration with respect to λ. (See note (c) to Table 3 below.) Whenever such a change of variables from ν to λ is made, we assume that the factor $-(v/\lambda^2)$ is suitably absorbed in the radiometric symbol, and the dimension of the radiometric concept, e.g., radiance, as far as the frequency component is concerned, is tacitly changed from "per unit frequency length" to "per unit wavelength".

Returning now to (4), we attempt to interpret (4) after the fashion of the interpretation of (3). A straightforward extension of the interpretation of (3) is the following: *for a given direction ξ, (4) is the amount of monochromatic radiance of wavelength $\lambda_0(a_i)$ which would produce an equivalent sensation of brightness in the brain of observer a_i as would the integrated radiance* $N(x,\xi,t,\Lambda)$, where Λ is the entire wavelength (or frequency) spectrum. In view of the preceding observations, in the definitions (4) of Sec. 2.5, one can replace "F" by "Λ" and have:

$$N(x,\xi,t,\Lambda) = \int_0^\infty N(x,\xi,t,\lambda)\ d\lambda \quad ,$$

by virtue of (4), Sec. 2.3. It is to be particularly noted that the preceding italicized interpretation is a *formal* interpretation with no known empirical basis--except for the single case where the given radiance distribution is monochromatic.

With the preceding interpretation of (4) in mind, we next return to (1) and emulate that definition in the present heterochromatic setting of (4). Thus, we write:

$$\text{"}\bar{y}(a_i)\text{"} \quad \text{for} \quad \frac{1}{N(\Lambda)}\int_0^\infty N(\lambda)\bar{y}(a_i,\lambda)\ d\lambda \qquad (5)$$

and where for brevity we have written:

$$\text{"}N(\lambda)\text{"} \quad \text{for} \quad N(x,\cdot,t,\lambda)$$

and

$$\text{"}N(\Lambda)\text{"} \quad \text{for} \quad N(x,\cdot,t,\Lambda) \quad .$$

We call $\bar{y}(a_i)$ the *relative luminosity* of the radiance $N(\cdot)$ over Λ for the observer a_i. In this way we come to one of the principal definitions of photometry:

Let $\mathcal{R}$ be any radiometric concept of geometrical radiometry (radiance, irradiance, radiant intensity, etc.), defined over the part M of the spectrum Λ. The *relative luminosity of* $\mathcal{R}$ *over* M *for an observer* a_i is the number $\bar{Y}(\mathcal{R},M,a_i)$ where we have written:

$$\text{"}\bar{Y}(\mathcal{R},M,a_i)\text{"} \quad \text{for} \quad \frac{\left[\int_M \mathcal{R}(\lambda)\bar{y}(a_i,\lambda)\, d\lambda\right]}{\int_M \mathcal{R}(\lambda)\, d\lambda} \qquad (6)$$

The Standard Luminosity Functions

We now re-examine the family of individual relative luminosity functions, depicted in Fig. 2.44, and attempt to define a single luminosity function which is representative of the entire set A of individual observers. There are several ways to go about this. For example in one method, we can go through the set of graphs of Fig. 2.44, note each $\lambda_0(a_i)$ and make a histogram, over λ in Λ, of the number of observers whose maximum luminosity occurred at wavelength λ. A typical histogram that would result is shown schematically in part (a) of Fig. 2.45. All indications in real experiments and theoretical considerations point to a gaussian distribution for the ideal limit of such histograms as the number of members in the set A increases indefinitely. The peak of the distribution is found in actual experiments to occur near a λ of 555 mμ. Next, a general wavelength λ is selected and the graphs of Fig. 2.44 are combed through with the specific goal in mind of finding the spread of values of $\bar{y}(a_i,\lambda)$ over the a_i in A. This spread of values is then split up into intervals. Part (b) of Fig. 2.45 depicts a typical histogram with the abscissas locating the observed values $\bar{y}(a_i,\lambda)$ occurring over the selected set of intervals, and the ordinates giving the number of a_i in each interval. Part (b) of Fig. 2.45 is adapted from Fig. 3.03a of Moon's treatise on Illuminating Engineering (Ref. [185]), which in turn is derived from actual experimental results by Coblentz and Emerson who gathered data from a set A of 125 observers. By means of (b) of Fig. 2.45, the relative luminosity value of 0.1750 is assigned to the standard observer for λ = 640 mμ.

By going through the entire spectrum in this way--i.e., by repeating the process summarized in (b) of Fig. 2.45, now

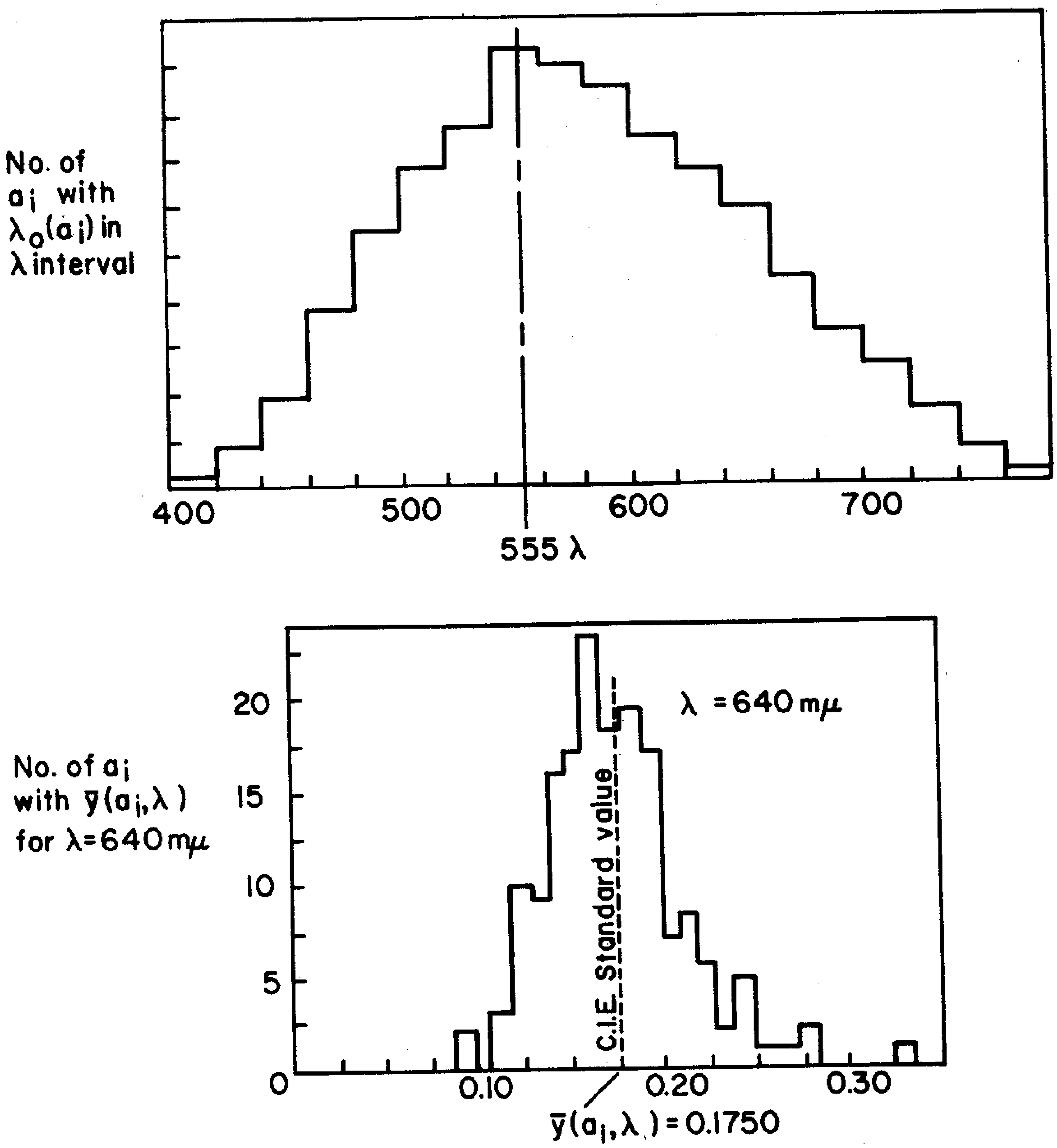

FIG. 2.45 Towards determining the standard luminosity function $\bar{y}$. (From [185], by permission)

for each λ in a selected range of λ's through Λ--the desired standard luminosity function is obtained. A graph, to scale, of the *standard photopic luminosity function* $\bar{y}(\cdot)$ is given in Fig. 2.46, and a tabulation of $\bar{y}(\cdot)$ is given in Table 1. A more detailed tabulation of the values $\bar{y}(\lambda)$ over the visible spectrum is given in Ref. [50].

Now, all that we did in the preceding discussion by means of individual luminosity functions $\bar{y}(a_i,\cdot)$ can be repeated line for line for the *standard observer* a. Thus, wherever "$\bar{y}(a_i,\cdot)$" appeared, we can write "$\bar{y}(a,\cdot)$" or, more simply, "$\bar{y}(\cdot)$", for the standard photopic luminosity function, and where "a" stands for the hypothetical standard observer (a creature who shares the same corner of conceptual reality with

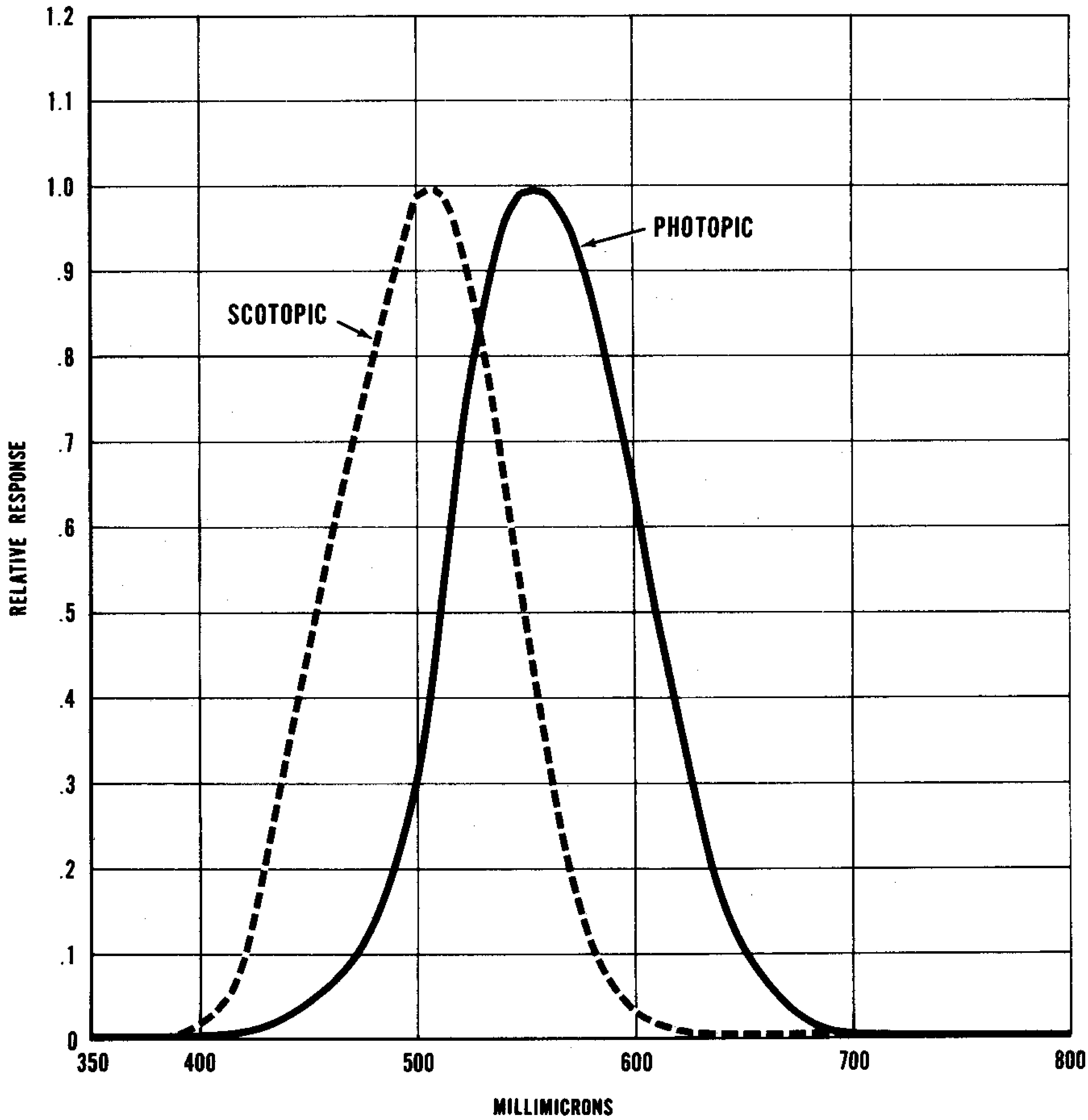

FIG. 2.46 The solid curve depicts the standard photopic luminosity function for daylight adaptation. The standard scotopic luminosity function (for dark adaptation) is shown dashed.

such entities as the "average American male, age 30"). Specifically, we can now make the following definition which is one of the principal definitions of photometry:

Let $\mathcal{R}$ be any radiometric concept of geometrical radiometry (radiance, irradiance, radiant intensity, etc.) defined on the part M of the spectrum Λ. The *relative luminosity of* $\mathcal{R}$ *over* M *for the standard observer* is the number $\bar{Y}(\mathcal{R}, M)$ where we have written:

$$"\bar{Y}(\mathcal{R}, M)" \quad \text{for} \quad \frac{\left[\int_M \mathcal{R}(\lambda)\bar{y}(\lambda)\ d\lambda\right]}{\int_M \mathcal{R}(\lambda)\ d\lambda} \ . \qquad (7)$$

TABLE 1

The Standard Photopic Luminosity Function $\bar{y}(\cdot)$
and its Indefinite Integral

λ(mμ)	$\bar{y}(\lambda)$	$\int_{390}^{\lambda} \bar{y}(\lambda') \, d\lambda'$
390	1. × 10^{-4}	0 × 10^{-3}
400	4.	1
410	12.	5
420	40.	17
430	116.	57
440	230.	173
450	380.	403
460	600.	783
470	910.	1,383
480	1,390.	2,293
490	2,080.	3,683
500	3,230.	5,763
510	5,030.	8,993
520	7,100.	14,023
530	8,620.	21,123
540	9,540.	29,743
550	9,950.	39,283
560	9,950.	49,233
570	9,520.	59,183
580	8,700	68,703
590	7,570.	77,403
600	6,310.	84,973
610	5,030.	91,283
620	3,810.	96,313
630	2,650.	100,123
640	1,750.	102,773
650	1,070.	104,523
660	610.	105,593
670	320.	106,203
680	170.	106,523
690	82.	106,693
700	41.	106,775
710	21.	106,816
720	11.	106,837
730	5.	106,848
740	3.	106,853
750	1.	106,856
760	0.	106,857

Photometric Bedrock: the Lumen

We now take the final step in the transition from radiometry to photometry. This step consists in reaching an agreement on how to assign to every sample of radiant flux of given spectral composition a quantitative measure of the sample's capability of producing within the standard observer the associated sensation of brightness. Now that the concept of the standard observer has been fixed, the remaining task consists in finding a suitable standard radiant-flux emitter whose wavelength dependence over the spectrum Λ is uniquely defined within a rigid experimental and theoretical framework and which is precisely reproducible in practice. Once such a standard is found it is assigned a preselected number of units of "brightness"-producing capability and all other radiant flux samples can then be given their amounts of brightness-producing capability relative to the standard.

Such candidates as laser beams of given monochromaticity, various flames of burning liquids or solids, incandescent gases of known spectral decomposition, the surface of the sun, the surfaces of various molten metals--all these are possible candidates which can serve as photometric standards. The traditional standard was a candle flame--the candle having been manufactured, set to burn, and observed in a rigidly controlled manner. The current standard is the surface of a pool of platinum which is at the precisely determinable temperature (see [51]) of its change from the solid to the liquid phase (2042° Kelvin). Once the metal has reached this temperature within some thermally stable enclosure, its radiant emittance W_b is precisely computable for each wavelength in the spectrum using the laws of blackbody thermal radiation. The surface radiance distribution of the platinum is uniform of magnitude N over all emergent directions from the surface at a point. Hence the relation $N_b = W_b/\pi$ exists between N_b and W_b (cf., closing remarks of Sec. 2.4).

The key step is now taken: it is agreed that the surface radiance of the surface of freezing platinum is to be assigned a brightness sensation producing capability, a *luminance*, of 6×10^5 *lumens per square meter per steradian*. Thus, the unit of brightness-producing capability of radiant flux is called a *lumen*. The lumen is the photometric counterpart to the radiometric *watt*. This convention is translated into practical working formulas as follows: we observe that if $N_b(\lambda)$ is the radiance of the standard platinum surface as given by the blackbody thermal radiation laws, then on the one hand, by (4), the *relative* (standard) luminance of the platinum surface is:

$$\int_0^\infty N_b(\lambda)\bar{y}(\lambda)\ d\lambda \tag{8}$$

and on the other hand, by fiat, the *absolute* (standard) luminance of the platinum surface is:

$$6 \times 10^5 \ \frac{\text{lumens}}{m^2 \times \text{steradian}} \qquad (9)$$

The units of (8) are *watts/(m² × steradian)*. Our agreement leads us to equate (8) and (9), after introducing a numerical constant which will balance units in the resulting equation. Let us denote this number by "K_m". Then we agree to write:

$$\boxed{6 \times 10^5 = K_m \int_0^\infty N_b(\lambda)\bar{y}(\lambda)\ d\lambda} \quad . \qquad (10)$$

The number K_m so defined has units: lumens/watt. Its magnitude is determined by explicitly introducing the functional form for the surface radiance N_b of the surface of a blackbody (a *complete* radiator or *Planckian* radiator) at temperature T:

$$N_b(\lambda) = c_1\lambda^{-5}\Big/\pi(e^{c_2/(\lambda T)} - 1)$$

in which we have set:*

$$c_1/\pi = 2c^2h = 1.1909 \times 10^{-16} \quad \text{watts m}^2/\text{steradian}$$

$$c_2 = hc/k = 1.4380 \times 10^{-2} \quad \text{m °(Kelvin)}$$

$$T = 2042° \quad \text{°(Kelvin)}$$

It follows, on numerical integration of $N_b(\lambda)\bar{y}(\lambda)$ over Λ, that:

$$\int_0^\infty N_b(\lambda)\bar{y}(\lambda)\ d\lambda = 884 \quad \text{watts}/(m^2 \times \text{steradian}) \quad .$$

Hence, from (10):**

$$K_m = 6 \times 10^5/884$$

$$= 680 \ \text{lumens/watt} \quad . \qquad (11)$$

---

*The units of c_1 are determined by specifically using the spectral density part of the dimensions of radiance. Thus dim [N] = watts/(m² × steradian × m), using wavelengths.

**Uncertainties in the measured values of c_1,c_2 and in the numerical integrations leading to the value 884 watts/(m²× steradian) lead to a corresponding uncertainty of K_m of about 5 or 6 units in the last digit. See, e.g., [51], [153].

The only seemingly arbitrary feature in this final step from radiometry to photometry is the choice of the magnitude 6×10^5. Actually, the choice of this particular magnitude is not completely arbitrary; it is tied to the historical precedent set for a lumen by the early international candle standard. The historical details of these matters may be found in the standard treatises on photometry, Refs. [185], [311], or in Ref. [50]. See also [206], [51]. The current standard unit of luminous intensity (defined formally below) is the *candela* which by definition is *1 lumen per steradian.* Hence the convention in (9) may be read as *600,000 candelas per square meter.*

Luminance Distributions

The magnitude of the transition factor K_m having been determined, we can go on to give a precise definition of the requisite "measure" of the capability of a given sample of radiant flux to evoke the brightness sensation. Thus let $N(x,\xi,t,\cdot)$ be the radiance function which assigns to every λ in the spectrum Λ a radiance $N(x,\xi,t,\lambda)$ at a fixed point x in the fixed direction ξ at given time t. Then we call the number:

$$K_m \int_0^\infty N(x,\xi,t,\lambda)\bar{y}(\lambda)\, d\lambda$$

the *luminance* associated with the radiance function $N(x,\xi,t,\cdot)$, and write:

$$\boxed{\text{"}B(x,\xi,t)\text{"} \quad \text{for} \quad K_m \int_0^\infty N(x,\xi,t,\lambda)\bar{y}(\lambda)\, d\lambda} \qquad (12)$$

If x and t are fixed but ξ allowed to vary in $N(x,\xi,t,\lambda)$ then the resultant function $B(x,\cdot,t)$ is called the *luminance distribution* at x, at time t. Often the time t, or x, or even ξ are understood (as occurred e.g., in the radiometric context) and so may be dropped from the notation provided no confusion results. Thus we agree that we can occasionally write:

$$\text{"}B(x,\xi)\text{"} \quad \text{or} \quad \text{"}B(\xi)\text{"} \quad \text{or "}B\text{"} \quad \text{for} \quad B(x,\xi,t) \quad .$$

These definitions serve to fix $B(x,\cdot)$ as the photometric counterpart to the radiometric function $N(x,\cdot)$ studied in earlier sections of this chapter. The units of $B(x,\cdot)$ are *lumens/ ($m^2 \times$ steradian).*

Table 1 of Sec. 2.4 can be used to construct a corresponding table of radiance by assuming that the surfaces S referred to in Table 1 of Sec. 2.4 have *uniform* radiance. Then the desired radiances are found by dividing each irradiance in the right hand column of that table by π. A similar table for the general order of magnitude of *luminance* of common natural objects can be constructed. A sample of such a table is given below which is partly constructed from Fig. 1.12.

TABLE 2

Source	Luminance (lumens/($m^2 \times$ steradian))
Surface of sun	2×10^9
Clear sky	3200
Surface of moon*	6200 (full) 1400 (half) 0 (new) } 2800 (average)

Further illustrative examples of luminance are easily constructed: suppose a source of monochromatic radiant flux has a radiance N of 1000 watts/($m^2 \times$ steradian) per meter wavelength for each λ over an interval $\Delta\lambda$ of 10 mμ centering on wavelength λ = 555 mμ, and of zero radiance outside this interval. What is the luminance B of this source? Returning to (12) we see that in this case:

$$B = K_m \int_0^\infty N(\lambda)\bar{y}(\lambda)\ d\lambda$$

$$= K_m N(555)\bar{y}(555)\Delta\lambda$$

$$= 680 \times 1000 \times 1 \times 10^{-8} \quad (1\ \text{m}\mu = 10^{-9}\text{m})$$

$$= 6.8 \times 10^{-3}\ \text{lumens}/(\text{m}^2 \times \text{steradian}) \quad .$$

As another example, consider a source of radiance N = 1000 watts/($m^2 \times$ steradian) per millimicron wavelength at λ = 450 mμ over an interval $\Delta\lambda_1$ of 10 mμ about this wavelength, and of radiance N = 500 watts/($m^2 \times$ steradian) per millimicron wavelength at λ = 600 mμ over an interval $\Delta\lambda_2$ of 5 mμ about the latter wavelength. What is the associated luminance of this source? By (12) we have:

*These luminances are computed directly from the full and half phase illuminances produced by the moon, as given in Fig. 1.12. For half phase, the solid angle of the luminous surface was taken as 3×10^{-5} steradians. Standard references give 2500-3000 lumens/($m^2 \times$ steradian) for the moon's luminance. The lighting geometry on the porous and craggy lunar surface is partly involved in this spread of values.

$$
\begin{aligned}
B &= K_m \int_0^\infty N(\lambda)\bar{y}(\lambda)\ d\lambda \\
&= K_m N(450)\bar{y}(450)\Delta\lambda_1 + K_m N(600)\bar{y}(600)\Delta\lambda_2 \\
&= 680(1000 \times .038 \times 10 + 500 \times .631 \times 5) \\
&= 680(380 + 1580) \\
&= 1.34 \times 10^6 \text{ lumens}/(\text{m}^2 \times \text{steradian}) \quad .
\end{aligned}
$$

Here $\bar{y}(450)$ and $\bar{y}(600)$ may be estimated from Fig. 2.46 or taken directly from Table 1. Note the two different ways of specifying the spectral density of radiance in these two examples.

As a final example, let a source be of constant radiance N (per millimicron wavelength) over the region of the spectrum from 390 mμ to 760 mμ (the part of the spectrum over which $\bar{y}(\lambda)$ is defined in Table 1, and zero outside this region. What is the luminance of the source? By (12) we have:

$$
\begin{aligned}
B &= K_m \int_{390}^{760} N\bar{y}(\lambda)\ d\lambda \\
&= K_m N \int_{390}^{760} \bar{y}(\lambda)\ d\lambda \\
&= 680 \times 107 \times N = 7.3 \times 10^4\ N
\end{aligned}
$$

Here we have integrated $\bar{y}(\cdot)$ over the range λ = 390 mμ, to λ = 760 mμ in steps of 10 mμ using the values of Table 1 above. The result is:

$$
\int_{390}^{760} \bar{y}(\lambda)\ d\lambda = 106.857 \quad \text{(millimicrons)}
$$

This value may, for all practical purposes, be taken as the integral of $\bar{y}(\cdot)$ from $\lambda = 0$ to $\lambda = \infty$.

Transition to Geometrical Photometry

The transition from geometrical radiometry to geometrical photometry has so far been made between two points, i.e., between the radiance and luminance concepts by means of (12), and with the help of (10) and (11). This choice of the radiance-luminance bridge rather than any other means was governed by the relative visualizability of these concepts as contrasted with other radiometric-photometric pairs, say with the visualizability of hemispherical irradiance and its counterpart hemispherical illuminance (to be defined below). But now that the bridge has been constructed with suitable attention to intuitive motivations and visualization, we return to its site and start anew with the purpose in mind of constructing the bridge once again, but now in a logically more satisfying way. By undertaking this reconstruction we are given the opportunity to re-emphasize and make formal the additivity assumption we had encountered on our way to the relative luminance distribution in (4). This formalized additivity assumption will subsequently take its place among the other basic assumptions of radiometry which we isolated for the radiant flux function in the discussions of Sec. 2.3.

The transition from radiance to luminance, as summarized in (12), may now be emulated systematically for each radiometric concept. That is, for every part M of the spectrum Λ we first define a general integral the *radiometric-photometric transition operator* by writing:

$$\text{"}Y(\cdot,M)\text{"} \quad \text{for} \quad K_m\int_M [\ ]\bar{y}(\lambda)\, d\lambda \quad . \tag{13}$$

Then it follows from (12) that:

$$B = Y(N,\Lambda) \quad , \tag{14}$$

where "B" and "N" are the abbreviated names for the given luminance and radiance functions in (12). But we need not stop at (12). Indeed, let us go on and write:

"$F^{\pm}(S,D,t)$"	for	$Y(P^{\pm}(S,D,t,\cdot),\Lambda)$	(*luminous flux*, (3) of Sec. 2.3; cf.,(17) of Sec. 2.4)	(15)
"$E(x,\xi,t)$"	for	$Y(H(x,\xi,t,\cdot),\Lambda)$	(*illuminance* (11), (17) of Sec. 2.4)	(16)
"$L(x,\xi,t)$"	for	$Y(W(x,\xi,t,\cdot),\Lambda)$	(*luminous emittance*, (22) of Sec. 2.4)	(17)
"$B^{\pm}(x,\xi,t)$"	for	$Y(N^{\pm}(x,\xi,t,\cdot),\Lambda)$	(*luminance*, (30), (31) of Sec. 2.5)	(18)
"$I^{\pm}(S,\xi,t)$"	for	$Y(J^{\pm}(S,\xi,t,\cdot),\Lambda)$	(*luminous intensity*, (7),(10) of Sec. 2.9)	(19)

These are the definitions of the first five principal photometric concepts under both the surface (+) and field (-) interpretations. The names of the concepts are given to the right of each definition and reference is made to the appropriate radiometric ancestor of each concept. Thus, e.g., surface luminous flux $F^{+}(S,D,t)$ is derived from surface radiant flux

$P^+(S,D,t)$, which in turn is defined in (3) of Sec. 2.3, and with the surface interpretation of $P^+(S,D,t)$ given in (17) of Sec. 2.4. In this way the logical ancestry of each of the preceding eight photometric concepts is traceable back to the primitive radiometric function Φ. The bridge to the ancestor in each case is the integral operator $Y(\cdot,\Lambda)$, defined in (13). This integral operator now permits, without the necessity of any further geometrical arguments, all the radiometric connections among P, H, W, N and J developed in the preceding sections, to be carried over directly to the photometric context. As an example, (8) of Sec. 2.5 is carried over to the photometric context by applying $Y(\cdot,\Lambda)$ to each side of that equation. Thus, by (16) above, (31) of Sec. 2.5, and (8) of Sec. 2.5:

$$E(x,\xi) = Y\big(H(x,\xi),\Lambda\big) = Y\left(\int_{\Xi(\xi)} N^-(x,\xi)\,\xi\cdot\xi'\,d\Omega(\xi'),\Lambda\right)$$

$$= \int_{\Xi(\xi)} Y\big(N^-(x,\xi'),\Lambda\big)\,\xi\cdot\xi'\,d\Omega(\xi')$$

$$= \int_{\Xi(\xi)} B^-(x,\xi')\,\xi\cdot\xi'\,d\Omega(\xi') \quad . \qquad (20)$$

Whenever either "+" or "-" is understood, or an equation is valid under both the field and surface interpretations, then these signs may be dropped, if desired. For example, in the case of (20), we know from (21) of Sec. 2.5 that H and N^- go together, so that dropping "-" on the right sides of the equations in (20), no confusion can result. Hence, every occurrence of the signs "-" may be dropped from (20) and left implicitly understood.

The roll-call of principal photometric concepts is continued as follows. We shall write:

$Q^{\pm}(X,t)$"	for	$Y\big(U^{\pm}(X,t,\cdot),\Lambda\big)$	(*luminous energy* in region X, at time t, (12) of Sec. 2.7)	(21)
"$Q^{\pm}(S,T)$"	for	$Y\big(U^{\pm}(S,T,\cdot),\Lambda\big)$	(*luminous energy* across surface S over time interval T, (17) of Sec. 2.7)	(22)
"$q^{\pm}(x,t)$"	for	$Y\big(u^{\pm}(x,t,\cdot),\Lambda\big)$	(*luminous energy density*, (2) of Sec. 2.7)	(23)
"$e(x,t)$"	for	$Y\big(h(x,t,\cdot),\Lambda\big)$	(*scalar illuminance*, (3) of Sec. 2.7)	(24)

"l(x,t)" for $Y(w(x,t,\cdot),\Lambda)$ (*scalar luminous emittance*, (19) of Sec. 2.7) (25)

We illustrate again the fact that any linear relation between two radiometric quantities has a carbon copy in the photometric context. Thus, consider (14) of Sec. 2.7; assuming v is independent of λ in X and applying the operator $Y(\cdot,\Lambda)$ to each side we have:

$$Q(X,t) = Y(U(X,t,\cdot),\Lambda) = Y((u/v)V(X),\Lambda)$$

$$= (V(X)/v)Y(u,\Lambda)$$

$$= (q/v)V(X) \quad . \tag{26}$$

There remains to be defined certain of the photometric concepts such as the vector counterpart **E** to **H**,**I** to **J**, etc. However, instead of going on to explicitly exhaust all these transitions, which are quite numerous, we state below a general definition-scheme which covers all transitions just made, and any yet unmade.

Let $\mathscr{R}$ be a radiometric function defined on Λ. Then $Y(\mathscr{R},\Lambda)$ is the *photometric counterpart* to $\mathscr{R}$. Let "$\mathscr{P}$" denote this photometric counterpart. Then the following statement is a definitional identity:

$$\boxed{\mathscr{P} = Y(\mathscr{R},\Lambda)} \tag{27}$$

A *definitional identity* is a statement of the form "A = B" where "A" and "B" are the names of one and the same object arising from a definition. Thus, e.g., "Q(X,t)" and "$Y(U(X,t,\cdot),\Lambda)$" are names of one and the same object, namely the number:

$$\int_0^\infty U(X,t,\lambda)\bar{y}(\lambda)\ d\lambda$$

and so:

$$Q(X,t) = Y(U(X,t,\cdot),\Lambda)$$

and alternatively:

$$Q(X,t) = \int_0^\infty U(X,t,\lambda)\bar{y}(\lambda)\ d\lambda$$

are definitional identities. For example, definitional identities were used to start and end the series of deductions summarized in (20) and (26). The significance of (27) is simply this: every photometric concept $\mathscr{P}$ is the image, under $Y(\cdot,\Lambda)$, of some radiometric concept $\mathscr{R}$; thus, to define a new photometric concept, first find its radiometric progenitor $\mathscr{R}$. Then $\mathscr{P}$ is the desired concept $Y(\mathscr{R},\Lambda)$.

General Properties of the Radiometric-Photometric Transition Operator

The integral operator $Y(\cdot,M)$ defined in (13) has several properties built into it which are of critical importance in establishing the science of theoretical photometry. To recognize and understand these properties is to recognize and understand the role of photometry as a descriptive science. Therefore we devote some attention to the isolation of these properties.

Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be any two radiometric functions defined on a subset M of Λ and let c_1 and c_2 be any two real numbers such that the sum $(c_1 \mathscr{R}_1 + c_2 \mathscr{R}_2)$ is defined. Then by (13) and the linearity of the mathematical integration process:

$$Y(c_1 \mathscr{R}_1 + c_2 \mathscr{R}_2,\Lambda) = c_1 Y(\mathscr{R}_1,\Lambda) + c_2 Y(\mathscr{R}_2,M) \tag{28}$$

This is the *linearity property* of $Y(\cdot,M)$, the formal vestige of the associated property of $\bar{y}(a_i,\lambda)$ discussed in (2).

Next, for every radiometric function $\mathscr{R}$ defined on Λ,

$$Y(\mathscr{R},M_1 \cup M_2) = Y(\mathscr{R},M_1) + Y(\mathscr{R},M_2) \tag{29}$$

for every pair of disjoint subsets M_1 and M_2 of Λ. This is the *additive property* of $Y(\mathscr{R},\cdot)$ and is the formal vestige of the property of $\bar{y}(a_i,\lambda)$ discussed in (3). Finally, for every radiometric function $\mathscr{R}$ defined on Λ,

$$\text{If} \quad 1(M) = 0, \quad \text{then} \quad Y(\mathscr{R},M) = 0 \tag{30}$$

which is the M-*continuity property* of $Y(\mathscr{R},\cdot)$ for continuous spectra. The length measure 1 and its general use was defined in (4) of Sec. 2.3.

The Mathematical Basis for Geometrical Photometry

Properties (29) and (30) may be added to the set of six additivity and continuity properties of Φ discussed in Sec. 2.3. In fact, in an axiomatic development of the mathematical theory of photometry, statements (28), (29) and (30) would constitute the essential starting point of the construction of the theory, just as the properties of Φ in Sec. 2.3 constitute the essential starting point of the theory of geometrical radiometry. Indeed, for any radiometric function defined on Λ, we may deduce from (29) and (30) alone the existence of a function $\bar{y}'(\cdot)$ on Λ such that:

$$Y(\mathscr{R},\Lambda) = \int_{\Lambda} \mathscr{R}(\lambda)\bar{y}'(\lambda)\, d\lambda \quad . \tag{31}$$

Evidently $\bar{y}'(\cdot)$ will turn out to be $K_m\bar{y}(\cdot)$ discussed above. The complete details of the mathematical justification of this assertion lie beyond the scope of this work. Some of the mathematical background of (31) will be covered as a matter of course in Sec. 3.16. The requisite mathematical basis of the assertion may be found in part in Sec. 56, in particular theorem D, of Ref. [103]. The general measure-theoretic approach to foundations of radiative transfer theory, introduced in Ref. [216], can now, by (31), be systematically extended to the domain of photometry. Hence, as far as the mathematical structure of photometry is concerned, it rests on three pillars: (28), (29), and (30), and its framework can be erected by means of the theorems of modern measure theory and without the necessity of any further reference to physical constructs. In other words, the epistemological content of classical photometry rests in but three postulates, the statements of the linearity, M-additivity and M-continuity of Y introduced above. We note in closing that the preceding observations apply immediately to the representations of colors by the tristimulus procedure of colorimetry; all that has been said for the function $\bar{y}$, now applies, without essential change, to the other two tristimulus functions $\bar{x}$ and $\bar{z}$ (cf., Sec. 1.7). The mathematical setting in the colorimetric case would be a three-dimensional vector space, and the measure-theoretic aspects will be elevated from the scalar to the vector level.

Summary and Examples

The present discussion of radiometry and photometry will be brought to a close with a summary of the main concepts introduced in this chapter. The units and dimensions of the concepts will be tabulated, discussed and illustrated, and a few further illustrative examples will be given.

Table 3 lists the main radiometric concepts by name, symbol, units, dimensions, and reference to its definition in the present work. A similar Table 4 lists the main photometric concepts in an exactly analogous way, as far as possible. Explanatory notes are appended to each table.

TABLE 3

RADIOMETRIC CONCEPTS

NAME	BASIC SYMBOL	DIMENSIONS	MKS UNITS	DEFINITION REFERENCES
RADIANT FLUX (general)	Φ	$\mathbf{P}^{\pm}$	WATT	Sec. 2.1; (17) and (18) of Sec. 2.4
RADIANT FLUX (spectral)	P	$\mathbf{P}_{\lambda}^{\pm}$	WATT/mμ	(3) of Sec.2.3 (17) and (18) of Sec. 2.4
RADIANCE (all radiometric concepts here and below may be either *general* or *spectral*)	N	$\mathbf{P}^{\pm}\mathbf{A}^{-1}\Omega^{-1}$	WATT/(m²×sr) (See note (c) below)	(1), (4), of Sec. 2.5
IRRADIANCE	H	$\mathbf{P}^{-}\mathbf{A}^{-1}$	WATT/m²	(1), (17) of Sec. 2.4
VECTOR IRRADIANCE	**H**	$\mathbf{P}^{-}\mathbf{A}^{-1}$	WATT/m²	(2) of Sec.2.8
SCALAR IRRADIANCE	h	$\mathbf{P}^{-}\mathbf{A}^{-1}$	WATT/m²	(3) of Sec.2.7
RADIANT EMITTANCE	W	$\mathbf{P}^{+}\mathbf{A}^{-1}$	WATT/m²	(18), (22) of Sec. 2.4
VECTOR RADIANT EMITTANCE	**W**	$\mathbf{P}^{+}\mathbf{A}^{-1}$	WATT/m²	See note (d) below
SCALAR RADIANT EMITTANCE	w	$\mathbf{P}^{+}\mathbf{A}^{-1}$	WATT/m²	(19) of Sec. 2.7
RADIANT INTENSITY	J	$\mathbf{P}^{\pm}\Omega^{-1}$	WATT/sr	(1), (10) of Sec. 2.9
VECTOR RADIANT INTENSITY	**J**	$\mathbf{P}^{\pm}\Omega^{-1}$	WATT/sr	(22) of Sec. 2.9
SCALAR RADIANT INTENSITY	j	$\mathbf{P}^{\pm}\Omega^{-1}$	WATT/sr	See note (d) below

TABLE 3 (Continued)

NAME	BASIC SYMBOL	DIMENSIONS	MKS UNITS	DEFINITION REFERENCES
RADIANT ENERGY	U	$P^{\pm}T$	WATT-SECOND or JOULE	(12), (17) of Sec. 2.7
RADIANT DENSITY	u	$P^{\pm}TV^{-1}$	WATT-SECOND/m^3 or JOULE/m^3	(2) of Sec.2.7
(RADIANT) PATH FUNCTION	N_*	$P^{\pm}V^{-1}\Omega^{-1}$	WATT/(m^3×sr) or HERSCHEL/m	(2) of Sec.3.12 (8) of Sec.3.14 (3) of Sec.13.3
PATH RADIANCE	N^*	$P^{\pm}A^{-1}\Omega^{-1}$	WATT/(m^2×sr) or HERSCHEL	(1) of Sec.3.12 (15) of Sec.3.12 (2) of Sec.13.3

Explanatory Notes for Table 3

(a) The names and basic symbols are drawn, as far as possible, from the current standard in nomenclature, namely that recommended in 1953 by the American Standards Association Section Committee Z-58, sponsored by the Optical Society ([4], [49], also cf., p. 229, Ref. [50]). The basic symbols are used to construct names for various radiometric functions by placing various modifiers after them. Thus, e.g., $\Phi(S,D,t,F)$ is the value of the function Φ which assigns to each set F of frequencies the radiant flux incident on collecting surface S through the set D of directions at time t. Further examples are found throughout the preceding sections of this chapter. It might be well to observe here that the symbols and names for the concepts in such a venerable subject as geometrical radiometry are still in a state of change. However, there is currently some effort being made in the direction of establishing an international standard of terminology in radiometry and photometry (see, e.g., Ref. [130]). It may be noted that the terminology and notation listed in Tables 3 and 4 have withstood the severe tests of use in courses and research studies by the author and his colleagues over the past twenty years, and have been found adequate for the purposes of radiative transfer studies in natural optical media. (See also p. 6, [177].)

It now appears possible to attain a systematic and basic terminology for radiometry and photometry by combining the best features of Table 3 and Table 4 (below) and the suggestions by Jones in Ref. [130]. Toward this end we observe that

Jones extracts the idea of *flux*, as the basic concept whose task is to describe the flow of a generalized 'substance'. The 'substance' may be radiant energy, luminous energy (the photometric counterpart to radiant energy) or even entropy. The suggested term for the 'ometry' which studies general flows is 'phluometry' ("phluo" = "to flow"). There are five such 'ometries' suggested at present:

Name of the Phluometry	Phluometric Modifier	Unit of Flux
Radiometry	Radiant	Watt
Photometry	Luminous	Lumen
Ergometry	Energic	Joule
Ergophotometry	Ergolumic	Lumen-second
Entropometry	Entropic	Watt/degree

(b) The basic radiant flux dimensions P^+, P^- are associated with flux leaving and incident on a surface, respectively. The idea of 'radiant flux' is the central physical idea of geometrical radiometry. However, it is found useful in theory and practice to distinguish between emitted and incident radiant flux. This distinction has been placed into the dimensions for appropriate use, if needed, and its geometrical significance is summarized in Fig. 2.47. (See also Fig. 2.12.) If the distinction is not needed, or is understood, the occurrences of "+" or "-" may be omitted. Further discussion of dimensions is made in note (h).

(c) The spectral radiant flux P has units of WATT/mμ if wavelengths in millimicrons (mμ) are used, or has units of

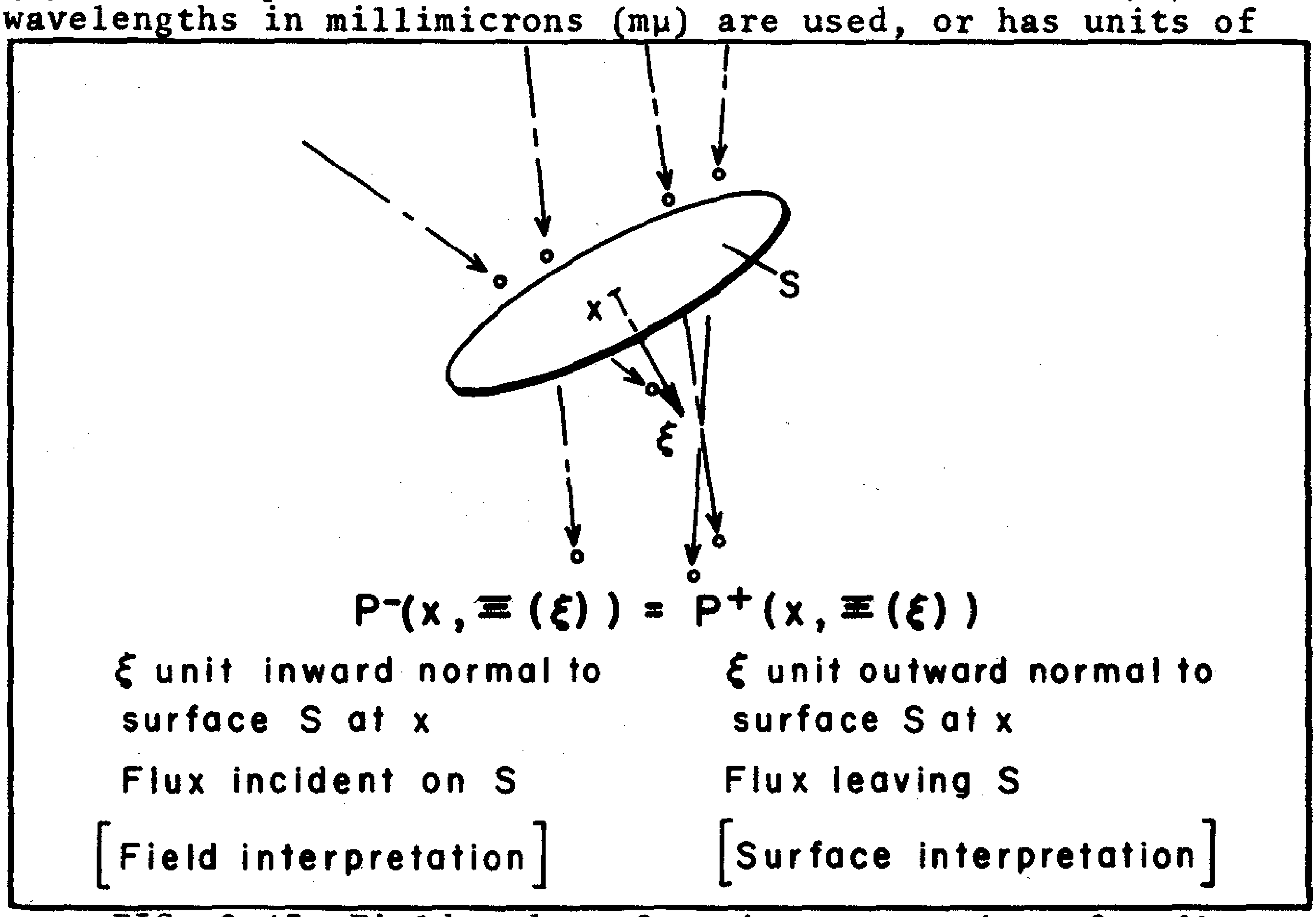

FIG. 2.47 Field and surface interpretation of radiant flux.

WATT/sec^{-1}, if frequency is in units of sec^{-1}. The dimensions "WATT/mμ" are used often in practice; hence their inclusion in the table. The radiant flux dimension **P** of any radiometric quantity below P in the table can be either spectral (hence $\mathbf{P}_\lambda$) or general (hence **P**). For simplicity, only the general radiant flux dimension is given. When working with spectral radiant flux, it is occasionally necessary to explicitly use, during a given discussion, both wavelength and frequency dimensions for radiant flux. The radiometric quantities can then be given a "λ" or a "ν" subscript for the duration of such discussions. In general, however, such explicitness is not needed and the dimension of the spectral flux is understood implicitly, and (except for specific numerical examples) will so be understood throughout this work. In theoretical radiative transfer discussions, e.g., the frequency dimension is usually preferred over wavelength (and this preference is implicit in the notation) because frequency of radiation is invariant along a path with variable index of refraction. The general (definitional) connection between P_ν and P_λ is obtained by writing:

$$\text{"}P_\nu\text{"} \quad \text{for} \quad \frac{d\Phi_\nu}{d\nu}$$

$$\text{"}P_\lambda\text{"} \quad \text{for} \quad \frac{d\Phi_\lambda}{d\lambda} \quad .$$

Whence:

$$P_\lambda = P_\nu \left|\frac{d\nu}{d\lambda}\right| = \frac{\nu}{\lambda^2} P_\nu \quad . \tag{32}$$

(d) Table 3 is divided into five natural groupings of concepts. First in order are the three main concepts--Φ, P, N. Then comes the irradiance group, the radiant emittance group, and the radiant intensity group. These are followed by the energy group, and the radiative transfer group consisting of N_* and N^*. In principle, the irradiance group and the radiant emittance groups may be coalesced into a single group by using explicitly the surface (+) and (-) concepts. However, historical precedent has fixed the distinction between these groups by means of the generic letters "H" and "W", and we see no reason at the present time to change such established notation to "H^+" for W and "H^-" for H merely on the grounds of esthetic reasons. However, esthetic reasons (in particular the desire for symmetry) are responsible for the inclusion of two concepts in Table 3 which--if the practical photometric worker had a say--would normally be omitted. These are the two concepts **W** and j. The distinction between **W** and **H** is very fine conceptually and non-existent vectorially. For we define **W**(x) as follows. We write:

$$\text{"}\mathbf{W}(x)\text{"} \quad \text{for} \quad \int \xi N^+(x,\xi)\, d\Omega(\xi) \tag{33}$$

where, as noted, the integral uses the surface radiance. We call **W**(x) the *vector radiant emittance* at x. Thus (33), by

(32) of Sec. 2.5, and (2) of Sec. 2.8, yields the equality $W(x) = H(x)$. Further, whenever S is a surface and $N(x)$ is a uniform radiance distribution at x on S (either field or surface) then we may write:

$$\text{"}j(S)\text{"} \quad \text{for} \quad \int_S N(x)\, dA(x) \tag{34}$$

and call j(S) the *scalar radiant intensity*. By including (33) and (34) we round out each of the radiant emittance and radiant intensity families to a full threesome, and maintain the interesting duality between irradiance and radiant intensity brought out in the main discussions above (where j(S) is now the dual to h(x)).

(e) The unit name "herschel" for radiance is adopted from a suggestion by Moon (Ref. [184]). However, the unit, as used here, is left "unrationalized." This means simply that for uniform radiance distribution, we have $H = \pi N$. Hence if $N = 1$ herschel, then $H = \pi$ watts/m^2, and numerical computations are not jeopardized by not remembering what to do with "π". Furthermore, the π serves to keep tabs on the dimensions of H and N in calculations. It is clear that something other than the relatively lengthy "WATT/(m^2 × sr)" is desirable, at least in verbal discussions, where "sr" stands for "steradian", and "m^2" as usual denotes "square meters".

(f) The final group consisting of path radiance N^* and path function N_* is included for convenience of reference. These are the only two additional radiometric concepts needed in the general studies of radiative transfer in natural optical media. Actually these concepts are mutually dependent and only one is needed. The full discussion of this matter is reserved for Chapter III.

(g) The only radiometric concepts omitted from Table 3 and which are of some importance, are the spherical and hemispherical scalar irradiances defined in Sec. 2.7. These concepts, especially the latter, are primarily indigenous to plane-parallel (or one parameter) geometries, whereas all the listed concepts pertain to general geometries. Not defined at all were the spherical and hemispherical scalar emittances. For the sake of completeness (cf., (6) of Sec. 2.7), we write:

$$\text{"}w_{4\pi}(x,t)\text{"} \quad \text{for} \quad \frac{1}{4}\, w(x,t) \tag{35}$$

and (cf. (7) of Sec. 2.7):

$$\text{"}w(x,\xi,t)\text{"} \quad \text{for} \quad \int_{\Xi(\xi)} N(x,\xi',t)\, d\Omega(\xi') \tag{36}$$

and (cf. (8) of Sec. 2.7):

$$\text{"}w_{4\pi}(x,\xi,t)\text{"} \quad \text{for} \quad \frac{1}{4}\, w(x,\xi,t) \tag{37}$$

and these are called, respectively, *spherical radiant* *emittance, hemispherical scalar radiant emittance,* and *hemispherical radiant emittance.*

(h) The theory of dimensions of radiometric and photometric concepts has received relatively little systematic attention. We shall devote a few comments to this matter in the present note. The dimensional system chosen for Tables 3 and 4 is constructed from two basic physical dimensions and one basic geometrical dimension. These are the dimensions of radiant flux **P**, time **T**, and length **L**. The general radiant flux function Φ is assigned the dimension **P**; *this dimension is considered irreducible in the radiometric context.* In other contexts, **P** need not be irreducible. Thus, in the electromagnetic context **P** is representable in terms of the dimensions of force, length and time: (force) × (length) × (time)$^{-1}$, or as (mass) × (length)^{2} × (time)$^{-3}$. The "+" and "-" superscripts on "**P**" do not change its dimension; they merely serve as convenient mnemonics for the surface and field interpretations of radiant flux.

As already made clear in note (c) above, the dimensions $\mathbf{P}_\nu$ or $\mathbf{P}_\lambda$ are reducible to **PT** or $\mathbf{PL}^{-1}$, respectively. Specifically, in Table 3 we have implicitly written:

$$"\mathbf{P}_\nu^{\pm}" \quad \text{for} \quad \mathbf{P}^{\pm}\mathbf{T}$$

and for the wavelength case we have explicitly written:

$$"\mathbf{P}_\lambda^{\pm}" \quad \text{for} \quad \mathbf{P}^{\pm}\mathbf{L}^{-1} \quad .$$

Now just as we find it convenient to append "+" and "-" to the basic symbol "**P**" to denote the geometric sense of the flow of radiant flux, so too is it helpful to distinguish between two types of length in geometrical radiometric discussions. Following Moon [184], we write: "$\mathbf{L}_t$" to denote the dimension of length measured in a direction transverse (i.e., perpendicular) to a given direction ξ; and "$\mathbf{L}_r$" to denote the dimension of length along the given (radial) direction ξ. As in the case of $\mathbf{P}^{\pm}$, attaching "t" and "r" to "**L**" does not change the dimension; rather it serves as a conceptual reminder of the transverse and radial interpretations of length. Then in the table we have written:

$$"\mathbf{A}" \quad \text{for} \quad \mathbf{L}_t^{2}$$

$$"\mathbf{V}" \quad \text{for} \quad \mathbf{L}_t^{2}\mathbf{L}_r$$

$$"\mathbf{\Omega}" \quad \text{for} \quad \mathbf{L}_t^{2}\mathbf{L}_r^{-2}$$

Thus in the present dimensional system, area has the dimensions of transverse length squared--a most natural dimension within radiometry since we perceive areas as two-dimensional extensions of space in the transverse directions to a line of sight. Volume has dimensions of $\mathbf{AL}_r$, i.e., (transverse) area times (radial) length--again a most natural combination of dimensions for the radiometrist. Finally, solid angles are measured using the steradian concept. In the present system the dimension of solid angles does not vanish from view, but rather is expressed as the product of $\mathbf{L}_t^2$ and $\mathbf{L}_r^{-2}$, as indicated above. Since $\mathbf{L}_t$ and $\mathbf{L}_r$ are conceptually distinct, this product is conceptually not dimensionless. In this way the occasionally bothersome problem of the vanishing dimensions of solid angle can be solved. (Ordinarily the dimensions vanish, but like the smile of the Cheshire cat, the units remain.) It should be noted that the conventional dimensions of the radiometric concepts are recovered by dropping "+", "-", "t", and "r" from **P** and **L** wherever they occur.

As an illustration of the use of these dimensions, observe that the dimension of path function can be written as $(\mathbf{P}^{\pm}\mathbf{A}^{-1}\Omega^{-1})\mathbf{L}_r^{-1}$, so that the path function concept is seen to have the dimensions of radiance per unit of radial length. The full significance of this interpretation will become clear in Sec. 3.12, wherein the path function concept is formally introduced. On the other hand, we may rearrange the path function dimensions as follows: $(\mathbf{P}^{\pm}\Omega^{-1})\mathbf{V}^{-1}$, and thereby discern another facet of this concept, namely that it may be viewed as a radiant intensity per unit volume (cf. (7), (10) of Sec. 13.6). The radiance concept itself may be viewed via the dimensional arrangement $(\mathbf{P}^{\pm}\mathbf{A}^{-1})\Omega^{-1}$ as irradiance (-) or radiant emittance (+) per unit solid angle on the one hand, and via the arrangement $(\mathbf{P}^{\pm}\Omega^{-1})\mathbf{A}^{-1}$ as field (-) or surface (+) radiant intensity per unit area, on the other hand.

A general guide to the fixing of dimensions of radiometric concepts and their manifold derivates in practice is as follows. Let us refer to "area", "length", "time", etc. by the generic term "measure", and use the generic symbol "m" for a measure. Let us write "dim(m)" for the dimension of m. Thus if A is an area measure, then A(S) is the area of a surface S, and dim(A) = $\mathbf{L}_t^2$. Further, if l is a length measure along paths of sight, then l(p) is the length of a path p, and dim(l) = $\mathbf{L}_r$. Now, according to our development of geometrical radiometry in this chapter, every radiometric concept $\mathscr{R}$ is definable first on the empirical level and then on the theoretical level. The empirical level of definition is simply the level on which the measures are used directly. Thus, e.g., recall that empirical irradiance H(S,D) is P(S,D)/A(S), i.e., the quotient of incident radiant flux over a surface S by the area of S. The corresponding theoretical definition is obtained by going to the appropriate limit (e.g., S→{x} in the case of irradiance). In going from the empirical level to the theoretical level, it is desirable to have the dimensions remain unchanged. Hence the definition on the empirical level already fixes the dimension of a radiometric concept. Suppose then that $\mathscr{R}$ is a radiometric concept and its empirical definition is such that we write:

$$\text{"}\mathscr{R}\text{"} \quad \text{for} \quad \frac{\Phi m_1' \ldots m_a'}{m_1 \ldots m_b}$$

where "m_i'", i = 1,...,a, and "m_j", j = 1,...,b, denote measures and "Φ" denotes the radiant flux function, which is also a measure with dimension dim(Φ) = **P**. Then the *dimension of*

ℛ is:

$$\frac{(\dim \Phi) \times \dim(m_1') \times \ldots \times \dim(m_a')}{\dim(m_1) \times \ldots \times \dim(m_b)} .$$

The preceding reduction of a dimension to simpler terms is facilitated by adopting the following conventions for the dimention operator dim. Let x and y be any two measures or physical concepts. Then:

(i) $\dim(xy) = \dim(x) \dim(y)$

(ii) $\dim(x/y) = \dim(x)/\dim(y)$

(iii) If $\dim(x) = \dim(y)$, then $\dim(x) = \dim(x+y)$

(iv) If $\{x_n\}$ is a sequence of terms of common dimension d, and if $\lim_n x_n = y$, then $\dim(y) = d$.

In our development of radiometry, the basic dimensions are **P**, **L**, and **T**. In order to use rules (i)-(iv), we agree that these dimensions obey the same rules of addition and multiplication as real numbers. This is implicitly assumed in the tables and in the various manipulations above. In addition to the four dimensions above, we introduce one more, namely **1**, which has the property that:

$$\mathbf{d1} = \mathbf{1d} = \mathbf{d}$$

for every dimension **d**, and

$$\mathbf{d}_1/\mathbf{d}_2 = \mathbf{1}$$

for every pair of dimensions $\mathbf{d}_1$ and $\mathbf{d}_2$ such that $\mathbf{d}_1 = \mathbf{d}_2$. Thus "**1**" denotes the dimensionless concept.

Explanatory Notes for Table 4

(a) The notes and comments for Table 3 apply also to this table except where explicit references to frequency or wavelength concepts are made. Observe that Tables 3 and 4 correspond item for item, except that there is naturally no luminous counterpart to the general radiant flux function Φ, the primitive radiometric function from which all others spring. The unit of luminance, the (unrationalized) *blondel*, is adapted from a suggestion by Moon (ref. [184]). The luminous counterparts to (35)-(37) are obtained by means of the general definition scheme of (27). In (35) and (37) "w" is replaced by "l", and "radiant" replaced by "luminous", to effect the definitions. We assign to the lumen the basic dimension **F**. Hence, in particular, $\dim(K_m) = \mathbf{FP}^{-1}$. By (15) and property (iv) of the operator dim in note (h) for Table 3, we have, e.g., $\dim(F^{\pm}(S,D,t)) = \mathbf{F}^{\pm}$. In this case, the limit

TABLE 4

PHOTOMETRIC CONCEPTS

NAME	BASIC SYMBOL	DIMENSIONS	MKS UNITS	DEFINITION REFERENCES
LUMINOUS FLUX	F	$\mathbf{F}^{\pm}$	LUMEN	(15)
LUMINANCE	B	$\mathbf{F}^{\pm}\mathbf{A}^{-1}\Omega^{-1}$	LUMEN/($m^2 \times$ sr)	(18)
ILLUMINANCE	E	$\mathbf{F}^{-}\mathbf{A}^{-1}$	LUMEN/m^2	(16)
VECTOR ILLUMINANCE	**E**	$\mathbf{F}^{-}\mathbf{A}^{-1}$	LUMEN/m^2	(2) of Sec. 2.8 and (27)
SCALAR ILLUMINANCE	e	$\mathbf{F}^{-}\mathbf{A}^{-1}$	LUMEN/m^2	(24)
LUMINOUS EMITTANCE	L	$\mathbf{F}^{+}\mathbf{A}^{-1}$	LUMEN/m^2	(17)
VECTOR LUMINOUS EMITTANCE	**L**	$\mathbf{F}^{+}\mathbf{A}^{-1}$	LUMEN/m^2	(33), (27)
SCALAR LUMINOUS EMITTANCE	l	$\mathbf{F}^{+}\mathbf{A}^{-1}$	LUMEN/m^2	(25)
LUMINOUS INTENSITY	I	$\mathbf{F}^{\pm}\Omega^{-1}$	LUMEN/sr or CANDELA	(19)
VECTOR LUMINOUS INTENSITY	**I**	$\mathbf{F}^{\pm}\Omega^{-1}$	LUMEN/sr or CANDELA	(22) of Sec. 2.9 and (27)
SCALAR LUMINOUS INTENSITY	i	$\mathbf{F}^{\pm}\Omega^{-1}$	LUMEN/sr or CANDELA	(34), (27)
LUMINOUS ENERGY	Q	$\mathbf{F}^{\pm}\mathbf{T}$	LUMEN-SECOND or TALBOT	(21), (22)
LUMINOUS DENSITY	q	$\mathbf{F}^{\pm}\mathbf{T}\mathbf{V}^{-1}$	LUMEN-SECOND/m^3 or TALBOT/m^3	(23)
(LUMINOUS) PATH FUNCTION	B_*	$\mathbf{F}^{\pm}\mathbf{V}^{-1}\Omega^{-1}$	LUMEN/($m^3 \times$ sr) or BLONDEL/m	(2) of Sec. 3.12 and (27)
PATH LUMINANCE	B^*	$\mathbf{F}^{\pm}\mathbf{A}^{-1}\Omega^{-1}$	LUMEN/($m^2 \times$ sr) or BLONDEL	(1) of Sec. 3.12 and (27)

operation is that used in the definition of the integral operator (13).

We conclude this discussion of the photometric concepts with a few examples.

Example 1. Using the luminance of the sun as given in Table 2, compute the corresponding illuminance on a plane normal to the rays of the sun. To find the requisite illuminance, recall from (2) of Sec. 2.5 that we can write:

$$H(S,D) = N(S,D)\Omega(D) \quad .$$

Applying the transition operator $Y(\cdot,\Lambda)$, as defined in (13), to each side of this equation, we obtain:

$$Y\big(H(S,D),\Lambda\big) = Y\big(N(S,D)\Omega(D),\Lambda\big)$$

$$= \Omega(D)\ Y\big(N(S,D),\Lambda\big)$$

Using (16) and (18) and the general definition scheme (27) to define the empirical counterparts of radiance and irradiance the preceding equation yields:

$$E(S,D) = B(S,D)\Omega(D) \quad ,$$

which is the desired connection between *empirical luminance* and *empirical illuminance*.

From Table 2,

$$B(S,D) = 2\times 10^9 \text{ blondels or candelas/m}^2$$
$$\text{or lumens/m}^2\times\text{sr}$$

and from Example 1 of Sec. 2.11:

$$\Omega(D) = 6.78\times 10^{-5} \quad \text{steradians}$$

Hence*:

$$E(S,D) = 2\times 10^9 \times (6.78\times 10^{-5})$$

$$= 136{,}000 \text{ lumens/m}^2$$

Example 2. If the sun in the context of Example 1 is at $\theta = 50°$ from the zenith, and surface S' is the projection on a horizontal plane of the surface S used in Example 1, what is the illuminance E(S',D) produced by the sun's rays on S'? To find this illuminance, recall (15) of Sec. 2.4:

$$H(S',D) = H(S,D)\cos\vartheta \quad ,$$

where the symbols are explained in detail in Sec. 2.4, and

*A relatively recent estimate (Ref. [128]) of E(S,D) is 136,000 lumens/m^2. See also [296] for a survey of measurements of the solar constant.

which now apply directly to the present context. In particular now $\vartheta = 50°$, and D is the cone of directions subtended by the sun's disk. Applying the operator $Y(\cdot,\Lambda)$, as defined in (13) to this equation, we have:

$$Y\big(H(S',D),\Lambda\big) = Y\big(H(S,D)\cos\vartheta,\Lambda\big) \quad .$$

Using the definition scheme (27) now for empirical illuminance, this becomes:

$$E(S',D) = E(S,D)\cos\vartheta \quad .$$

Hence:

$$\begin{aligned} E(S',D) &= 136{,}000 \times \cos 50° \\ &= 136{,}000 \times .643 \\ &= 87{,}500 \text{ lumens/m}^2 \quad . \end{aligned}$$

Example 3. Using the illuminance of Example 1, compute the illuminance produced by the sun's rays normally incident on surfaces in the orbits of the planets Venus and Mars. The requisite illuminances can be found by means of the inverse square law for irradiance deduced in Example 4 of Sec. 2.11. Thus, from (8) of Sec. 2.11, if r and s are two distances from the sphere's center, we deduce that:

$$H_r r^2 = H_s s^2 \quad .$$

Using the operator $Y(\cdot,\Lambda)$ and the definition scheme (27), this equation becomes:

$$E_r r^2 = E_s s^2 \quad .$$

Let E_r be the illuminance at the earth as given in the form E(S,D) in Example 1. Hence $r = 93 \times 10^6$ miles. In the case of Venus, $s = 67 \times 10^6$ miles. Hence:

$$\begin{aligned} E_s &= E_r (r/s)^2 \\ &= 136{,}000 \times (93/67)^2 \\ &= 136{,}000 \times 1.93 \\ &= 262{,}000 \text{ lumens/m}^2 \end{aligned}$$

In the case of Mars, $s = 142 \times 10^6$ miles. Hence:

$$\begin{aligned} E_s &= E_r (r/s)^2 \\ &= 136{,}000 \times (93/142)^2 \\ &= 136{,}000 \times .430 \\ &= 58{,}500 \text{ lumens/m}^2 \end{aligned}$$

Example 4. Compute the number of lumens F incident on a plane surface S of area A(S), every point of which is illuminated by a luminance distribution of constant magnitude B incident over directions within a conical solid angle D of half angle θ and whose axis is normal to S. The requisite relation for the lumens incident on S is obtained by beginning with (14) in Example 6 of Sec. 2.11:

$$H = \frac{N}{2} \int_0^{2\pi} \sin^2\theta(\phi)\, d\phi \quad ,$$

and applying the operator Y(·,Λ) to each side to get:

$$Y(H,\Lambda) = H\left(\frac{N}{2} \int_0^{2\pi} \sin^2\theta(\phi)\, d\phi, \Lambda\right)$$

$$= \frac{1}{2}\left(\int_0^{2\pi} \sin^2\theta(\phi)\, d\phi\right) Y(N,\Lambda) \quad ,$$

which, via the definition scheme (27), can be written:

$$E = \frac{B}{2} \int_0^{2\pi} \sin^2\theta(\phi)\, d\phi \quad .$$

In the present case, $\theta(\phi) = \theta$ for every ϕ, $0 \le \phi \le 2\pi$. Hence:

$$E = \pi B \sin^2\theta \quad .$$

Next, from (6) of Sec. 2.4 we have:

$$P(S,D,t,\nu) = H(S,D,t,\nu)\, A(S) \quad .$$

Applying the operator Y(·,Λ), to each side of this equation, we have:

$$Y\big(P(S,D,t,\cdot),\Lambda\big) = Y\big(H(S,D,t,\cdot)A(S),\Lambda\big) \quad .$$

From (15) and the definition scheme (27) applied to empirical irradiance, we consequently have:

$$F(S,D,t) = E(S,D,t)A(S) \quad .$$

Considering references to S,D, and t as understood for the present discussion, we distill this to:

$$F = EA \quad .$$

Thus we are led to the desired relation:

$$F = \pi BA \sin^2\theta \quad .$$

As a specific example, let $\theta = 30°$, $B = 120$ blondels, and $A = 4\ m^2$. Then:

$$F = (3.14) \times (120) \times 4 \times (1/2)^2$$

$$= 378 \text{ lumens.}$$

Example 5. A red-orange appearing filter is known to have a band pass of 10 mμ, but it is not known precisely what wavelengths of radiant flux it transmits. An experiment is suggested and tried in which it is inferred that an irradiance of 2 watts/m^2 over the transmission interval $\Delta\lambda = 10$ mμ produces an illuminance of 1360 lumens. Can the transmission wavelengths of the filter also be inferred from this information? To answer this, consider the following observations.

From the definitional identity:

$$E = Y(H,\Lambda)$$

$$= K_m \int_0^\infty H(\lambda)\bar{y}(\lambda)\, d\lambda$$

and the fact that $H(\lambda) = 0$ outside the interval $\Delta\lambda$ about the unknown λ, we have very nearly:

$$E = K_m H(\lambda)\bar{y}(\lambda)\Delta\lambda \quad .$$

Hence:

$$\bar{y}(\lambda) = E/\big(K_m H(\lambda)\Delta\lambda\big)$$

$$= 1360/(680 \times 2 \times 10)$$

$$= 1/10 = .10 \quad .$$

From Table 1, by linear interpolation, we infer that $\lambda = 472$ mμ or 652 mμ. From the given general appearance of the filter's color, we infer that $\lambda = 652$ mμ.

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2.13 Generalized Photometries

We conclude this chapter with a few observations on the necessary forms of certain generalized photometries which arise in an attempt to extend the salient ideas of classical photometry. The directions of extension to which we subject the ideas of photometry in this discussion are toward a more general class of 'luminosity' functions. The class we envision here is to contain not only the classical luminosity functions of human eyes, as briefly discussed in 2.11, but also irradiation-response functions describing photographic, phototransmissive, photovoltaic, photoemissive, and photocurrent phenomena. In short, we attempt to sketch in broad terms certain possible generalizations of the 'lumen' concept