Unless specifically noted otherwise, we shall henceforth mean by "optical medium" any three-dimensional part X of Euclidean three-dimensional space. This then will automatically set the dimensionality of S in the various interpretations of the interaction principle. (A formal definition of optical media, as they are studied in radiative transfer theory, is given in Sec. 9.1.)

3.3 Reflectance and Transmittance Operators for Surfaces

In this section we begin the sequence of constructions of the concepts needed for the description of the manifold radiative transfer phenomena encountered in the practice of radiative transfer theory. In particular in this section we shall use the interaction principle as a base for the construction of the more commonly used surface reflectance and transmittance concepts. Some work has already been done in this direction in Sec. 3.1. In fact the empirical reflectance function was defined in that section as a necessary prerequisite for the construction of the preliminary example of the interaction principle. We now return to that setting for the purpose of establishing systematic definitions for the family of reflectance and transmittance operators for surfaces.

Geometrical Conventions

Figure 3.3 (a) depicts a general surface Y in an optical medium X and a relatively small part S of Y about point x on Y. We are interested in the reflectance and transmittance of Y in the region S about x. Now the terms "transmittance" and "reflectance" become meaningful only after adequate reference frames have been established at given points x of Y within which one can unambiguously establish conventions about the notions of "inwardness", "outwardness", "upwardness", "downwardness", "forwardness", "backwardness", etc. Suppose then we affix to point x of Y a unit vector $\mathbf{k}(x)$ and call it the *unit outward normal* to Y at x. Perhaps some readers would prefer to call $-\mathbf{k}(x)$ the unit outward normal to Y at x. This is perfectly admissible for our present purposes, and the reader may therefore turn around the arrows in parts (a)-(d) of Fig. 3.3 and read the following discussion as it stands. The point being made here is that what one calls "outward", etc., is immaterial. What does matter is what one subsequently does with the concept and that, within a given discussion, a measure of consistency is sustained in the use of the concept once the convention is made.

During the present discussion, let "D'" and "D" denote narrow circular conical solid angles of central directions ξ' and ξ, respectively. S is a small collecting surface on Y, and x is a point of Y in S. Let "S'" denote the projection of S on a plane normal to ξ'. (See parts (c) and (d) of Fig. 3.3.) D' is the set of *incident directions*; D is the set of *response directions*. Both D' and D will always lie completely within $\Xi_+(\mathbf{k}(x))$ or $\Xi_-(\mathbf{k}(x))$ where $\Xi_+(\mathbf{k}(x))$ is the set of all directions ξ' such that $\xi'\cdot\mathbf{k}(x) > 0$, and $\Xi_-(\mathbf{k}(x))$ is the set

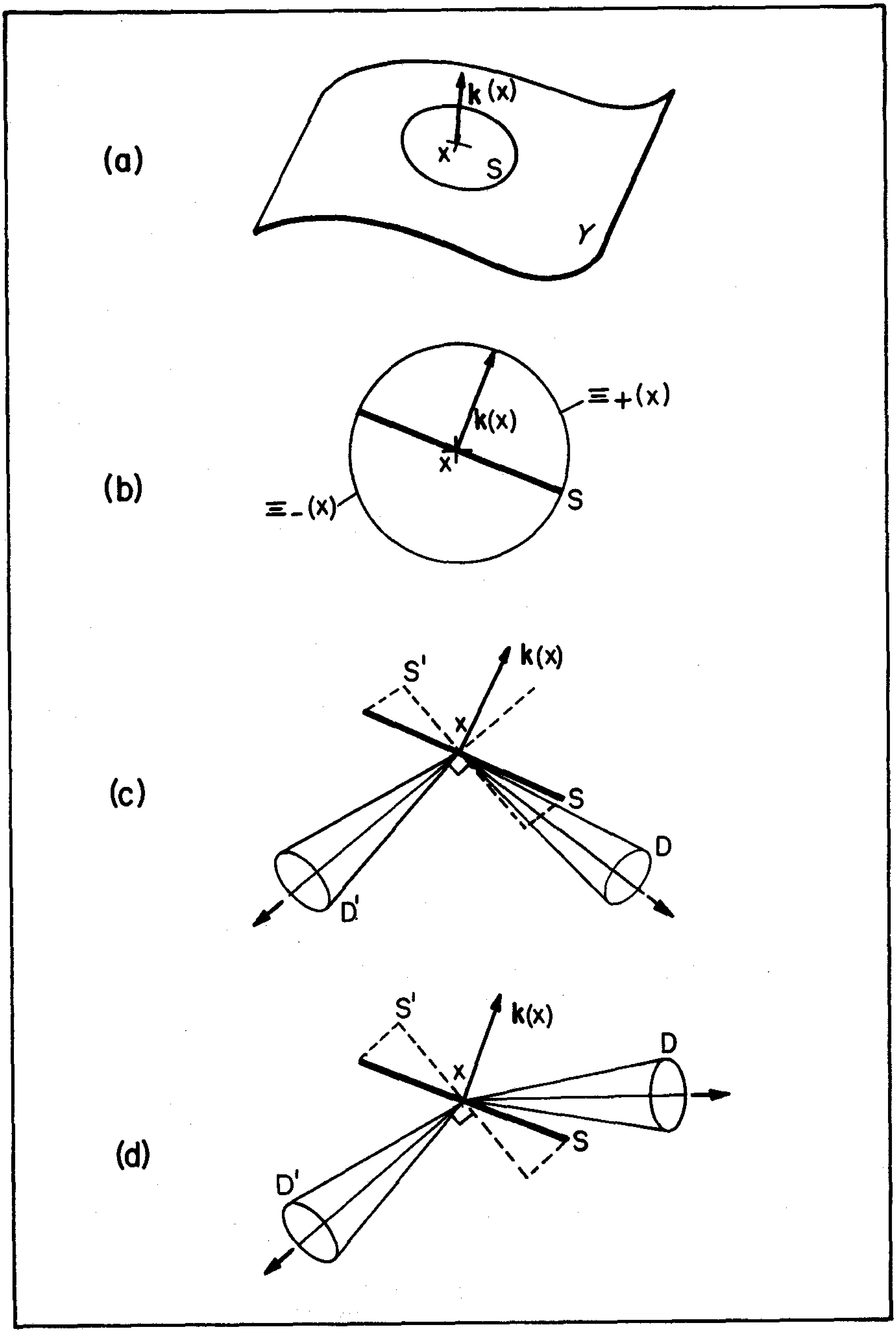

FIG. 3.3 Setting for reflectance and transmittance operators for surfaces.

of all directions ξ' such that $\xi' \cdot \mathbf{k}(x) < 0$. (See part (b) of Fig. 3.3 and compare with Sec. 2.4, so that $\Xi_+(\mathbf{k}(x)) = \Xi(\mathbf{k}(x))$ and $\Xi_-(\mathbf{k}(x)) = \Xi(-\mathbf{k}(x))$. We shall also write for brevity:

$$\text{"}\Xi_\pm(x)\text{"} \quad \text{for} \quad \Xi_\pm(\mathbf{k}(x)) \quad .$$

The notation "$\Xi_\pm(x)$" finds its best use when specific surfaces are under consideration, while the notation "$\Xi(\pm\mathbf{k}(x))$" finds its greatest use when (as in Sec. 2.4) purely radiometric arguments are in effect as no specific surfaces are being discussed.

The Empirical Reflectances and Transmittances

With these preliminaries established we can define with some measure of precision the empirical reflectance and transmittance function. Emulating (1) of Sec. 3.1 we write:

$$\text{"}s(S',D';S,D)\text{"} \quad \text{for} \quad \frac{N(S',D';S,D)}{N(S',D')\Omega(D')} \tag{1}$$

where all terms on the right side of the definition are as described in Sec. 3.1, but now with x, ξ', ξ, S', D', S, and D as specified above. The notation in (1) does not tell us specifically on which side of S the sets D' and D lie. By specifying this information, the values $s(S',D';S,D)$ take on the characteristics of reflectances and transmittances. Thus let us write:

$$\text{"}r_+(S',D';S,D)\text{"} \quad \text{for} \quad s(S',D';S,D), \text{ if } D' \subset \Xi_+(x) \text{ and } D \subset \Xi_-(x) \tag{2}$$

$$\text{"}r_-(S',D';S,D)\text{"} \quad \text{for} \quad s(S',D';S,D), \text{ if } D' \subset \Xi_-(x) \text{ and } D \subset \Xi_+(x) \tag{3}$$

$$\text{"}t_+(S',D';S,D)\text{"} \quad \text{for} \quad s(S',D';S,D), \text{ if } D' \subset \Xi_+(x) \text{ and } D \subset \Xi_+(x) \tag{4}$$

$$\text{"}t_-(S',D';S,D)\text{"} \quad \text{for} \quad s(S',D';S,D), \text{ if } D' \subset \Xi_-(x) \text{ and } D \subset \Xi_-(x) \tag{5}$$

Here "$D' \subset \Xi_+(x)$" is an inclusion statement which means that D' is contained in $\Xi_+(x)$. Similar interpretations hold for the other three inclusion statements. For example, part (c) of Fig. 3.3 depicts the geometrical arrangement for $t_-(S',D';S,D)$, and part (d) of Fig. 3.3 depicts the arrangement for $r_-(S',D';S,D)$. Definitions (2) and (4) cover the *outward* (or *upward* or *forward*) empirical *reflectance* and *transmittance* of Y over S. Properties (i) and (ii) in Sec. 3.1 hold for the $r_\pm$ and $t_\pm$ just defined.

We could have arrived at the preceding four empirical reflectance and transmittance functions just above by direct appeal to the interaction principle. Thus, with X and S as given, let $m = n = 1$ and A be the set of all outward directed incident radiances $N(S',D')$ on S (i.e., D' contained in $\Xi_+(x)$), and let B be the set of all inward directed response radiances of S (i.e., D contained in $\Xi_-(x)$). Then the interaction principle asserts the existence of a linear interaction operator s_{11}--call it "$r_+(\cdot,\cdot;\cdot,\cdot)\Omega(\cdot)$"--with the property that for every N(S',D') in A there exists an N(S,D) in B such that:

$$N(S,D) = N(S',D')r_+(S',D';S,D)\Omega(D') \quad .$$

Hence the interaction operator in this instance is a real valued function of four variables (S',D',S,D) which assigns to each choice of these variables a number--the reflectance of Y over S under the indicated conditions. If instead of incident radiances, we chose incident scalar irradiances over D' for the set A, then $r_+(\cdot,\cdot;\cdot,\cdot)$ itself would have been obtained. If we had chosen incident irradiances instead, then $r_+(\cdot,\cdot;\cdot,\cdot)/\xi'\cdot\mathbf{k}$ would have been obtained. This shows the potential flexibility of the principle in supplying a great variety of "reflectances", depending on what set of radiometric quantities are chosen for A and for B.

The Theoretical Reflectances and Transmittances

By letting S approach $\{x\}$, D' approach $\{\xi'\}$, and D approach $\{\xi\}$ in the limit, definitions (2)-(5) yield definitions of the corresponding *theoretical reflectances and transmittances* of Y at x. Thus by performing the indicated limit operations, we arrive at:

$$r_+(x;\xi';\xi) \quad \text{if} \quad \xi'\varepsilon\,\Xi_+(x) \quad \text{and} \quad \xi\varepsilon\,\Xi_-(x) \tag{6}$$

$$r_-(x;\xi';\xi) \quad \text{if} \quad \xi'\varepsilon\,\Xi_-(x) \quad \text{and} \quad \xi\varepsilon\,\Xi_+(x) \tag{7}$$

$$t_+(x;\xi';\xi) \quad \text{if} \quad \xi'\varepsilon\,\Xi_+(x) \quad \text{and} \quad \xi\varepsilon\,\Xi_+(x) \tag{8}$$

$$t_-(x;\xi';\xi) \quad \text{if} \quad \xi'\varepsilon\,\Xi_-(x) \quad \text{and} \quad \xi\varepsilon\,\Xi_-(x) \tag{9}$$

Here "$\xi'\varepsilon\,\Xi_+(x)$" means that ξ' is a direction in $\Xi_+(x)$, etc.

It is a simple matter to show how these theoretical reflectance and transmittance functions for surfaces follow directly from the interaction principle. The technique of obtaining $r_\pm$ or $t_\pm$ is similar to that discussed in Sec. 2.13 for obtaining the generalized luminosity function $\bar{z}(\cdot)$. Specifically, we would use the interaction principle to supply a positive linear function with the property that it acts on incident radiance distributions and yields reflected or transmitted radiance distributions. Interested mathematical readers may pursue this matter further in Sec. 3.16. To develop this application of the interaction principle in the present section would be to digress too far from the chosen scope of the present discussions. We give only the results of such an excursion into measure theory. Thus, we write:

$$\text{"}r_\pm(Y)\text{"} \quad \text{for} \quad \int_{\Xi_\pm(Y)} [\;]\; r_\pm(\cdot;\xi';\xi)\,d\Omega(\xi') \tag{10}$$

$$\text{"}t_\pm(Y)\text{"} \quad \text{for} \quad \int_{\Xi_\pm(Y)} [\;]\; t_\pm(\cdot;\xi';\xi)\,d\Omega(\xi'). \tag{11}$$

These are the general *reflectance* and *transmittance* integral operators associated with an arbitrary surface Y with outward unit normal **k**(x) at each point x of Y. The domain of integration in each operator is of the form $\Xi_+(Y)$ or $\Xi_-(Y)$ and is known once x in Y is specified. Thus, if $N(x,\cdot)$ is an inward incident radiance distribution at x in Y, then:

$$\int_{\Xi_-(x)} N(x,\xi')r_-(x;\xi';\xi)\,d\Omega(\xi') \tag{12}$$

is the outward reflected radiance at x in the direction ξ in response to $N(x,\cdot)$. In general, if $N(x,\cdot)$ and $r_\pm$ and $t_\pm$ are defined over just part a of Y, then we use "$N_-(a)$" to denote inward incident or response radiance distributions over part a, and "$N_+(a)$" to denote outward incident or response radiance distributions over part a. For example, if x is a point of a and ξ is an outward direction, then $N_+(a)$ assigns to x and ξ the response radiance $N(x,\xi)$. If we let x in (12) range over all points of part a of Y, then we see that (12) defines the response function $N_+(a)$ of a. Hence $N_+(a)$ in this instance is a general reflected radiance distribution resulting from operating on $N_-(a)$ by $r_-(a)$. This fact we write in the form:

$$N_+(a) = N_-(a)r_-(a) \tag{13}$$

where we have written:

$$\text{"}N_-(a)r_-(a)\text{"} \quad \text{for} \quad \int_{\Xi_-(a)} N(\cdot,\xi')r_-(\cdot;\xi',\xi)\,d\Omega(\xi') \tag{14}$$

The radiance distribution appearing in the integral is (by noting that the range of integration is $\Xi_-(a)$) an inward radiance distribution incident on a at a general (unspecified) point. The definition (14) can be repeated for the three other general cases associated with a, namely $N_+(a)r_+(a)$, $N_-(a)t_-(a)$, $N_+(a)t_+(a)$. Equation (13) gives the integration operation an algebraic appearance, a feature which, as we shall see, is most conducive to rapid and creative manipulations during theoretical radiative transfer computations. This algebraization of radiative transfer theory is fostered by the interaction principle whose salient character is itself basically algebraic (rather than, say, analytic or geometric).

Variations of the Basic Theme

Some attention will next be given to the possible variations the preceding definitions of $r_\pm$ and $t_\pm$ may undergo as shifts are made in the choice of types of radiometric incident and response quantities. A few specific instances will suffice to show the potentially great number of variations possible.

To begin, suppose that the radiometric quantities in the incident set A are to be *irradiances* and those in the response set B to be *radiances*. Then, e.g., in the expanded rendition of (13):

$$N(x,\xi) = \int_{\Xi_-(x)} N(x,\xi')\, r_-(x;\xi';\xi)\, d\Omega(\xi') \quad ,$$

we rearrange matters so:

$$N(x,\xi) = \int_{\Xi_-(x)} N(x,\xi')\, |\xi'\cdot\mathbf{k}(x)|\, \frac{r_-(x;\xi';\xi)}{|\xi'\cdot\mathbf{k}(x)|}\, d\Omega(\xi') \qquad (15)$$

with the result that the new reflectance operator has a kernel with values of the form:

$$\frac{r_-(x;\xi';\xi)}{|\xi'\cdot\mathbf{k}(x)|} \quad . \qquad (16)$$

We shall not devise notation to cover this case or the multitude of alternate cases possible. The notation is best settled by those who must work repeatedly with the specialized concepts. A semblance of order and universality is attained in such matters, however, if some set of functions such as those defined via (6)-(9) is taken as a fixed base of operations from which to proceed to new territory.

Reflectance functions of the form displayed in (16) are used in practice where the surfaces under study are often considered ideally or nearly *uniform* (or *lambert*) *reflectors*. For suppose a surface Y at x has the property that there is a real number r_- such that:

$$\frac{r_-(x;\xi';\xi)}{|\xi'\cdot\mathbf{k}(x)|} = \frac{r_-}{\pi} \qquad (17)$$

for every ξ' in $\Xi_-(x)$ and every ξ in $\Xi_+(x)$. Then (15) becomes:

$$N(x,\xi) = \frac{r_-}{\pi} \int_{\Xi_-(x)} N(x,\xi')\, |\xi'\cdot\mathbf{k}(x)|\, d\Omega(\xi')$$

$$= \frac{r_-}{\pi}\, H(x, \Xi_-(x)) \quad .$$

Hence the reflected radiance distribution $N(x,\cdot)$ is uniform (independent of ξ) of magnitude $N(x)$, say. Then the associated radiant emittance is:

$$\begin{aligned} W(x,\Xi_-(x)) &= \pi N(x) \\ &= r_- H(x,\Xi_-(x)) \quad , \end{aligned}$$

as one would expect by the way r_- is defined. If the incident radiance distribution itself was uniform, of magnitude $N'(x)$ then

$$H(x,\Xi_-(x)) = \pi N'(x) \quad .$$

From this and the preceding equation we have:

$$N(x) = r_- N'(x) \quad ,$$

again as one would expect of the new version of the reflectance function and a lambert reflector.

As another example, suppose that the incident radiometric quantities in A are radiances and those in B are radiant emittances. Specifically, let (15) be used as starting point and operate on each side of (15) with an integration of the kind:

$$\int_{\Xi_+(x)} N(x,\xi)\,\xi\cdot\mathbf{k}(x)\,d\Omega(\xi) =$$

$$= \int_{\Xi_+(x)} \left[\int_{\Xi_-(x)} N(x,\xi')r_-(x;\xi';\xi)\,d\Omega(\xi')\right]\xi\cdot\mathbf{k}(x)\,d\Omega(\xi) \quad . \tag{18}$$

It is clear that the integral on the left yields the requisite radiant emittance $W(x,\Xi_+(x))$ (cf. (22) of Sec. 2.4) which thus is obtained by operating on the incident radiance distribution $N(x,\cdot)$ with the integral operator

$$\int_{\Xi_+(x)} \left[\int_{\Xi_-(x)} [\ \ ]r_-(x;\xi';\xi)\,d\Omega(\xi')\right]\xi\cdot\mathbf{k}(x)\,d\Omega(\xi) \quad .$$

Now it is quite natural when using irradiance and radiant emittance in this way for us to assign to the quotient

$$W(x,\Xi_+(x))/H(x,\Xi_-(x))$$

the meaning of a reflectance (an *albedo*) of the surface Y at x. Thus if we write:

"$r_-(x)$" for

$$\frac{1}{H(x,\Xi_-(x))}\cdot\int_{\Xi_+(x)}\left[\int_{\Xi_-(x)} N(x,\xi')r_-(x;\xi';\xi)\,d\Omega(\xi')\right]\xi\cdot\mathbf{k}(x)\;d\Omega(\xi)$$

then we can go on to rearrange (18) into the form:

$$W(x,\Xi_+(x)) = H(x,\Xi_-(x))r_-(x) \quad . \qquad (19)$$

This definition of $r_-(x)$ (and the three analogous definitions $r_+(x)$, $t_\pm(x)$) is motivated by the need for working with numerical irradiances and radiant emmittances, and numerical reflectances rather than the analogous functional and operatorial concepts which must be used in certain full treatments of interreflection problems. In the next section, we shall illustrate in more detail the use of (13) and (19).

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3.4 Applications to Plane Surfaces

In this section we shall illustrate the application of the reflectance and transmittance operators for surfaces, constructed in Sec. 3.3, for several types of frequently encountered plane-surface settings in radiative transfer theory. Throughout this section and, indeed, the remainder of this chapter, one of the principal goals is the demonstration of the systematic use to which the interaction principle may be put in formulating the concepts and problems of radiative transfer theory.

Example 1: Irradiances on Two Infinite Parallel Planes

Let "a" and "b" denote two infinite parallel plane surfaces separated by a vacuum, as in Fig. 3.4. The coordinate system used is the terrestrial system defined in Sec. 2.4. Each plane has assigned reflectance and transmittance functions as developed in Sec. 3.3 which are to be constant over a and b. However, the directional structures of the reflectance and transmittance functions are otherwise arbitrary. An interreflection process between a and b is initiated and sustained by a steady downward field radiance distribution $N^0_-(b)$ on plane b which has the same structure at all points of b. Our present goal is to compute the resultant steady state irradiances on a and b, that is the upward irradiance $H_+(a)$ on a and the downward irradiance $H_-(b)$ on b.

The interaction principle applied to a and b in turn yields the requisite irradiance reflectance operators. Thus for a the set A of incident radiometric functions consists of irradiances like $H_+(a)$, and the set B of response radiometric functions of a consists of downward radiant emittances $W_-(a)$ which by the hypothesized vacuum between a and b have magnitudes equal to $H_-(b)$. (Cf. also Example 12 of Sec. 2.11 showing independence of H_r with r in the case of infinite