"$r_-(x)$" for

$$\frac{1}{H(x,\Xi_-(x))}\cdot\int_{\Xi_+(x)}\left[\int_{\Xi_-(x)} N(x,\xi')r_-(x;\xi';\xi)d\Omega(\xi')\right]\xi\cdot\mathbf{k}(x)\ d\Omega(\xi)$$

then we can go on to rearrange (18) into the form:

$$W(x,\Xi_+(x)) = H(x,\Xi_-(x))r_-(x) \quad . \tag{19}$$

This definition of $r_-(x)$ (and the three analogous definitions $r_+(x)$, $t_\pm(x)$) is motivated by the need for working with numerical irradiances and radiant emmittances, and numerical reflectances rather than the analogous functional and operatorial concepts which must be used in certain full treatments of interreflection problems. In the next section, we shall illustrate in more detail the use of (13) and (19).

3.4 Applications to Plane Surfaces

In this section we shall illustrate the application of the reflectance and transmittance operators for surfaces, constructed in Sec. 3.3, for several types of frequently encountered plane-surface settings in radiative transfer theory. Throughout this section and, indeed, the remainder of this chapter, one of the principal goals is the demonstration of the systematic use to which the interaction principle may be put in formulating the concepts and problems of radiative transfer theory.

Example 1: Irradiances on Two Infinite Parallel Planes

Let "a" and "b" denote two infinite parallel plane surfaces separated by a vacuum, as in Fig. 3.4. The coordinate system used is the terrestrial system defined in Sec. 2.4. Each plane has assigned reflectance and transmittance functions as developed in Sec. 3.3 which are to be constant over a and b. However, the directional structures of the reflectance and transmittance functions are otherwise arbitrary. An interreflection process between a and b is initiated and sustained by a steady downward field radiance distribution $N_-^o(b)$ on plane b which has the same structure at all points of b. Our present goal is to compute the resultant steady state irradiances on a and b, that is the upward irradiance $H_+(a)$ on a and the downward irradiance $H_-(b)$ on b.

The interaction principle applied to a and b in turn yields the requisite irradiance reflectance operators. Thus for a the set A of incident radiometric functions consists of irradiances like $H_+(a)$, and the set B of response radiometric functions of a consists of downward radiant emittances $W_-(a)$ which by the hypothesized vacuum between a and b have magnitudes equal to $H_-(b)$. (Cf. also Example 12 of Sec. 2.11 showing independence of H_r with r in the case of infinite

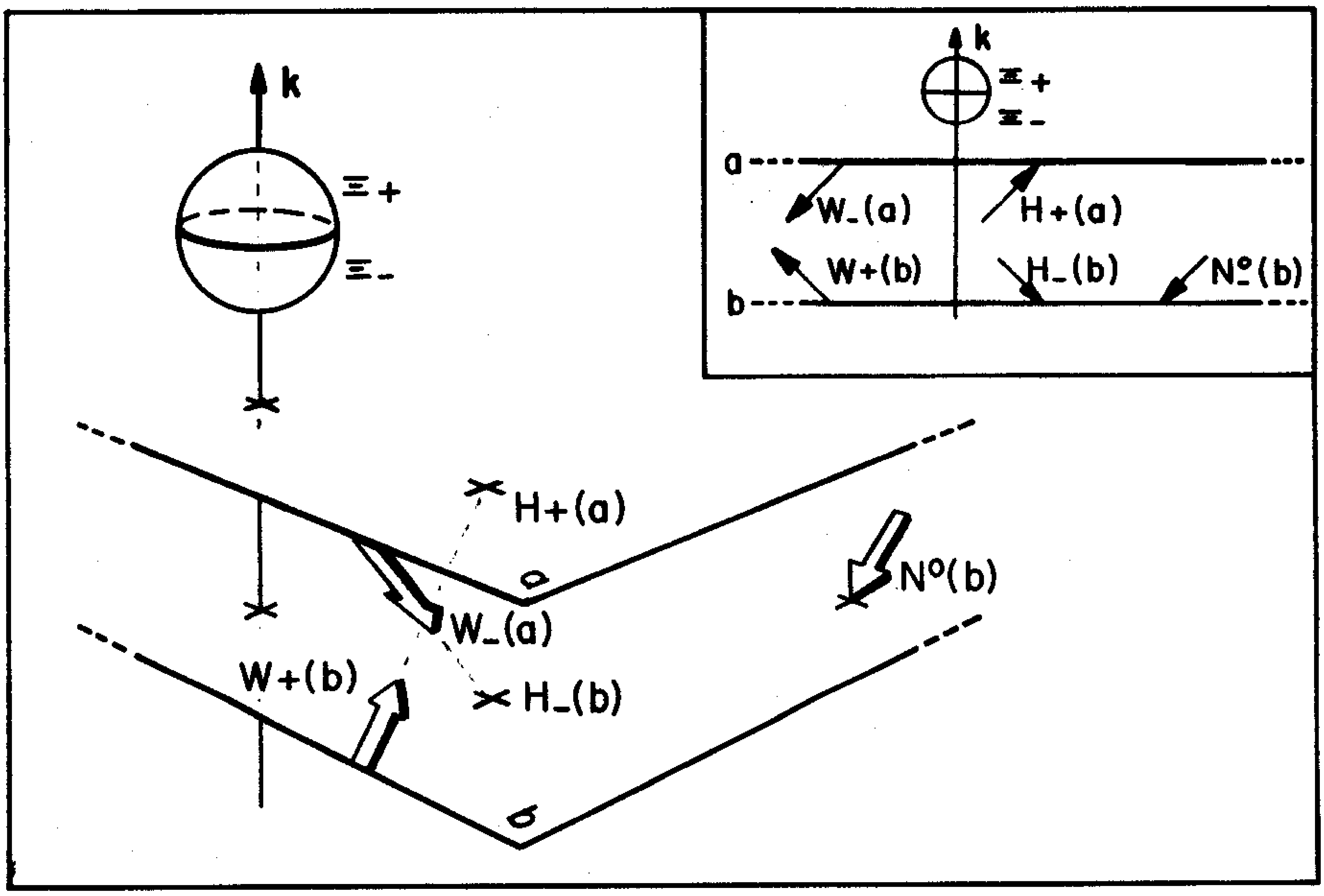

FIG. 3.4 Two interreflecting parallel planes.

planes). The interaction principle then asserts the existence of a reflectance (a number) $r_+(a)$ such that:

$$W_-(a) = H_-(b) = H_+(a)r_+(a)$$
$$= W_+(b)r_+(a) \qquad (1)$$

The last equality uses the radiance invariance law which implies that $W_+(b) = H_+(a)$. The closing example of Sec. 3.3 shows the necessary form of $r_+(a)$. Thus, following the pattern (19) of Sec. 3.3 we have written:

$$\text{"}r_+(a)\text{"} \quad \text{for} \quad \frac{1}{H_+(a)}\int_{\Xi_-}\left[\int_{\Xi_+} N(x,\xi')r_+(x;\xi';\xi)\,d\Omega(\xi')\right]\xi\cdot(-\mathbf{k})\,d\Omega(\xi)$$

and:

$$\text{"}H_+(a)\text{"} \quad \text{for} \quad \int_{\Xi_+} N(x,\xi')\xi'\cdot\mathbf{k}\;d\Omega(\xi')$$

where $N(x,\cdot)$ is now the upward surface radiance distribution of b at x which, with $r_+(x;\xi';\xi)$, is independent of x. By noting that the iterated integration amounts to finding $W_-(a)$, the downward radiant emittance of a, we see that we are simply writing:

$$\text{"}r_+(a)\text{"} \quad \text{for} \quad \frac{W_-(a)}{H_+(a)} \quad .$$

However, $r_+(a)$ is now precisely determinable as shown in the iterated integration whenever $N(x,\xi)$ and $r_+(x;\xi';\xi)$ are known for every x,ξ', and ξ. Even if the surface radiances $N(x,\xi)$ of b (and also a) are not known in absolute magnitude, but only in relative magnitude (i.e., its shape but not the size is known) the present goal can be attained, as we shall see.

Continuing, we apply the interaction principle to plane b, which has two sets of incident functions and one set of response functions. For, the given downward surface radiance $N_-^o(b)$ on b gives rise to a known incident irradiance $H_-^o(b)$. Irradiances like $H_-^o(b)$ comprise the set A_1 of incident radiometric functions for b. Irradiances like $H_-(b)$ comprise the set A_2 of incident radiometric functions for b. The set B_1 of response functions of b consists of radiant emittances $W_+(b)$ numerically equal to $H_+(a)$,(via the radiance invariance law once again). The interaction principle then yields two reflectances (numbers) $r_-^o(b)$ (for A_1 and B_1) and $r_-(b)$ (for A_2 and B_1) such that:

$$W_+(b) = H_+(a) = H_-^o(b)r_-^o(b) + H_-(b)r_-(b) \quad . \tag{2}$$

The numbers $r_-^o(b)$ and $r_-(b)$ are defined exactly analogously to $r_+(a)$. Equations (1) and (2) together determine $H_+(a)$. Thus, using (1) to eliminate $H_-(b)$ from (2), we have:

$$H_+(a) = H_-^o(b)r_-^o(b) + (H_+(a)r_+(a))r_-(b) \quad ,$$

whence:

$$\boxed{W_+(b) = H_+(a) = H_-^o(b)r_-^o(b)/[1-r_+(a)r_-(b)]} \tag{3}$$

and so:

$$\boxed{W_-(a) = H_-(b) = H_-^o(b)r_-^o(b)r_+(a)/[1-r_+(a)r_-(b)]} \tag{4}$$

These solutions exist provided that the product $r_+(a)r_-(b)$ is less than 1. This provision is reminiscent of a similar provision for Σ_{121} and Σ_{212} encountered in the preliminary example of Sec. 3.1, and may also be handled via the energy conservation law if desired. It is clear that (3) and (4) are usable in practice once reasonable estimates of $r_+(a)$ and $r_-(b)$ are made. Such estimates can be based either on empirical data in the form of measured ratios such as $W_-(a)/H_+(a)$, or by means of integral computations knowing the values $r_+(x;\xi';\xi)$ and the *shape* of the reflected radiance distributions. For example one can assume the perennial favorite: a uniform radiance distribution, or other readily integrated products of the form $N(x,\xi')r_+(x;\xi';\xi)$.

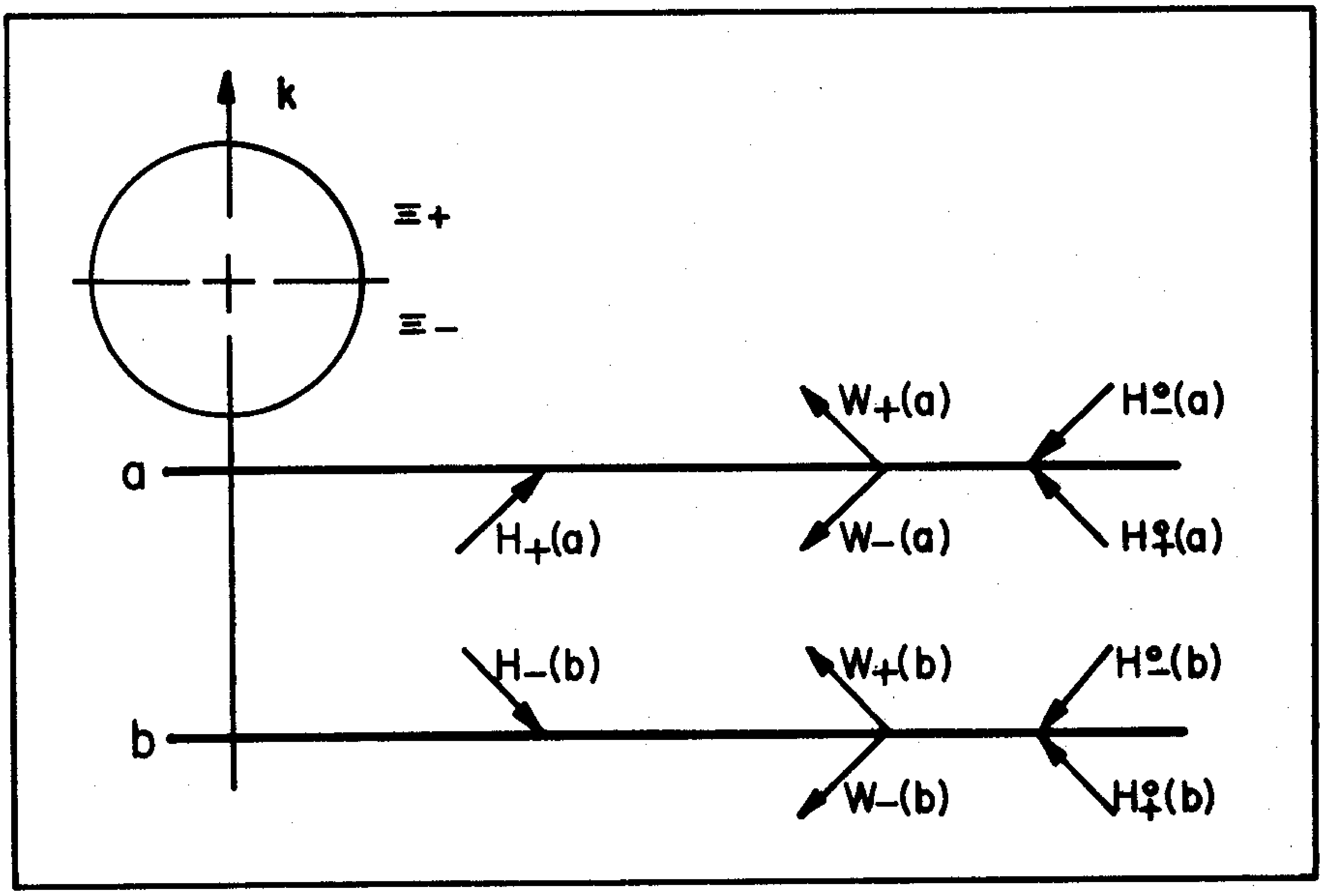

FIG. 3.5 Systematic details for an interreflection calculation between two parallel planes.

Example 2: Irradiances on Two Infinite Parallel Planes, Reexamined

In this example, we systematize the procedure and results of Example 1. In that example the radiometric details were kept at an absolute minimum so that the algebraic workings of the interaction principle could be readily followed. Now that the algebraic details of the interaction formulation have been demonstrated, we return to that simple setting and pull out nearly all the radiometric stops and turn on all the lights--so to speak. Specifically, we now let plane a be irradiated by two external sources, (i.e., origins of flux other than a and b) which produce downward $H_-^o(a)$ and upward $H_+^o(a)$ irradiances; similarly, b is irradiated by two external sources which produce $H_-^o(b)$ and $H_+^o(b)$ as schematically shown in Fig. 3.5. Our present goal is to use the interaction principle to formulate the equations governing the four quantities: $W_\pm(a)$, $W_\pm(b)$, i.e., the upward (+) and downward (-) radiant emittances of a and b induced by the interreflection interaction between a and b and the incident external sources on a and b. We direct attention first to plane a and list all possible incident radiometric quantities on a:

A_1: all irradiances like $H_-^o(a)$

A_2: all irradiances like $H_+^o(a)$

A_3: all irradiances like $H_+(a)$

A_1 and A_2 are self explanatory; A_3 is the set of irradiances induced by the presence of plane b below a. Next, the set of all response radiometric quantities of a are enumerated as follows:

B_1: all radiant emittances like $W_+(a)$

B_2: all radiant emittances like $W_-(a)$

Thus, in the case of plane a, m = 3, n = 2, and the six abstract interaction operators s_{ij} supplied by the interaction principle are in the form of reflectance and transmittance numbers as follows:

$$s_{11} \text{ -- } r^o_-(a)$$

$$s_{12} \text{ -- } t^o_-(a)$$

$$s_{21} \text{ -- } t^o_+(a)$$

$$s_{22} \text{ -- } r^o_+(a)$$

$$s_{31} \text{ -- } t_+(a)$$

$$s_{32} \text{ -- } r_+(a)$$

The six numbers $r^o_-(a),\ldots,r_+(a)$ are defined exactly analogously to $r_+(a)$ in Example 1 and come ultimately from the interaction principle as outlined in Sec. 3.3. The superscripts "o" set off the external incident sources from the internal sources. Then, according to the interaction principle $W_+(a)$ and $W_-(a)$ are given by:

$$W_+(a) = H^o_-(a)r^o_-(a) + H^o_+(a)t^o_+(a) + H_+(a)t_+(a) \tag{5}$$

$$W_-(a) = H^o_-(a)t^o_-(a) + H^o_+(a)r^o_+(a) + H_+(a)r_+(a) \quad . \tag{6}$$

By repeating this process of application of the interaction principle to plane b we arrive at the analogous pair of statements:

$$W_+(b) = H^o_+(b)t^o_+(b) + H^o_-(b)r^o_-(b) + H_-(b)r_-(b) \tag{7}$$

$$W_-(b) = H^o_+(b)r^o_+(b) + H^o_-(b)t^o_-(b) + H_-(b)t_-(b) \quad . \tag{8}$$

When we append the following two equations:

$$W_+(b) = H_+(a) \tag{9}$$

$$W_-(a) = H_-(b) \quad , \tag{10}$$

which follow from the hypothesized vacuum between a and b and the radiance invariance law, the resulting system (5)-(10) is self-contained and in principle solvable. In particular when $W_-(a)$ and $W_+(b)$ in (6) and (7) are eliminated via (9) and (10), the resultant pair of equations is autonomous:

$$H_-(b) = A_- + H_+(a)r_+(a) \tag{11}$$

$$H_+(a) = B_+ + H_-(b)r_-(b) \tag{12}$$

where we have written:

"A_-" for $H^o_-(a)t^o_-(a) + H^o_+(a)r^o_+(a)$

"B_+" for $H^o_+(b)t^o_+(b) + H^o_-(b)r^o_-(b)$.

Using (11) eliminate $H_-(b)$ from (12):

$$H_+(a) = B_+ + [A_- + H_+(a)r_+(a)]r_-(b) ,$$

we readily solve for $H_+(a)$:

$$\boxed{H_+(a) = \left[B_+ + A_-r_-(b)\right] \Big/ \left[1 - r_+(a)r_-(b)\right]} . \tag{13}$$

Then by (9) we obtain $W_+(b)$. From (13) and (11) we find $H_-(b)$ and so by (10), $W_-(a)$. Equations (5) and (8) then yield $W_+(a)$ and $W_-(b)$. In this way all four radiant emittances $W_\pm(a)$ and $W_\pm(b)$ are determined.

A First Synthesis of the Interaction Method

This example is valuable in pointing up the systematic use to which the interaction principle may be put in formulating and solving a radiative transfer problem associated with a subset S of an optical medium X. The essential steps of this method exhibited by the preceding example are as follows:

(i) Isolate the subset S of the optical medium X.

(ii) Enumerate the incident radiometric quantities a_i on S. This determines $A_1,\ldots,A_m$.

(iii) Enumerate the requisite response radiometric quantities b_j of S. This determines $B_1,\ldots,B_n$.

(iv) Enumerate the mn operators s_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n$ supplied by the interaction principle.

(v) Write the interaction equation $b_j = \sum_{i=1}^{m} a_i s_{ij}$ for $j = 1,\ldots,n$.

(vi) Append auxiliary equations connecting various chosen a_i and b_j, in as much detail as required to solve the system in (v) for the b_j.

Step (vi) in the present example occurred in (9) and (10) above. Invariably, the additional auxiliary equations in (vi) are equations which match radiances on adjoining subsets of X and use one or the other of the following laws:

(a) The radiance invariance law

(b) The equality of field and surface radiance at a given point and for a given direction.

The six steps (i)-(vi) together with (a) and (b) above will be used time and again in the following examples. These steps appear to lead to systematic formulations of radiative transfer problems in a manner similar to that used in the formulation of the problems of statics and dynamics in mechanics, i.e., by using the technique which begins with the time-honored injunction to: "isolate the body", then categorically adding up all forces on the isolated body, and finally applying one or all of the three basic Newtonian laws of mechanics to the isolated system. It is somewhat amusing and perhaps of interest to observe that the three Newtonian laws even appear to have their explicit radiometric counterparts in the form of (a) above for the First Law, (v) above for the Second law, and (b) above for the Third law. We shall call the method of formulation summarized in (i)-(vi) and (a), (b) above the *method of the interaction principle,* or simply the *interaction method.**

Example 3: Irradiances on Finitely Many Infinite Parallel Planes

What we have done above for two plane surfaces we can in principle do again for any finite number and even an infinite number of plane surfaces. We now consider the case of finitely many parallel planes mainly for the novel problems of solution it presents subsequent to the invocation of the interaction principle. This will serve to show that the

*In studies of linear hydrodynamics subsequent to the completion of the present work, I have found that the interaction method is capable of unifying this field in an elegant and practical manner, and that it leads to detailed numerical descriptions of scattered fields of surface water waves. Further studies in water wave-guide theory show similarities to the scattering matrix method in e.m. wave-guide propagation. All of this is not surprising, as the wording of the interaction principle is quite wide, and will apply to these other contexts by changing "radiometric" along with "optical medium" appropriately. See, e.g., Preisendorfer, R.W., "Surface-wave transport in nonuniform canals," Report NOAA/JTRE-80, Hawaii Institute of Geophysics, 1972.

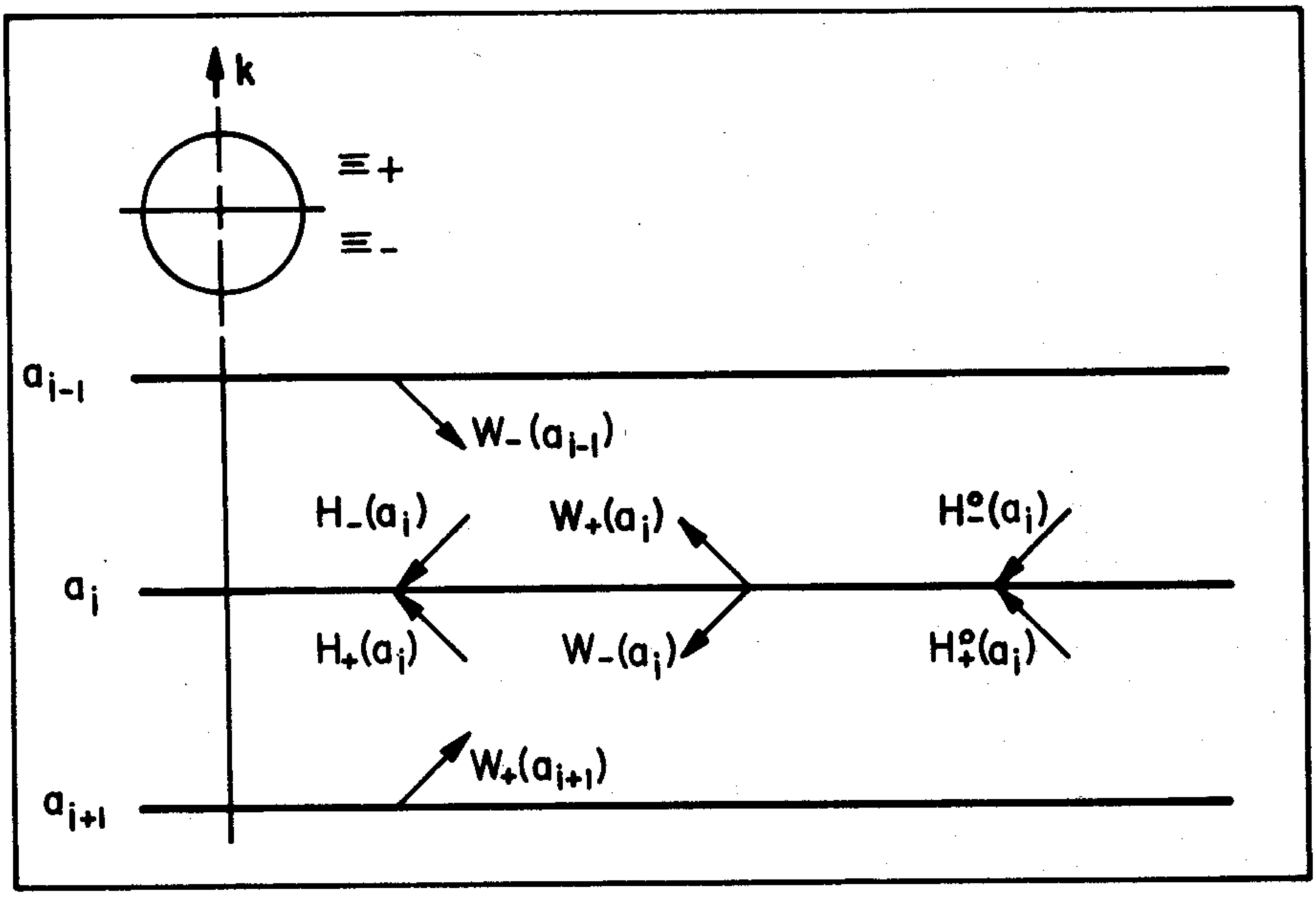

FIG. 3.6 Interaction calculation details for finitely many parallel planes.

interaction principle can lead even the most assiduous investigator only so far: there will always be a need for effective solution procedures of the more complex formulations supplied by the method of the interaction principle.

Figure 3.6 depicts three adjacent, parallel planes a_{i-1}, a_i, and a_{i+1} in a family of p parallel planes separated by vacua. Hence $1 < i < p$ in the Figure. For the moment, p may be either finite or infinite. Each plane a_i, $i = 1,\ldots,p$ is generally irradiated by an external source and the irradiation is constant over the extent of each a_i. The reflectance and transmittance functions of the a_i are also independent of x over a_i. Let "$H_\pm^o(a_i)$" denote the upward (+) and downward (-) external source irradiance on a_i. Furthermore, for every internal plane a_i, $1 < i < p$, there are irradiances $H_\pm(a_i)$ produced by flux from its lower neighbor (a_{i+1}) and its upper neighbor (a_{i-1}). Therefore for each isolated internal plane a_i there are four sets of incident radiometric quantities:

$$1 < i < p \quad \begin{cases} A_1: & \text{all irradiances like } H_-^o(a_i) \\ A_2: & \text{all irradiances like } H_+^o(a_i) \\ A_3: & \text{all irradiances like } H_-(a_i) \\ A_4: & \text{all irradiances like } H_+(a_i) \end{cases}$$

The set of all response radiometric quantities of a_i are enumerated as follows:

$1 < i < p$ $\quad B_1$: all radiant emittances like $W_+(a_i)$

$\quad B_2$: all radiant emittances like $W_-(a_i)$

Thus for the case of subset a_i of X, $1 < i < p$, we have $m = 4$ and $n = 2$, and the eight abstract interaction operators s_{ij} supplied by the interaction principle are specifically of the following forms:

$$s_{11} \text{ -- } r_-^o(a_i)$$
$$s_{12} \text{ -- } t_-^o(a_i)$$
$$s_{21} \text{ -- } t_+^o(a_i)$$
$$s_{22} \text{ -- } r_+^o(a_i)$$
$$s_{31} \text{ -- } r_-(a_i)$$
$$s_{32} \text{ -- } t_-(a_i)$$
$$s_{41} \text{ -- } t_+(a_i)$$
$$s_{42} \text{ -- } r_+(a_i)$$

The eight numbers $r_-^o(a_i),\ldots,r_+(a_i)$ for a given a_i are defined exactly as $r_+(a)$ was defined in Example 1. Then, according to step (2) of the method of the interaction principle (as outlined in Example 2), we have the following two equations for $W_\pm(a_i)$:

$$W_+(a_i) = H_-^o(a_i)r_-^o(a_i) + H_+^o(a_i)t_+^o(a_i) + H_-(a_i)r_-(a_i) + H_+(a_i)t_+(a_i) \tag{14}$$

$$W_-(a_i) = H_-^o(a_i)t_-^o(a_i) + H_+^o(a_i)r_+^o(a_i) + H_-(a_i)t_-(a_i) + H_+(a_i)r_+(a_i) \tag{15}$$

$1 < i < p$

The interaction principle is now applied to planes a_1 and a_p in turn. In the case of a_1, the five steps of the method of the interaction principle yield:

$$W_+(a_1) = H_-^o(a_1)r_-^o(a_1) + H_+^o(a_1)t_+^o(a_1) + H_+(a_1)t_+(a_1) \tag{16}$$

$$W_-(a_1) = H_-^o(a_1)t_-^o(a_1) + H_+^o(a_1)r_+^o(a_1) + H_+(a_1)r_+(a_1) \tag{17}$$

which are (and should be) identical in form to (5) and (6) of Example 2. As might now be expected the radiant emittance equations for a_p, are identical in form to (7) and (8) of Example 2:

$$W_+(a_p) = H_+^o(a_p)t_+^o(a_p) + H_-^o(a_p)r_-^o(a_p) + H_-(a_p)r_-(a_p) \quad (18)$$

$$W_-(a_p) = H_+^o(a_p)r_+^o(a_p) + H_-^o(a_p)t_-^o(a_p) + H_-(a_p)t_-(a_p) \quad (19)$$

In this way, for finite p, we arrive at p pairs of equations for $W_\pm(a_i)$, $i = 1,\ldots,p$. Step (vi) of the interaction principle method (in particular law (a)) yields the following 2(p-1) auxiliary equations:

$$W_+(a_i) = H_+(a_{i-1}) \ , \quad i = 2,\ldots,p \quad (20)$$

$$W_-(a_i) = H_-(a_{i+1}) \ , \quad i = 1,\ldots,p-1 \quad (21)$$

Using (20) and (21) in (14), (15), (17), and (18) we arrive at the following set of 2(p-1) equations in 2(p-1) unknowns $H_\pm(a_i)$:

$$H_-(a_2) = A_-(a_1) + H_+(a_1)r_+(a_1) \quad (22)$$

$$1<i<p \begin{cases} H_+(a_{i-1}) = A_+(a_i) + H_-(a_i)r_-(a_i) + H_+(a_i)t_+(a_i) & (23) \\ H_-(a_{i+1}) = A_-(a_i) + H_-(a_i)t_-(a_i) + H_+(a_i)r_+(a_i) & (24) \end{cases}$$

$$H_+(a_{p-1}) = A_+(a_p) + H_-(a_p)r_-(a_p) \quad (25)$$

where for every i, $1 \le i \le p$, we have written:

"$A_+(a_i)$" for $H_-^o(a_i)r_-^o(a_i) + H_+^o(a_i)t_+^o(a_i)$

"$A_-(a_i)$" for $H_-^o(a_i)t_-^o(a_i) + H_+^o(a_i)r_+^o(a_i)$

The system of equations (22)-(25) is very nearly a diagonal system, so can be solved relatively easily by successive elimination, or by other well-known methods of solution for such a system. However, it is interesting to note that a general recursion procedure for solving the system is suggested by the following physical observations. Suppose we conceptually view the system of p planes as a new system of two interacting subsets namely a_1 and the remaining set $\{a_2,\ldots,a_p\}$ of p-1 planes. Considering $\{a_2,\ldots,a_p\}$ as a single unit is reminiscent of considering the two surfaces S_1 and S_2 in the preliminary example of Sec. 3.1 as a single unit. By applying the interaction principle to a_1 we obtain the equations for $W_\pm(a_1)$ whose forms are precisely those of (5), (6). By applying the interaction method to the set $\{a_2,\ldots,a_p\}$, which we will also call by the new name "b", we obtain equations for $W_\pm(b)$ which are precisely those in (7), (8) provided that all internal sources on $\{a_2,\ldots,a_p\}$ are shut off. We defer treatment of the internal-source case for the moment. Then $\{a_2,\ldots,a_p\}$ reacts radiometrically to irradiation as would a single plane. That is, $\{a_2,\ldots,a_p\}$ has its own reflectance and transmittance functions assigned to it by the interaction principle and hence its associated reflectance and transmittances for irradiance are presumed known. It follows that equation (13) yields $H_+(a_1)$ and hence $W_+(b)$, etc., provided that $r_-(b)$ --i.e., the reflectance of the set $\{a_2,\ldots,a_p\}$ for downward irradiance is known. We do not know $r_-(b)$ as such, but at any rate we have gone down the ladder of complexity one rung: our original task was to find the reflectance and transmittance properties of a set of p parallel planes. Now we have only to find them for a set of p-1 planes. The course of action now before us is clear: we use (13) and its related equations applied to a_{p-1} and a_p to obtain the reflectance and transmittance of the set $\{a_{p-1},a_p\}$. Then we use (13) again to find the reflectance and transmittance of $\{a_{p-2},a_{p-1},a_p\}$, and so on, adding layer by layer until we return to the system comprised of a_1 and $\{a_2,\ldots,a_p\}$ and perform the final step. The preceding technique has been hastily sketched rather than developed in detail because it is only a special case of a more general problem, the complete details of which have been worked out elsewhere using the interaction principle and thus need not be repeated here. The reader may consult Chapters IX and X of Ref. [251] for a categorical analysis of this internal-source problem. Furthermore, for several succinct formulas summarizing the preceding source-free analysis of the system $\{a_1,\ldots,a_p\}$, see Example 6 of Sec. 8.7.

Despite the fact that the system (22)-(25) can be subsumed under a more general completed analysis, the reader should try his own hand at solving the system *de novo*, first for the case of all internal sources zero (i.e., $H_+^o(a_i) = 0$, $i = 1,\ldots,p-1$ and $H_-^o(a_i) = 0$, $i = 2,\ldots,p$), and then with a general distribution of internal sources. The continuous version of the present problem is considered later (see (44)-(66) of Sec. 8.5). A general solution of the internal-source problem is given in Ex. 3 of Sec. 3.9.

Example 4: Irradiances on Infinitely Many Infinite Parallel Planes

We now consider an infinite set A of infinite parallel planes, sharing common reflectance and transmittance functions. These functions are constant over each plane but may have arbitrary directional structure. This infinite set is depicted schematically in Fig. 3.7. Our purpose in exhibiting this example is to point up again the fact that the interaction principle carries one only as far as to permit a meticulous formulation of the problem. However, the variations in the incisiveness and pertinence of a given formulation is thereafter limited only by the ingenuity of the wielder of the principle.

As a case in point, let the set A be irradiated at its upper level only and in an amount $H_-^o(A)$. It is required to find $W_+(A)$, the resultant upward radiant emittance of the upper surface of the set A. A straightforward application of the interaction method to the subset A yields the equation:

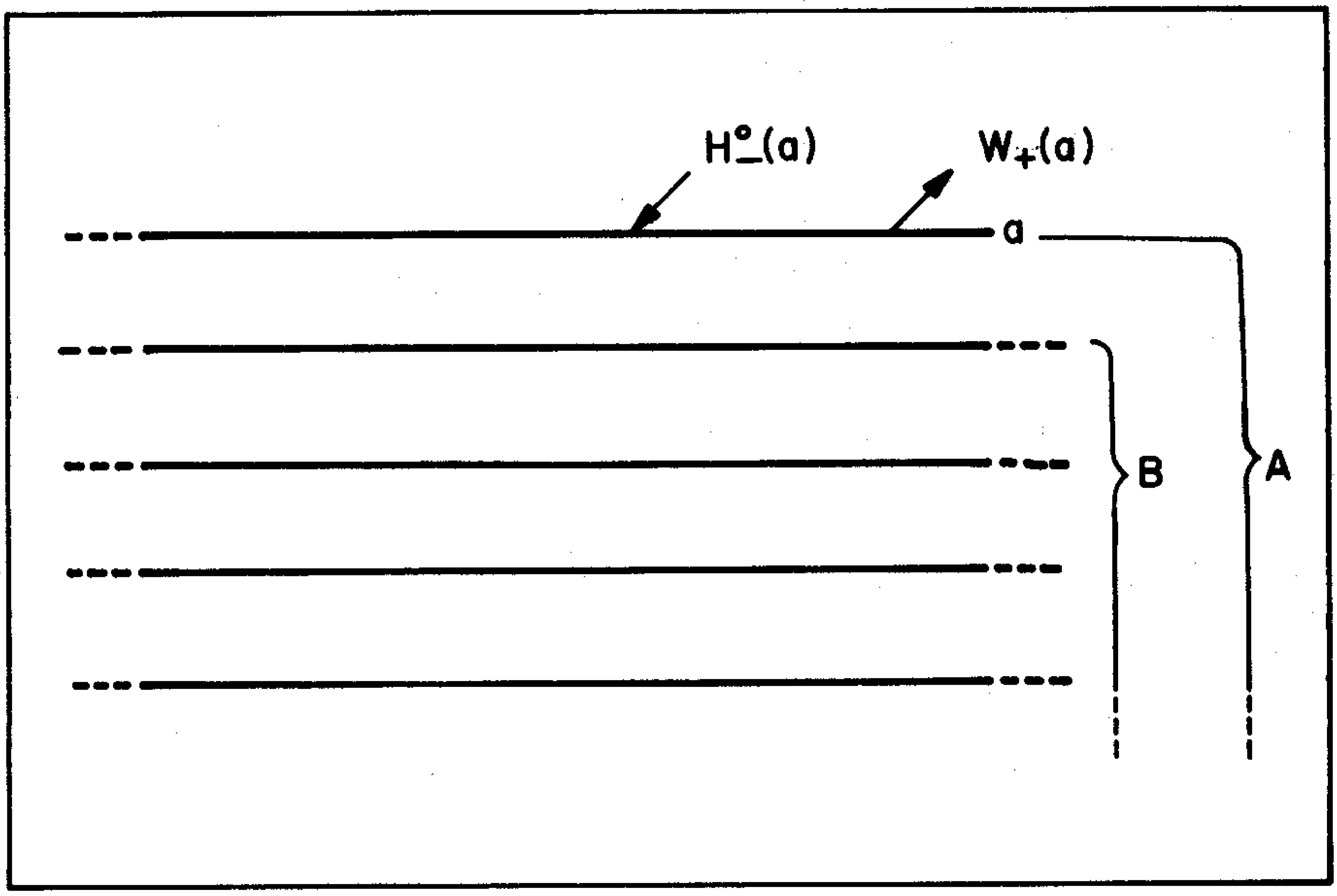

FIG. 3.7 An interaction of infinitely many planes, which illustrates Ambarzumian's principle.

$$W_+(A) = H^o_-(a)r_-(A) \quad . \tag{26}$$

Without any further insight into the problem on the part of a student of the subject, the interaction principle can carry the discussion no further. A significant advance toward the solution occurs, however, if the first plane a of the set A is removed from A and it is noticed that the remaining set B is in all particulars just like A except that it starts one layer lower than its predecessor. This may be viewed alternatively as if A were shoved bodily down one notch and otherwise left unchanged. Thus if a radiance distribution of the same directional structure impinges on B as that associated with $H^o_-(A)$, our hypothesized conditions imply that the reflectance $r_-(B)$ of B is equal to the reflectance $r_-(A)$ of A. Let us explicitly make this assumption to see where such an insight into the physics of the problem leads us.

The system A is now imagined to consist of two interacting subsets a and B. By invoking the interaction principle for each of these subsets, interaction equations are obtained which are formally identical to those for $W_\pm(a)$ and $W_\pm(b)$ (with "b" replaced by "B") in Example 2. The only salient difference is that in the present case $H^o_+(a) = H^o_-(B) = H^o_+(B) = 0$. When these incident lighting conditions are put into (13) we obtain:

$$H_+(a) = H^o_-(a)t^o_-(a)r_-(B)/[1-r_+(a)r_-(B)] \tag{27}$$

By (5) we have:

$$W_+(a) = H_-^o(a)r_-^o(a) + \frac{H_-^o(a)t_-^o(a)t_+(a)r_-(B)}{[1-r_+(a)r_-(B)]} \quad (28)$$

From a simple operational argument (i.e., imagine taking instrument readings) it is clear that:

$$W_+(a) = W_+(A) \quad . \quad (29)$$

Now, comparing (26) and (28), and using the physically-based insight:

$$r_-(A) = r_-(B) \quad (30)$$

we have:

$$r_-(A) = r_-^o(a) + \frac{t_-^o(a)t_+(a)r_-(A)}{1-r_+(a)r_-(A)} \quad (31)$$

Equation (31) governs the requisite reflectance $r_-(A)$ (for irradiance) of an infinite system A of parallel plane surfaces of common optical properties. In order to arrive at (31) two uses of physical intuition in the form of (29), (30) were needed. We assumed in particular that the structure of the downward incident radiance distribution on B was of the same angular structure as that on A. In the present adventurous spirit we can go on to assume that all radiance distributions within A have the same angular structure (say uniform) so that:

$$r_-^o(a) = r_+(a)$$

$$t_-^o(a) = t_+(a)$$

Let us write "r" for this common reflectance value for each plane in A and "t" for this common transmittance value for each plane in A. Then (31) reduces to:

$$r_-(A) = r + \frac{t^2 r_-(A)}{1 + rr_-(A)} \quad . \quad (32)$$

This yields the following quadratic equation governing $r_-(A)$:

$$rr_-^2(A) + (1-r^2-t^2)r_-(A) - r = 0 \quad (33)$$

which has the physically meaningful solution:

$$r_-(A) = \frac{-(1-r^2-t^2) + \sqrt{(1-r^2-t^2)^2 + 4r^2}}{2r} \quad . \quad (34)$$

To help choose between the two root signs, observe that if $t = 0$, then $r_-(A)$ should be r. By using the sign "+", (34) yields this limiting answer. As an example of $r_-(A)$ for a particular value of r, let $r = 1/2$ and $t = 1/2$, then:

$$r_{-}(A) = .618 \quad .$$

The lesson provided by this particular case of many interacting plane surfaces may be summarized as follows: while the interaction principle always yields the correct formulation corresponding to the wielder's method of analysis of a given radiometric system, there are nevertheless some choices of physical analyses of that system which are more pertinent than others. The methods of devising such analyses defy systematic description and retain radiative transfer theory, in this respect, in the ranks of the arts. The particular insight which was decisive in the present case was that summarized in (30), and is originally due to Ambarzumian. (See Ref. [1], [2].) This insight was the basis for the formulation of the first of the principles of invariance of modern radiative transfer theory (for a historical sketch, see Sec. 49 of Ref. [251]). Now that hindsight and formal principles (such as the interaction principle) are available, we can mechanically reproduce Ambarzumian's insight (30), (see, e.g., (30) of Sec. 7.3). Hence the arguments leading to (34) can be made formally rigorous without superfluous physical assumptions.

Example 5: The Algebra of Reflectance and Transmittance Operators for Planes

In this example we show how the reflectance and transmittance operators $r_{\pm}(Y)$, $t_{\pm}(Y)$, as given in (10), (11) of Sec. 3.3, are used in computations leading to reflected and transmitted radiance distributions on plane surfaces. Furthermore, some important technical concepts, such as the radiometric norm of an integral operator, will be developed, along with the rudiments of the algebra of reflectance and transmittance operators.

We consider first a plane surface a with unit upward normal **k**, as in Fig. 3.8. As usual, "Ξ_+" and "Ξ_-" will denote the set of all *upward* and *downward* directions with respect to **k**. Let "$N_-^+(x,\cdot)$" denote the downward (-) surface radiance distribution at x in a. Let "$N_+^+(x,\cdot)$" denote the upward surface radiance (+) distribution at x in a. In general throughout this example, a signed subscript on "N" will tell whether the radiance distribution is upward (+) or downward (-). A signed superscript on "N" will tell whether the radiance distribution is a surface (+) or field (-) radiance. This convention will serve to help us keep track of the comings and goings of radiant flux over the surface a.

For the moment we are interested in relating the downward field radiance distribution $N_-^-(x,\cdot)$ to its reflected and transmitted surface radiance distribution at x on surface a. Our goal is to establish the concept of the radiometric norm of radiance distributions and of the reflectance and transmittance operators. According to (10) and (13) of Sec. 3.3:

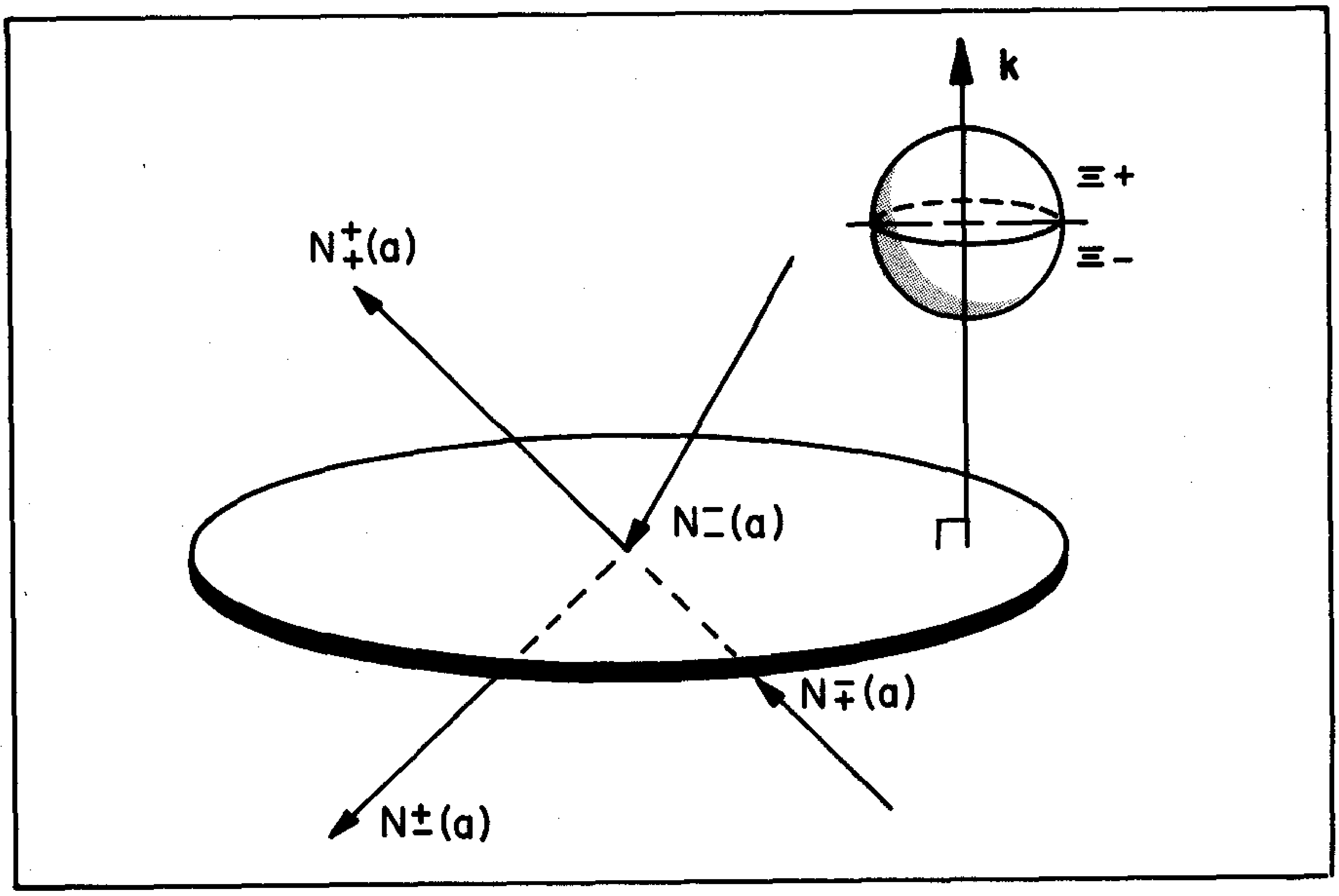

FIG. 3.8 Depicting the notation conventions for incident and response radiances of the field and surface type.

$$N^+_+(x,\xi) = \int_{\Xi_-} N^-_-(x,\xi')r_-(x;\xi';\xi)\ d\Omega(\xi') \quad , \tag{35}$$

and similarly, according to (11) of Sec. 3.3:

$$N^+_-(x,\xi) = \int_{\Xi_-} N^-_-(x,\xi')t_-(x;\xi';\xi)\ d\Omega(\xi') \quad . \tag{36}$$

Here $N^+_+(x,\cdot)$ is the *reflected* and $N^+_-(x,\cdot)$ the *transmitted* radiance distributions at x on a. Integral (35) shows us how to find the value of $N^+_+(x,\xi)$ for every x in a and ξ in Ξ_+, given the values $N^-_-(x,\xi)$ for every x in a and ξ in Ξ_-. Thus (35) leads to the operator equation:

$$N^+_+(a) = N^-_-(a)r_-(a) \quad , \tag{37}$$

where $r_-(a)$ is $r_-(Y)$ for Y = a, and $N^+_+(a)$ is the upward surface radiance distribution over a. Thus, to every x in a and ξ in Ξ_+, the value of $N^+_+(a)$ at (x,ξ) is $N^+_+(x,\xi)$ as given in (35). Similarly, $N^-_-(a)$ is the downward field radiance distribution over a. Equation (36) gives rise to the analogous operator equation:

$$N^+_-(a) = N^-_-(a)t_-(a) \tag{37a}$$

There are two more operator equations in addition to (37) and (37a), which round out the family of reflectance and transmittance operations over surface a:

$$N_+^+(a) = N_+^-(a)t_+(a) \tag{38}$$

$$N_-^+(a) = N_+^-(a)r_+(a) \tag{39}$$

which are the respective operator condensations of:

$$N_+^+(x,\xi) = \int_{\Xi_+} N_+^-(x,\xi')t_+(x;\xi';\xi)\ d\Omega(\xi') \tag{40}$$

and

$$N_-^+(x,\xi) = \int_{\Xi_+} N_+^-(x,\xi')r_+(x;\xi';\xi)\ d\Omega(\xi') \tag{41}$$

When working with the operators $r_\pm(a)$, $t_\pm(a)$ on one surface only, such as plane surface a, it is clear that the "+" and "-" superscripts on the radiance symbol are redundant (since field radiances are always operated on to yield surface radiances) and therefore may be dropped. However, when working with two or more interacting surfaces, the "+" and "-" superscripts often must be retained to avoid ambiguity during certain manipulations, as we shall see in later examples.

Radiometric Norm

We go on now to define the concept of a *radiometric norm* of radiance distributions over plane surfaces and of reflectance and transmittance operators. This concept is of central importance in both theoretical and practical computations of radiance distributions resulting from reflections or transmissions over surfaces. For every plane surface of finite area, let us write:

$$\text{"}|N_\pm(a)|\text{"} \quad \text{for} \quad \frac{1}{A(a)} \int_a \int_{\Xi_\pm} N(x,\xi)\ d\Omega(\xi)\ dA(x) \tag{42}$$

where "A(a)" denotes the area of surface a. If A(a) is infinite, then we write:

$$\text{"}|N_\pm(a)|\text{"} \quad \text{for} \quad \lim_{a'\to a} |N_\pm(a')| \tag{42a}$$

where a' is one of a family of subsets of a of finite area which equals a in the limit. We call $|N_\pm(a)|$ the *radiometric norm* of $N_\pm(a)$ over a. It is reminiscent of (and related to) the radiometric norm used in Example 15 of Sec. 2.11. The extension of (42) to curved surfaces is immediate. $N_\pm(a)$ in (42) can be either a field or surface radiance distribution for each direction (+) or (-). We observe that the radiometric norm is linear in the sense that:

$$|cN_{\pm}(a) + c'N'_{\pm}(a)| = c|N_{\pm}(a)| + c'|N'_{\pm}(a)| \qquad (43)$$

for every pair $N_{\pm}(a)$, $N'_{\pm}(a)$ of radiance distributions and nonnegative real numbers c and c'. Next, we write:

$$\text{"}\gamma_{\pm}(x;\xi')\text{"} \quad \text{for} \quad \int_{\Xi_{\mp}} r_{\pm}(x;\xi';\xi)\, d\Omega(\xi) \qquad (44)$$

and

$$\text{"}\delta_{\pm}(x;\xi')\text{"} \quad \text{for} \quad \int_{\Xi_{\pm}} t_{\pm}(x;\xi';\xi)\, d\Omega(\xi) \qquad (45)$$

For a given (field or surface) radiance distribution $N_{\pm}(a)$ with non zero radiometric norm over a, we write:

$$\text{"}\gamma_{\pm}(x,N)\text{"} \quad \text{for} \quad \frac{\int_{\Xi_{\pm}} N(x,\xi')\,\gamma_{\pm}(x,\xi')\, d\Omega(\xi')}{\int_{\Xi_{\pm}} N(x,\xi')\, d\Omega(\xi')} \qquad (46)$$

and

$$\text{"}\delta_{\pm}(x,N)\text{"} \quad \text{for} \quad \frac{\int_{\Xi_{\pm}} N(x,\xi')\,\delta_{\pm}(x,\xi')\, d\Omega(\xi')}{\int_{\Xi_{\pm}} N(x,\xi')\, d\Omega(\xi')} \qquad (47)$$

Finally, we write:

$$\text{"}\gamma_{\pm}(a,N)\text{"} \quad \text{for} \quad \frac{\int_a \gamma_{\pm}(x,N)\left[\int_{\Xi_{\pm}} N(x,\xi')\, d\Omega(\xi')\right] dA(x)}{\int_a \int_{\Xi_{\pm}} N(x,\xi')\, d\Omega(\xi')\, dA(x)} \qquad (48)$$

$$\text{"}\delta_{\pm}(a,N)\text{"} \quad \text{for} \quad \frac{\int_a \delta_{\pm}(x,N)\left[\int_{\Xi_{\pm}} N(x,\xi')\, d\Omega(\xi')\right] dA(x)}{\int_a \int_{\Xi_{\pm}} N(x,\xi')\, d\Omega(\xi')\, dA(x)} \qquad (49)$$

The motivation behind this seemingly bizarre set of definitions will now become clear. Let $N_-(a)$ be an incident downward radiance distribution on surface a, and let $N_+(a)$ be its reflected radiance distribution. Then by (35):

$$N_+(x,\xi) = \int_{\Xi_-} N_-(x,\xi')r_-(x;\xi';\xi)\; d\Omega(\xi') \quad .$$

We now wish to relate the radiometric norms of $N_+(a)$ and $N_-(a)$. The preceding definitions have been formulated with precisely this task in mind. We begin by integrating each side of this equation over Ξ_+. The result is:

$$\int_{\Xi_+} N_+(x,\xi)\; d\Omega(\xi) = \int_{\Xi_-} N_-(x,\xi')\; \gamma_-(x;\xi')\; d\Omega(\xi')$$

Then we integrate this result over a and divide by A(a). The new result is:

$$|N_+(a)| = \frac{1}{A(a)} \int_a \left[\int_{\Xi_-} N_-(x,\xi')\gamma_-(x;\xi')\; d\Omega(\xi')\right] dA(x)$$

by definition of the radiometric norm of $N_+(a)$. Next, by (46):

$$|N_+(a)| = \frac{1}{A(a)} \int_a \gamma_-(x,N_-)\left[\int_{\Xi_-} N_-(x,\xi')\; d\Omega(\xi')\right] dA(x)$$

and by (48):

$$|N_+(a)| = \frac{\gamma_-(a,N_-)}{A(a)} \int_a \int_{\Xi_-} N_-(x,\xi')\; d\Omega(\xi')\; dA(x)$$

$$= \gamma_-(a,N_-)\,|N_-(a)| \quad .$$

The last equality follows from the definition of the radiometric norm of $N_-(a)$. This is the desired relation between the norm of $N_-(a)$ and that of $N_+(a)$. This type of computation can be repeated three times. The collected results are as follows:

$$|N_\pm(a)| = \gamma_\mp\left(a,N_\mp\right)|N_\mp(a)| \qquad (50)$$

$$|N_\pm(a)| = \delta_\pm\left(a,N_\pm\right)|N_\pm(a)| \qquad (51)$$

In these equations the norms of incident radiance distributions are on the right, and the norms of response radiance distributions are on the left. To denote this fact explicitly and to avoid possible ambiguities, superscripts "+" and "-" may be appropriately appended to "N". The numbers $\gamma_{\pm}(a,N)$, $\delta_{\pm}(a,N)$ are called the *special radiometric norms* of $r_{\pm}(a)$, $t_{\pm}(a)$ associated with N over a, respectively. These numbers are dependent on the incident radiance distributions as can be seen by inspection of (46)-(49). However, it is not difficult to show that, for every incident radiance distribution N over a and every plane surface a:

$$0 \le \gamma_{\pm}(a,N) \le 1 \tag{52}$$

$$0 \le \delta_{\pm}(a,N) \le 1 \quad . \tag{53}$$

These inequalities follow from the energy conservation principle as enunciated in Sec. 3.1. From this we conclude at once that the radiometric norms of response functions cannot exceed those of the incident functions. The proofs of (52) and (53) first bring out the facts that $0 \le \gamma_{\pm}(x;\xi') \le 1$ and $0 \le \delta_{\pm}(x;\xi') \le 1$ for every x in a and every ξ' in $\Xi_{\pm}$. An empirical form of these inequalities was established in Sec. 3.1 directly from the energy conservation principle and may be used as a basis for the present proofs. From these latter inequalities and (46) we see that $0 \le \gamma_{\pm}(x,N) \le 1$ and $0 \le \delta_{\pm}(x,N) \le 1$ for every x in a and every N. Inequalities (52) and (53) now follow for every a and N from (48) and (49). Finally, we can free the numbers $\gamma_{\pm}(a,N)$ and $\delta_{\pm}(a,N)$ from dependence on the incident radiance distributions N by taking their maxima (or suprema) as N varies over all possible incident radiance distributions in a. Thus let us write:

$$\text{"}\gamma_{\pm}(a)\text{"} \quad \text{for} \quad \max_N \gamma_{\pm}(a,N) \tag{54}$$

$$\text{"}\delta_{\pm}(a)\text{"} \quad \text{for} \quad \max_N \delta_{\pm}(a,N) \quad . \tag{55}$$

From this and (50)-(53) we have:

$$|N_{\pm}(a)| \le \gamma_{\pm}(a)\,|N_{\mp}(a)| \tag{56}$$

$$|N_{\pm}(a)| \le \delta_{\pm}(a)\,|N_{\pm}(a)| \tag{57}$$

$$0 \le \gamma_{\pm}(a) \le 1 \tag{58}$$

$$0 \le \delta_{\pm}(a) \le 1 \tag{59}$$

The response radiance distributions are on the left in (56), (57), and the incident radiance distributions are on the right. We call $\gamma_{\pm}(a)$ and $\delta_{\pm}(a)$ the *general radiometric norms* of $r_{\pm}(a)$ and $t_{\pm}(a)$ for surface a. The properties (56) and (57) will play an important role in the discussion of the existence of solutions of the radiative transfer formulations below. In particular the following properties will turn out to be sufficient conditions for the existence of many solutions. We shall say that $r_\pm(a)$ and $t_\pm(a)$ are *norm contracting* if and only if:

$$0 < \gamma_\pm(a) < 1 \tag{60}$$

$$0 < \delta_\pm(a) < 1 \tag{61}$$

Iterated Operators

We turn next to a systematic description of certain integration details arising in interreflection calculations. Familiarity with these details will help the reader to attain a working understanding of how to translate into numerical form the results of the algebraic manipulations of the reflectance and transmittance operators.

We have seen in the preceding examples how the four operators $r_\pm(a)$, $t_\pm(a)$ associated with a surface a serve to describe the reflection or transmission of incident radiance distributions on a. These response radiance distributions can subsequently go on to interact with another surface b, or conversely, the incident radiance distributions on a may have come to a after being reflected or transmitted in some surface b. Hence there arises the possibility of considering an operation like $r_+(a)$ followed by an operation like $r_-(b)$; or $t_-(a)$ followed by $t_-(b)$; or $r_-(b)$ by $t_+(a)$, and so on. These combined operations are called *iterations*. We now systematically consider all such iteration possibilities of two operators when a and b are two parallel plane surfaces. There are eight such possibilities. They are schematically depicted in parts (a)-(d) of Fig. 3.9. Actually, the four possibilities in (a) and (b) of Fig. 3.9 are exhaustive of the basic possible combinations. Turn them upside down to get types in parts (c) and (d) of the figure. However, in a terrestrially-based coordinate system, generally one that is fixed independent of the planes a and b, it is convenient to explicitly and independently list also the possibilities (c) and (d) of Fig. 3.9.

As a specific example of the iteration of the interaction operators $r_\pm(a)$, $t_\pm(a)$, let us consider the iteration of $r_+(a)$ and $r_-(b)$ in that order. That is, an upward field radiance distribution $N_+^-(a)$ is incident on plane a and is reflected in plane a. The reflected flux goes on to become an incident radiance distribution $N_-^-(b)$ on b and which is reflected in turn in plane b. This is shown in part (b) of Fig. 3.9. If $N_+^+(b)$ is the resultant upward surface radiance distribution over b, then the value of $N_+^+(b)$ at y_j on b in the direction ξ_j in Ξ_+ is given by:

$$N_+^+(y_j,\xi_j) = \int_{\Xi_-} N_-^-(y_j;\xi_j')\,r_-(y_j;\xi_j';\xi_j)\; d\Omega(\xi_j') \tag{62}$$

Now by the radiance invariance law and with the help of Fig. 3.10:

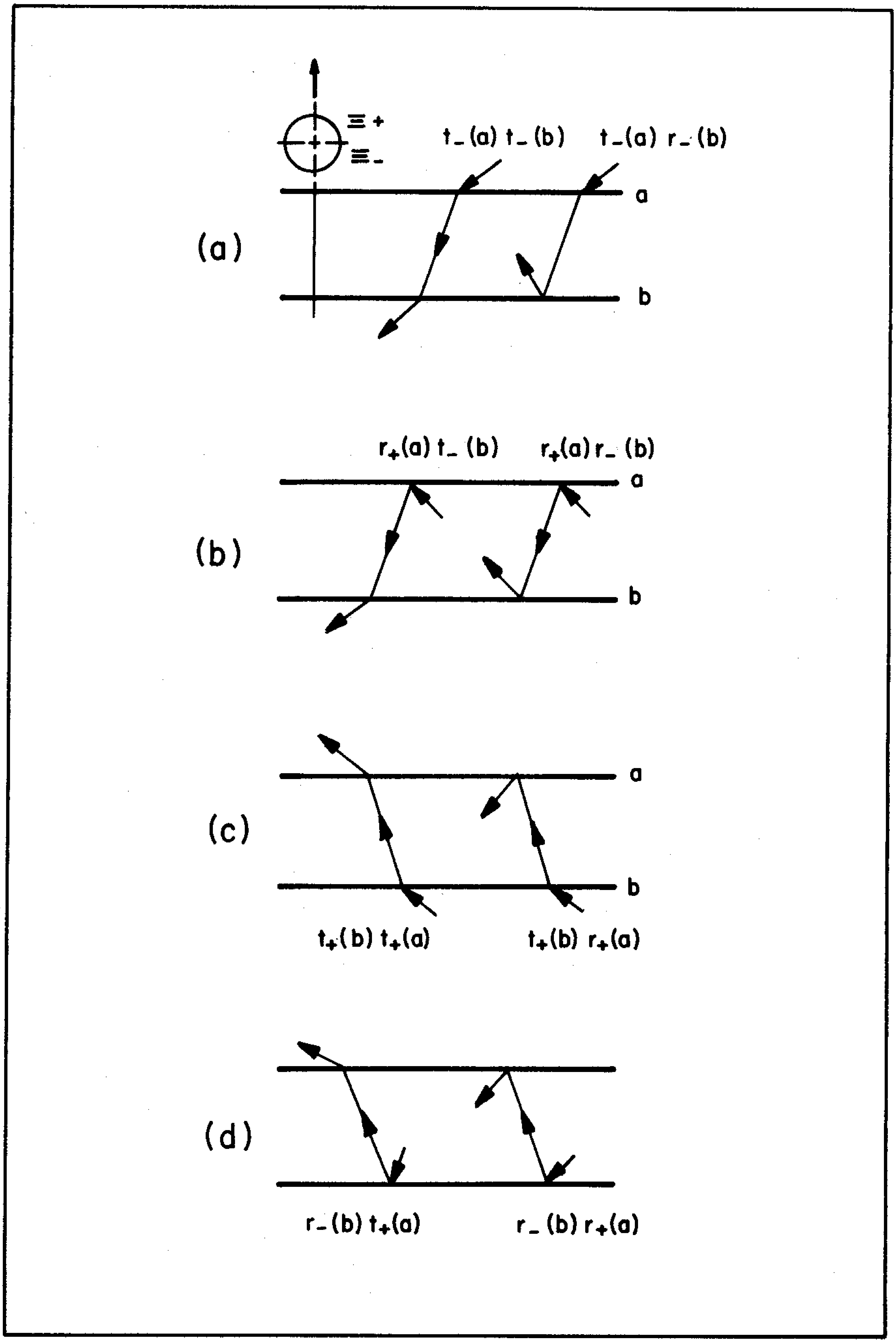

FIG. 3.9 Bookkeeping procedure in operator iterations: the eight possible types of interactions between two planes.

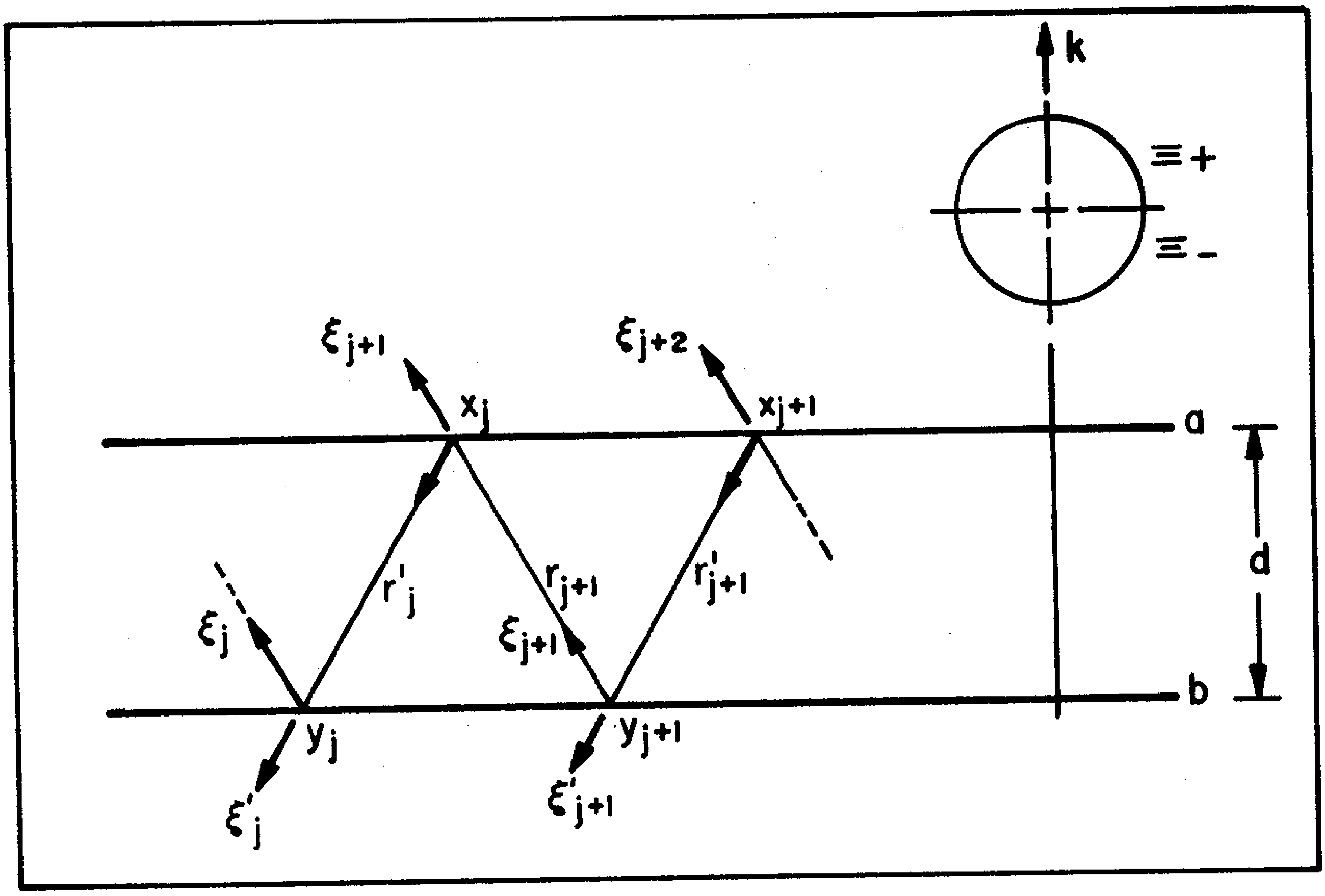

FIG. 3.10 Further detail of Diagram (b) in Fig. 3.9.

$$N_-^-(y_j,\xi'_j) = N_-^+(x_j,\xi'_j) \tag{63}$$

where:

$$y_j = x_j + r'_j\xi'_j$$

for every ξ'_j in Ξ_- at every y_j in b. But the value $N_-^+(x_j,\xi'_j)$ is the value of $N_-^+(a)$ at x_j and for ξ'_j, and this, by our present agreement, is given by:

$$N_-^+(x_j,\xi'_j) = \int_{\Xi_+} N_+^-(x_j,\xi_{j+1})r_+(x_j;\xi_{j+1};\xi'_j)\, d\Omega(\xi_{j+1}) \tag{64}$$

Combining (62), (63) and (64) we have:

$$N_+^+(y_j,\xi_j) =$$

$$= \int_{\Xi_-}\left[\int_{\Xi_+} N_+^-(x_j,\xi_{j+1})r_+(x_j;\xi_{j+1};\xi'_j)\, d\Omega(\xi_{j+1})\right] r_-(y_j;\xi'_j;\xi_j)\, d\Omega(\xi'_j) \tag{65}$$

which holds for every y_j in b and ξ_j in Ξ_+. In brief:

$$N_+^+(b) = N_+^-(a)r_+(a)r_-(b) \quad , \tag{66}$$

where we have written the left and right sides of (66) as abbreviations of the corresponding sides of (65). The variables x_j, y_j, ξ_j, ξ'_j, etc., are appropriately subscripted by integers j with $j = 0,1,2,\ldots$ simply in order to help the eye keep track of manipulations of variables in iterations of more than two operators.

To summarize, we have agreed to write:

"$r_+(a)r_-(b)$" for

$$\int_{\Xi_-}\left[\int_{\Xi_+}[\ ]r_+(x_j;\xi_{j+1};\xi'_j)\ d\Omega(\xi_{j+1})\right]r_-(y_j;\xi'_j;\xi_j)\ d\Omega(\xi'_j) \tag{67}$$

where x_j is in a, and $y_j = x_j + r_j\xi'_j$ and r'_j is the distance between x_j and y_j, i.e.,

$$r'_j = d/|\xi'_j \cdot \mathbf{k}| \tag{68}$$

and where d is the distance between a and b, and **k** is the unit upward normal to a and b. Equation (66) may be conveniently interpreted as follows: if $N_+^-(a)$ is a member of a set $A_1(a)$ of incident functions on a and $B_1(a)$ is a set of response functions containing $N_-^+(a)$, we view $r_+(a)$ as an operator which maps elements of $A_1(a)$ into $B_1(a)$. Similarly, $r_-(b)$ maps elements of a set $A_1(b)$ of incident functions on b into a set $B_1(b)$ of response functions on b. Therefore, the iterated operator $r_+(a)r_-(b)$ maps elements of $A_1(a)$ into $B_1(b)$.

As another example suppose $N_-^+(b)$ is the transmitted radiance distribution of the surface b in response to $N_-^-(b)$. If the space between surfaces a and b is a vacuum, then by the radiance invariance law, $N_-^-(b)$ is equal to $N_-^+(a)$, which in turn is the transmitted response to $N_-^-(a)$. See Fig. 3.11, which is a version of part (a) of Fig. 3.9. Then for every y_j in b and ξ_j in Ξ_-, we have:

$$N_-^+(y_j,\xi_j) = \int_{\Xi_-} N_-^-(y_j,\xi'_j)t_-(y_j;\xi'_j;\xi_j)\ d\Omega(\xi'_j) \tag{69}$$

By the radiance invariance law:

$$N_-^-(y_j,\xi'_j) = N_-^+(x_j,\xi'_j) \quad ,$$

where:

$$y_j = x_j + r'_j\xi'_j$$

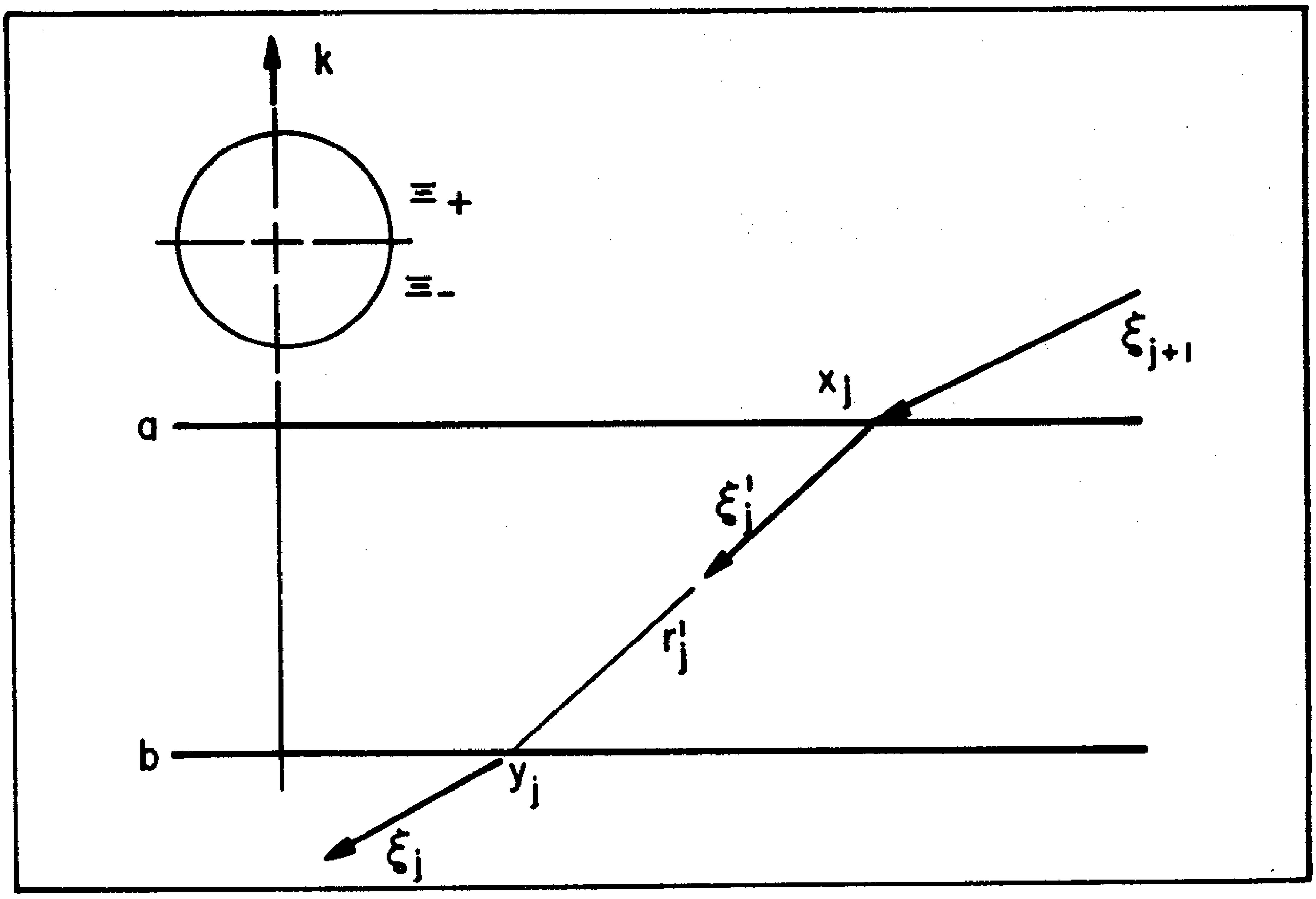

FIG. 3.11 Further detail of Diagram (a) in Fig. 3.9.

for every ξ'_j in Ξ_- at every y_j in b. But $N_-^+(x_j,\xi'_j)$ is the value of $N_-^+(a)$ at x_j and for ξ'_j and this, by our present agreement, is given by:

$$N_-^+(x_j,\xi'_j) = \int_{\Xi_-} N_-^-(x_j,\xi_{j+1})\, t_-(x_j;\xi_{j+1};\xi'_j)\, d\Omega(\xi_{j+1}) \qquad (70)$$

Combining (69) and (70) we have:

$$N_-^+(y_j,\xi_j) =$$

$$\int_{\Xi_-} \left[\int_{\Xi_-} N_-^-(x_j;\xi_{j+1};\xi'_j)\, t_-(x_j;\xi_{j+1};\xi'_j)\, d\Omega(\xi_{j+1}) \right] t_-(y_j;\xi'_j;\xi_j)\, d\Omega(\xi'_j) \qquad (71)$$

which holds for every y_j in b and ξ_j in Ξ_-. In brief:

$$N_-^+(b) = N_-^-(a)\, t_-(a)\, t_-(b) \qquad (72)$$

where we have written the left and right sides of (72) as abbreviations of the corresponding sides of (71). Thus we agree to write:

$$"t_-(a)t_-(b)" \quad \text{for}$$

$$\int_{\Xi_-}\left[\int_{\Xi_-} [\;]t_-(x_j;\xi_{j+1};\xi'_j)\, d\Omega(\xi_{j+1})\right]t_-(y_j;\xi'_j;\xi_j)\, d\Omega(\xi'_j) \tag{73}$$

It is clear that the iteration $t_-(a)t_-(b)$ maps elements in an incident set of surface a into the response set of surface b. Similar definitions can now be made of the remaining six types of iterations depicted in (a)-(d) of Fig. 3.9. These are left to the reader as exercises.

The radiometric norms of response functions under doubly iterated operators are related to the radiometric norms of the original incident function as follows. From (66) to which we apply (56) twice:

$$\begin{aligned}|N_+(b)| &= |N_+(a)r_+(a)r_-(b)| \\ &\leq \gamma_-(b)|N_+(a)r_+(a)| \\ &\leq \gamma_+(a)\gamma_-(b)|N_+(a)| \quad .\end{aligned}$$

Similarly from (72) to which we apply (57) twice:

$$|N_-(b)| \leq \delta_-(a)\delta_-(b)|N_-(a)| \quad .$$

Similar inequalities hold for the remaining six iteration possibilities.

Operator Algebras and Radiative Transfer

We close this discussion of Example 5 by observing that iterations of the operators $r_\pm(a)$ and $t_\pm(a)$ can be continued indefinitely as long as each iteration is meaningful. As an example of a meaningful continued iteration consider iterating $r_+(a)r_-(b)$ with itself. Thus let us write:

$$"(r_+(a)r_-(b))^1" \quad \text{for} \quad r_+(a)r_-(b) \tag{74}$$

and for every $j \geq 1$ we write:

$$"(r_+(a)r_-(b))^{j+1}" \quad \text{for} \quad (r_+(a)r_-(b))^j(r_+(a)r_-(b)). \tag{75}$$

Thus $(r_+(a)r_-(b))^j$ is an iterated integral, iterated 2j times using the reflectance functions $r_+(a)$ and $r_-(b)$. An example of a meaningless iteration is $t_+(a)r_-(b)$, since after a transmits flux upward, the flux does not go directly to b for reflection. Finally, we note that interaction operators can be added together. Thus "$r_-(b) + r_-(b)r_+(a)r_-(b)$" denotes the operator which acts on downward radiance distributions like

$N_-^-(b)$ to give upward radiance distributions like $N_-^-(b)r_-(b)$ + $N_-^-(b)r_-(b)r_+(a)r_-(b)$, etc. Hence we can speak of *sums* and *products* (i.e., iterations) of interaction operators and so handle the operators in many instances as if they were numbers, and as if they obeyed the same laws as numbers.

The properties of iterated and added operators arising in radiative transfer theory, when all assembled in a systematic list, turn out to be rather interesting from an algebraic point of view. Suppose that r, s, and t are any of the interaction operators defined above whose iterations rs, rt, st or sums r+s, r+t, s+t, are meaningful, then we can readily establish the following properties under suitable regularity conditions on the *reflectance*, or *transmittance* operators underlying the symbols. Let $\mathcal{A}$ be the set of all such operators. Then to every pair r,s of operators in $\mathcal{A}$ there corresponds a sum r+s in $\mathcal{A}$ of r and s such that

(i)	$r + s = s + r$	(commutativity)
(ii)	$r + (s+t) = (r+s) + t$	(associativity)
(iii)	$0 + r = r$	(identity)
(iv)	$r + (-r) = 0$	(inverse)

Furthermore, to every real number α an operator r in $\mathcal{A}$ there corresponds an operator αr, the *product* of α and r. (For example, if r is a reflectance operator, then αr is just the integral operator formed in the usual manner after multiplying $r_\pm(\cdot;\xi';\xi)$ in (10) of Sec. 3.3 by α.) The product αr has the general properties:

(v)	$\alpha(r+s) = \alpha r + \alpha s$	(operator distributivity)
(vi)	$(\alpha+\beta)r = \alpha r + \beta r$	(scalar distributivity)
(vii)	$(\alpha\beta)r = \alpha(\beta r)$	(scalar associativity)
(viii)	$0r = 0,\ 1r = r$	(zero, identity)

In the preceding equations α,β are real numbers, and 0,1 on the left sides in (viii) are the usual zero and unit real numbers. The 0 on the right in (viii) is the zero integral operator obtained by using $r_+ = 0$ or $r_- = 0$.

If the set $\mathcal{A}$ of operators satisfies all eight of the preceding conditions, then $\mathcal{A}$ is a *vector space* of operators. Now a vector space is a most useful object to work with because of its highly intuitive concepts and because of the rich body of computational theorems that exists for it. This gives one incentive to arrange matters so that, in suitably enlarged domains, the operator sums r+s and products rs are meaningful for every r, s in $\mathcal{A}$. We shall not pursue such matters in the present work, as it is too potentially vast a subject to compress into one section of a study devoted primarily to the immediate mathematical-physical foundations of the subject. However, before leaving this area of ideas, several more interesting algebraic aspects of the reflectance and transmittance operators will be brought out for future reference.

As observed above, we can assign meaning to the product rs of two reflectance or transmittance operators. Consider once again the collection $\mathcal{A}$ of all such operators associated with some optical medium. Then we have, in addition to properties (i)-(viii), the following:

(ix) $r(st) = (rs)t$

(x) $r(s+t) = rs + rt$

(xi) $(r+s)t = rt + st$

When $\mathcal{A}$ satisfies (i)-(xi) we call it an *operator ring*. If, moreover, we have

(xii) $\alpha(rs) = (\alpha r)s = r(\alpha s)$

for every real number α and pair of operators r, s, in $\mathcal{A}$, then $\mathcal{A}$ is an *operator algebra*. All of these twelve properties are readily shown to hold for the surface reflectance and transmittance operators of the form $r_{\pm}(a)$, $t_{\pm}(a)$, provided their products are definable. Unfortunately the commutative property, $rs = sr$, does not generally hold (except when certain reciprocity conditions are in force).

Going still further, let us write "$||r||$" for the *norm* of r, obtained as shown in (54), (55), depending on the nature of r. Then it is possible to show that, under suitable conditions:

(xiii) $0 \le ||r||$, and $||r|| = 0$ if and only if $r = 0$

(xiv) $||r+s|| \le ||r|| + ||s||$

(xv) $||\alpha s|| = |\alpha| \; ||s||$

(xvi) $||rs|| \le ||r|| \; ||s||$

(xvii) If $\mathcal{A}$ has an identity element I, then $||I|| = 1$

The first of these, namely (xiii), was essentially proved in (58), (59). Condition (xiv) is the triangle inequality, condition (xv) is trivial, and (xvi) is readily established using definitions of the kind (67), (73). The identity element I of $\mathcal{A}$ can be a suitably defined Dirac delta function. A set $\mathcal{A}$ of operators satisfying (i)-(xvii) is called a *normed operator algebra*. Unfortunately, for each r we do not generally have an s such that $rs = I$, so that operator division and hence the finding of inverses is generally not possible (however transmittance operators generally have inverses).

It appears that all of classical radiative transfer theory for surfaces and solids can be mathematically cast in terms of the theory of normed operator algebras. While this has not been proved in detail as yet, the preceding evidence is indicative of the validity of this conjecture. In this way radiative transfer theory could be completely algebraized.

The norm of an operator generally behaves like a measure of distance between two operators. For if r, s, and t are any three operators, then by (xiv):

$$||r-t|| \leqq ||r-s|| + ||s-t|| \quad ,$$

which is exactly analogous to the triangle inequality holding for the lengths of the sides of a triangle whose vertices are at points r, s, t in the plane. One very useful norm property of vectors is the following: If $r_1, r_2, \ldots,$ is an infinite sequence of vectors such that:

$$\lim_{m\to\infty} ||r_m - r_{m+d}|| = 0$$

for every d, then there exists a vector r such that:

$$\text{(xviii)} \qquad r = \lim_{m\to\infty} r_m \quad .$$

In other words, if a sequence of vectors eventually crowds together, then there is a vector toward which all the crowding is aimed.

Now if the set $\mathcal{A}$ of operators has the analogous property to (xviii), then $\mathcal{A}$ becomes a *Banach algebra*, a very rich concept in modern mathematics, with a highly developed theory [264]. One of the outstanding mathematical problems of modern radiative transfer theory is to develop the theory in terms of Banach algebras (cf., Problem X, in [251]), thereby providing a powerful, streamlined and abstract (i.e., economical) basis for the solutions of all manners of reflectance, transmittance, and general scattering problems.

Example 6: Radiances of Infinite Parallel Planes

The interaction principle will now be applied to the case of two radiometrically interacting (infinite) parallel planes a and b separated by a vacuum. Now that the meanings of operator products (iterations) and the associated algebraic concepts (i)-(xviii) of Example 5 have been explained for the operators $r_\pm(a)$ and $t_\pm(a)$ we can apply these concepts freely and concentrate mainly on the physical meanings of these operations.

Figure 3.12 depicts two parallel planes a and b over which reflectance and transmittance functions are defined and such that these functions may vary from point to point. Each plane is irradiated by two incident source distributions: $N_\pm^o(a)$ for a and $N_\pm^o(b)$ for b. These sources are considered to be external to a and b and initiate an interreflection process between a and b. The directional structure of these source distributions may vary from point to point. Our goal is to formulate and solve the radiometric interaction equations for

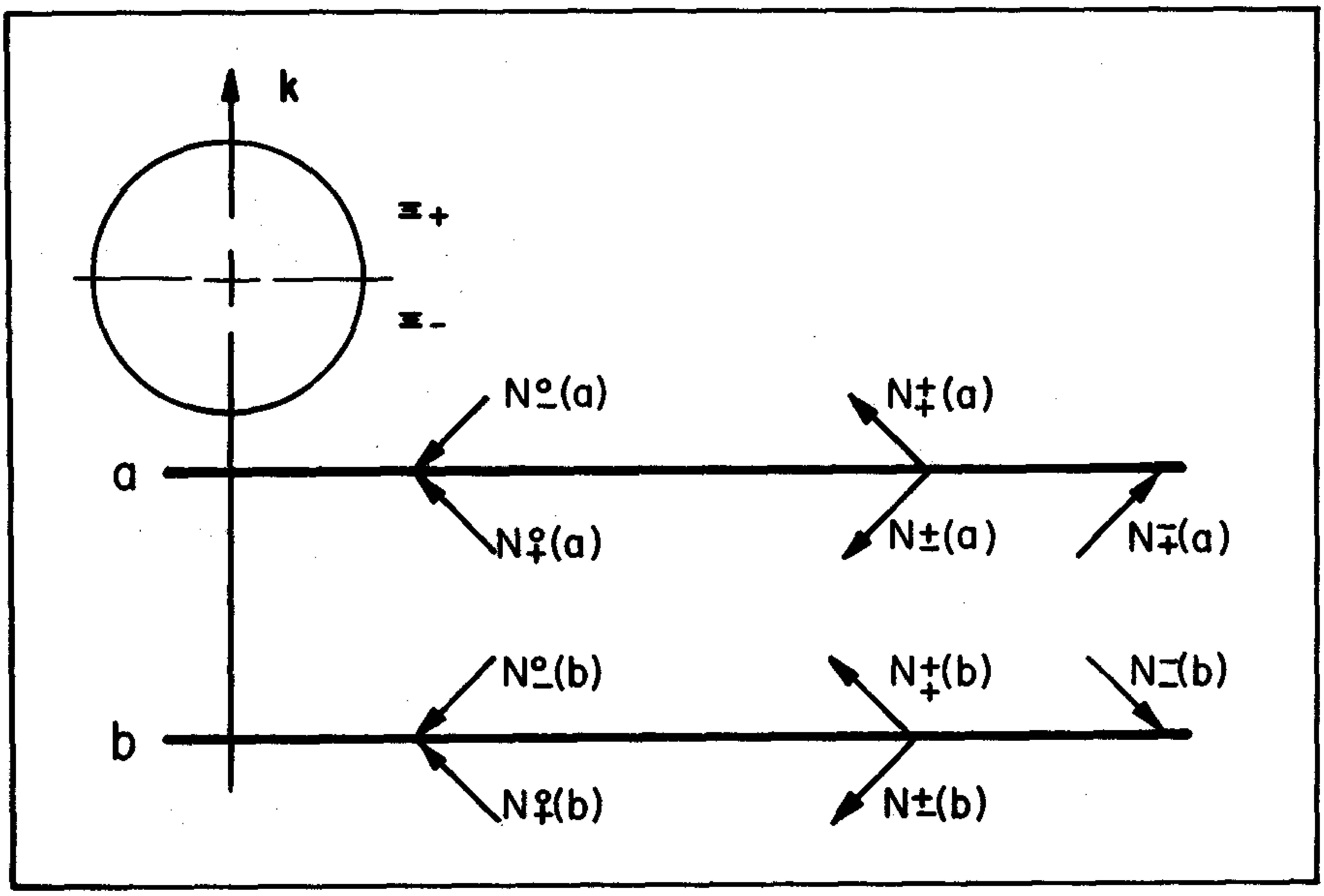

FIG. 3.12 Two interacting infinite plane surfaces.

the resultant response.

We first isolate plane a and enumerate the sets of incident radiometric functions on a. We obtain:

A_1: all field radiance distributions like $N^o_-(a)$

A_2: all field radiance distributions like $N^o_+(a)$

A_3: all field radiance distributions like $N^-_+(a)$.

Next, the set of all response radiometric quantities of a are enumerated as follows:

B_1: all surface radiance distributions like $N^+_+(a)$

B_2: all surface radiance distributions like $N^+_-(a)$.

Thus, in the case of plane a, m = 3, n = 2, and the six abstract interaction operators s_{ij} supplied by the principle are now in the form of reflectance and transmittance integral operators as follows:

s_{11}-- $r^o_-(a)$

s_{12}-- $t^o_-(a)$

s_{21}-- $t^o_+(a)$

s_{22}-- $r^o_+(a)$

$$s_{31} \;\text{--}\; t_+(a)$$

$$s_{32} \;\text{--}\; r_-(a)$$

The six operators $r_-^o(a),\ldots,r_-(a)$ are instances of definitions (10), (11) of Sec. 3.3. They are handled using the techniques illustrated in Example 5. Then, according to the interaction principle, $N_+^+(a)$ and $N_-^+(a)$ are given by:

$$N_+^+(a) = N_-^o(a)r_-^o(a) + N_+^o(a)t_+^o(a) + N_+^-(a)t_+(a) \qquad (76)$$

$$N_-^+(a) = N_-^o(a)t_-^o(a) + N_+^o(a)r_+^o(a) + N_+^-(a)r_+(a) \ . \qquad (77)$$

By repeating this process of application of the interaction principle to plane b, we arrive at the analogous pair of statements:

$$N_+^+(b) = N_+^o(b)t_+^o(b) + N_-^o(b)r_-^o(b) + N_-^-(b)r_-(b) \qquad (78)$$

$$N_-^+(b) = N_+^o(b)r_+^o(b) + N_-^o(b)t_-^o(b) + N_-^-(b)t_-(b) \ . \qquad (79)$$

This system of equations is self-contained when we append the following auxiliary equations:

$$N_+^+(b) = N_+^-(a) \qquad (80)$$

$$N_-^+(a) = N_-^-(b) \qquad (81)$$

where these equations follow from use of the radiance invariance law. It must be remembered that $N_+^+(b)$ and $N_+^-(a)$, e.g., are radiance distributions, i.e., *functions* which yield radiance values when given a specific point in b or a and a direction in Ξ_+. Detailed illustrations of (80) and (81) are given in Example 5; e.g., see (63) and Fig. 3.10.

The time has now come in the present analysis when it is most efficient to write the systems (76), (77), and (78), (79) in terms of field radiance or surface radiance only. Since Example 2 was written in terms of irradiances, we shall, for variety, write the present systems in terms of surface radiances. Thus using (80) and (81) to eliminate $N_+^-(a)$ in (77) and $N_-^-(b)$ in (78), and dropping the superscript "+", (77) and (78) become:

$$N_-(a) = A_- + N_+(b)r_+(a) \qquad (82)$$

$$N_+(b) = B_+ + N_-(a)r_-(b) \qquad (83)$$

where we have written:

"A_-" for $N_-^o(a)t_-^o(a) + N_+^o(a)r_+^o(a)$

"B_+" for $N_+^o(b)t_+^o(b) + N_-^o(b)r_-^o(b)$.

Using (82) to eliminate $N_-(a)$ from (83) we have:

$$N_+(b) = B_+ + [A_- + N_+(b)r_+(a)]r_-(b)$$

Using this, we solve for $N_+(b)$:

$$N_+(b) = (B_+ + A_-r_-(b))[I - r_+(a)r_-(b)]^{-1} \tag{84}$$

Then by (82) we obtain $N_-(a)$. Equation (76) now can yield $N_+(a)$, and equation (79) yields $N_-(b)$. In this way all four surface radiance distributions $N_\pm(a)$ and $N_\pm(b)$ are determined.

The preceding solution should be compared term by term with (13) in Example 2. In the case of (84) the upward surface radiance distribution $N_+(b)$ is determined by operating on $N_\pm^o(a)$ and $N_\pm^o(b)$ as shown. The meaning of the term "$[I - r_+(a)r_-(b)]^{-1}$" in (84) is now that of the inverse of an integral operator. In (13), "$[1 - r_+(a)r_-(b)]$" is interpreted as the *numerical* inverse of the number $[1 - r_+(b)r_-(b)]$, defined in the setting of (13). The numerical inverse exists provided that at least one of the numbers (reflectances) $r_+(a)$ or $r_-(b)$ is less than 1. *In the present case the inverse of the integral operator* $[I-r_+(a)r_-(b)]$ *exists provided that at least one of the operators* $r_+(a)$, $r_-(b)$ *is norm contracting* (cf. (60)), i.e., provided that $0< \gamma_+(a)< 1$ or $0< \gamma_-(b)< 1$. In this case the norm contracting theorem (cf. Example 15 of Sec. 2.11, and Sec. 40 of Ref. [251]) of vector space theory asserts that the inverse $[I-r_+(a)r_-(b)]^{-1}$ of the integral operator $[I-r_+(a)r_-(b)]$ exists and that:

$$[I - r_+(a)r_-(b)]^{-1} = \sum_{j=0}^{\infty}(r_+(a)r_-(b))^j , \tag{85}$$

where $(r_+(a)r_-(b))^o = I$, and $(r_+(a)r_-(b))^j$ is defined for all positive integers j in (75).

The reader may by now have detected in the form of (85) the abstract vestige of the natural mode of solution of the present interreflection problem (re Sec. 3.1). As our discussions progress through this and succeeding chapters, we shall see even more clearly that: *the natural mode of solution of radiative transfer problems is as indigenous to radiative transfer problems as, e.g., the method of power series solutions is to classical differential equation theory.* The natural mode of solution is basically a recursive process, i.e., a systematically repetitive process, one which is meaningful not only physically but also in the context of the logical foundations of the mathematics underlying radiative transfer theory (cf., Sec. 22 of Ref. [251]).

Example 7: Terminable and Non Terminable Interreflection Calculations

It will be an instructive exercise in the use of the reflectance and transmittance operators of Example 6 to consider one of the simpler types of radiance interreflection problems--one that can be solved in finite terms. In general we say that an interreflection calculation for a response function of subset S of an optical medium X is *terminable* if it can be completed in a finite number of algebraic operations, i.e., operator products (iterations) and additions on the external incident source terms. Otherwise the calculation is *non-terminable*. In this example we shall exhibit a simple terminable calculation and also provide a means of evaluating truncation errors arising from stopping a non-terminable (or terminable) calculation at some intermediate stage.

A Terminable Calculation

As an example of a terminable interreflection calculation, consider once again the geometry of Fig. 3.12. Planes a and b are now assumed lambert reflectors and transmitters. This means that they will have uniform radiance distributions. That is, for plane a, e.g., we assume that:

$$r_+(x;\xi';\xi) = \frac{r_+}{\pi}\,|\xi'\cdot\mathbf{k}|$$

$$t_-(x;\xi';\xi) = \frac{t_-}{\pi}\,|\xi'\cdot\mathbf{k}|$$

for every x in a and ξ' in $\Xi_\pm$ as the case may be. Here "r_+" and "t_-" denote real numbers not exceeding 1. Similarly, for plane b we assume:

$$r_-(x;\xi';\xi) = \frac{r_-}{\pi}\,|\xi'\cdot\mathbf{k}| \quad .$$

Of r_- and r_+ we assume at least one of these is positive and less than 1. Furthermore, we assume $N_-^o(b) = 0$ and $N_+^o(a) = 0$. The incident source radiance distribution $N_-^o(a)$ is of arbitrary directional structure but independent of x on a. Our goal is to determine the formulas for the surface radiance distributions $N_-(a)$ and $N_+(b)$ under the present conditions.

Now, if $N_+(a)$ is any member of the incident set A_3 for a, as defined in Example 6, then $N_+(a)r_+(a)$ is a member of response set B_2 for a and

$$N_+(a)r_+(a) = \frac{r_+}{\pi}\,H_+(a)$$

where "$H_+(a)$" denotes the upward irradiance on a produced by $N_+(a)$. Similarly, if $N_-(a)$ is in the incident set A_1 for a, then:

$$N_-(a)t_-(a) = \frac{t_-}{\pi} H_-(a)$$

where $H_-(a)$ is the downward irradiance on a produced by $N_-(a)$. Furthermore, if $N_+(a)$ above is of uniform directional structure and of magnitude N_+, then*

$$N_+(a)r_+(a) = r_+N_+ \quad .$$

A similar set of equalities hold for plane b.

We are ready to evaluate $N_+(b)$. From (84) we have, with the present external source conditions:

$$N_+(b) = N_-^o(a)t_-^o(a)r_-(b)[I - r_+(a)r_-(b)]^{-1} \ .$$

From our preceding observations:

$$N_-^o(a)t_-^o(a) = \frac{t_-}{\pi} H_-^o(a) \quad ,$$

so that this transmitted radiance distribution is directionally uniform. Hence,

$$\left[\frac{t_-}{\pi} H_-^o(a)\right] r_-(b) = \frac{t_-r_-}{\pi} H_-^o(a)$$

again by our preceding observations. We have reduced the computation of $N_+(b)$ to:

$$N_+(b) = \frac{t_-r_-}{\pi} H_-^o(a)[I - r_+(a)r_-(b)]^{-1}$$

$$= \frac{t_-r_-}{\pi} H_-^o(a) \sum_{j=0}^{\infty} [r_+(a)r_-(b)]^j$$

where the last equality follows from (85) and our hypothesis about the norm contraction of either $r_-(a)$ or $r_-(b)$. By repeated application of the preceding results, the uniform radiance distribution with magnitude $t_-r_-H_-^o(a)/\pi$ is again mapped into uniform radiance distributions such that, for each j, $j = 0,1,2,\ldots$,

$$\left[\frac{t_-r_-H_-^o(a)}{\pi}\right] (r_+(a)r_-(b))^j = \frac{t_-r_-}{\pi} H_-^o(a)(r_+r_-)^j \quad .$$

*Strictly, $N_+(a)r_+(a)$ is a constant *function* which assigns to every x in a and ξ in Ξ_- the value r_+N_+, where $N_+ = H_+(a)/\pi$. Thus we cannot really set $N_+(a)r_+(a)$ equal to the *single number* r_+N_+. However, when no confusion will result, we will occasionally resort to such abbreviated notation, as indicated.

The radiance distribution $N_+(b)$ is therefore of uniform directional structure and of magnitude:

$$N_+(b) = \frac{t_-r_-}{\pi} H_-^o(a) \sum_{j=0}^{\infty} (r_+r_-)^j = \frac{H_-^o(a)}{\pi} \frac{t_-r_-}{[1 - r_+r_-]} \quad (86)$$

From (82):

$$\begin{aligned} N_-(a) &= N_-^o(a)t_-^o(a) + N_+(b)r_+(a) \\ &= \frac{t_-}{\pi} H_-^o(a) + \frac{1}{\pi} \frac{r_+r_-t_-H_-^o(a)}{[1 - r_+r_-]} \\ &= \frac{H_-^o(a)}{\pi} \left[t_- + \frac{r_+r_-t_-}{(1 - r_+r_-)} \right] \\ &= \frac{H_-^o(a)}{\pi} \frac{t_-}{[1 - r_+r_-]} \\ &= \frac{N_+(b)}{r_-} \end{aligned} \quad (87)$$

This completes the general calculation of $N_+(b)$ and $N_-(a)$. By adopting sufficiently severe assumptions, the calculation has been reduced to a finite number of arithmetic operations. Nevertheless, these results constitute useful rules of thumb for estimating the order of magnitudes of interreflected radiance distributions between two extensive parallel surfaces.

As a specific example of the use of (86) and (87), let $r_+ = r_- = t_- = 1/2$. Hence

$$N_+(b) = \frac{1}{\pi} \frac{(1/4)H_-^o(a)}{1 - (1/4)} = \frac{1}{3\pi} H_-^o(a)$$

$$N_-(a) = \frac{N_+(b)}{r_-} = \frac{2}{3\pi} H_-^o(a)$$

Truncation Error Estimates

The matter of truncating the series expansion of $[I - r_+(a)r_-(b)]^{-1}$ and estimating the resultant truncation error will be taken up next. The technique we use for the present special case is indicative of what can be done in the general truncation processes.

Suppose that planes a and b in Fig. 3.12 have arbitrary reflectance and transmittance operators subject only to the condition that at least one of $r_+(a)$ or $r_-(b)$ is norm contracting. For simplicity in manipulation let us suppose that

$N_+^o(b) = 0$, and $N_+^o(a) = 0$, with $N_-^o(a)$ arbitrary. Then from (84):

$$N_+(b) = N_-^o(a)t_-^o(a)r_-(b)[I - r_+(a)r_-(b)]^{-1}$$

$$= N_-^o(a)t_-^o(a)r_-(b)\sum_{j=0}^{\infty}(r_+(a)r_-(b))^j \quad .$$

Let us truncate the infinite series at term k and denote by "$N_+(b;k)$" the associated radiance distribution. Thus:

$$N_+(b;k) = N_-^o(a)t_-^o(a)r_-(b)\sum_{j=0}^{k}(r_+(a)r_-(b))^j \quad .$$

From this, we have

$$N_+(b) - N_+(b;k) = N_-^o(a)t_-^o(a)r_-(b)\sum_{j=k+1}^{\infty}(r_+(a)r_-(b))^j \quad .$$

We are interested in the radiometric norm of the difference $N_+(b) - N_+(b;k)$, for this difference is evidently the component of the actual radiance distribution omitted by the truncation process. To facilitate the estimate of the norm, let us write *ad hoc*:

$$\text{"}N^o\text{"} \quad \text{for} \quad N_-^o(a)t_-^o(a)r_-(b) \quad .$$

Then

$$N_+(b) - N_+(b;k) = \left| N^o \sum_{j=k+1}^{\infty} (r_+(a)r_-(b))^j \right|$$

$$= \left| \sum_{j=k+1}^{\infty} N^o(r_+(a)r_-(b))^j \right|$$

$$= \sum_{j=k+1}^{\infty} \left| N^o(r_+(a)r_-(b))^j \right|$$

The last equation follows by repeated use of the linearity of the radiometric norm (42) (with c = c' = 1). The next step will be facilitated by examining a typical term of the preceding sum of norms. Thus observe that:

$$\left|N^o(r_+(a)r_-(b))^j\right| = \left|[N^o(r_+(a)r_-(b))^{j-1}](r_+(a)r_-(b))\right|$$
$$= \left|[N^o(r_+(a)r_-(b))^{j-1}r_+(a)]r_-(b)\right|$$
$$\leq \gamma_-(b)\left|N^o(r_+(a)r_-(b))^{j-1}r_+(a)\right|$$
$$\leq \gamma_+(a)\gamma_-(b)\left|N^o(r_+(a)r_-(b))^{j-1}\right|$$

The last two inequalities follow from the definition of the general radiometric norms given in Example 6. The pattern of reductions arising in the estimate of $|N_+(b) - N_+(b;k)|$ is now clear. We have:

$$|N_+(b) - N_+(b;k)| \leq \sum_{j=k+1}^{\infty} (\gamma_+(a)\gamma_-(b))^j |N^o|$$
$$= \frac{|N^o|\,[\gamma_+(a)\gamma_-(b)]^{k+1}}{[1 - \gamma_+(a)\gamma_-(b)]} \quad .$$

In addition:

$$|N^o| \leq |N^o_-(a)|\,\delta_-(a)\gamma_-(b) \quad .$$

Since $|N^o_-(a)| < |N_+(b)|$ in many reflection problems,* we find:

$$\frac{|N_+(b) - N_+(b;k)|}{|N_+(b)|} < \frac{\delta_-(a)\gamma_-(b)\left(\gamma_+(a)\gamma_-(b)\right)^{k+1}}{[1 - \gamma_+(a)\gamma_-(b)]} \quad . \qquad (88)$$

This gives an estimate of the relative error arising from truncation of the natural mode of solution of $N_+(b)$ at the kth term.

As an example of the use of (88) suppose $\gamma_+(a) = \gamma_-(b) = 1/2$ and $\delta_-(a) = 1/2$. Then:

$$\frac{|N_+(b) - N_+(b;k)|}{|N_+(b)|} < \frac{4}{3}\left(\frac{1}{4}\right)^{k+2} = \frac{1}{3}\left(\frac{1}{4}\right)^{k+1}$$

By terminating the series of iterated integral operators at $k = 2$, we have

*Otherwise, simply divide through by $|N^o_-(a)|$ instead of $|N_+(b)|$.

$$\frac{|N_+(b) - N_+(b;2)|}{|N_+(b)|} < \frac{1}{3}\left(\frac{1}{4}\right)^3 = \frac{1}{192} = .0052 \quad .$$

Thus the resultant relative error of truncation in this case is on the order of 1/2 percent.

By using the calculation in Example 6 for the response radiances of Lambert surfaces a and b with r_+, r_- and t_- in that example now replaced by $\gamma_+(a)$, $\gamma_-(b)$, $\gamma_-(a)$, an estimate of $|N_+(b)|$ can be made. From this and the preceding inequality an estimate of the absolute error of truncation can be made. Thus, as an illustration, we use the result (84) of Example 6 to find:

$$|N_+(b)| = \frac{1}{3\pi} H_-^0(a) \quad .$$

Hence:

$$|N_+(b) - N_+(b;k)| < 5.5 \times 10^{-4} H_-^0(a) \quad .$$

It follows that if $H_-^0(a)$ is on the order of 10^3 watts/m^2 then:

$$|N_+(b) - N_+(b;k)| < .55 \text{ watts}/(\text{m}^2 \times \text{steradian}) \quad .$$

This estimate of $H_-^0(a)$ is a reasonable one for natural light fields as may be seen by an inspection of Table I of Sec. 2.4.

One can occasionally profitably reverse the preceding error estimate calculation as follows. We agree on an error $\varepsilon > 0$ at the outset and then solve for the k which will yield that ε. Thus, from (88), we set:

$$\frac{\delta_-(a)\gamma_-(b)\left(\gamma_+(a)\gamma_-(b)\right)^{k+1}}{[1 - \gamma_+(a)\gamma_-(b)]} = \varepsilon$$

whence we find k by means of the relation:

$$k+1 = \frac{\ln\left[\dfrac{\varepsilon\left(1 - \gamma_+(a)\gamma_-(b)\right)}{\delta_-(a)\delta_-(b)}\right]}{\ln\left(\gamma_+(a)\gamma_-(b)\right)}$$

This formula is associated with the particular geometric arrangements of the present example. It is a relatively simple matter to extend this result to other formulas in connection with related problems, one of which will be discussed next.

Quantum-Terminable Calculations

In closing Example 7 we remarked that the preceding method of determining the value of k, which goes with a particular ε, may be extended to certain interesting extreme cases. For example, suppose that the average number n of photons of a given frequency ν incident per second per unit area per unit solid angle on a surface falls below some number n_0, say $n_0 = 10^{-2}$, or $n_0 = 10^{-3}$, etc. Suppose that this magnitude of n_0 is so small that it is operationally meaningless to theorize about or experiment with the radiance N_0 produced by n_0. That is, N_0 is not measurable using available radiance meters because it is below their threshold of sensitivity. Suppose $\varepsilon = N_0/N$, where N is some fiducial magnitude for radiance--say that of the order of magnitude of the sun's maximum spectral radiance. This value of ε will then determine a corresponding finite value of k, say k(ε), after the manner illustrated for the special case above. This value k(ε) in turn can evidently be used in defining a terminable response radiance calculation. For example:

$$N_+(b;k(\varepsilon)) = N_-^0(a)t_-^0(a)r_-(b)\sum_{j=0}^{k(\varepsilon)} \left(r_+(a)r_-(b)\right)^j$$

would define a terminable calculation for $N_+(b;k(\varepsilon))$. This would in turn give rise to a terminable calculation for $N_-(a;k(\varepsilon))$.

Such calculations, which are terminable by introducing quantum concepts in the way just indicated, are called *quantum-terminable calculations* and provide a basis for a strong physical argument in favor of the study of terminable calculations in radiative transfer theory. Terminations therefore need not be arbitrary; but can be based on real physical limitations of the apparatus on which rest the phenomenological foundations of the discipline. A systematic study of quantum-terminable calculations appears to hold certain interesting theoretical challenges (for example, can a consistent finite algebra of operators be developed on the basis of quantum-terminable calculations?). This study, however, is beyond the scope of the present work and is left for the interested reader to pursue.

Example 8: Two Interacting Finite Plane Surfaces

In the present example we return to the setting of the preliminary example in Sec. 3.1 and reformulate the problem of that section using now theoretical radiances and the method of the interaction principle. Fig. 3.13 reconstructs the essential features of the setting of Fig. 3.2 in anticipation of the use of the appropriate forms of the integral operators $r_\pm(Y)$ and $t_\pm(Y)$. The unit *outward* normals $\mathbf{k}_1$ and $\mathbf{k}_2$ for the two plane surfaces S_1 and S_2 fix the outward $\Xi_+(S_i)$ and

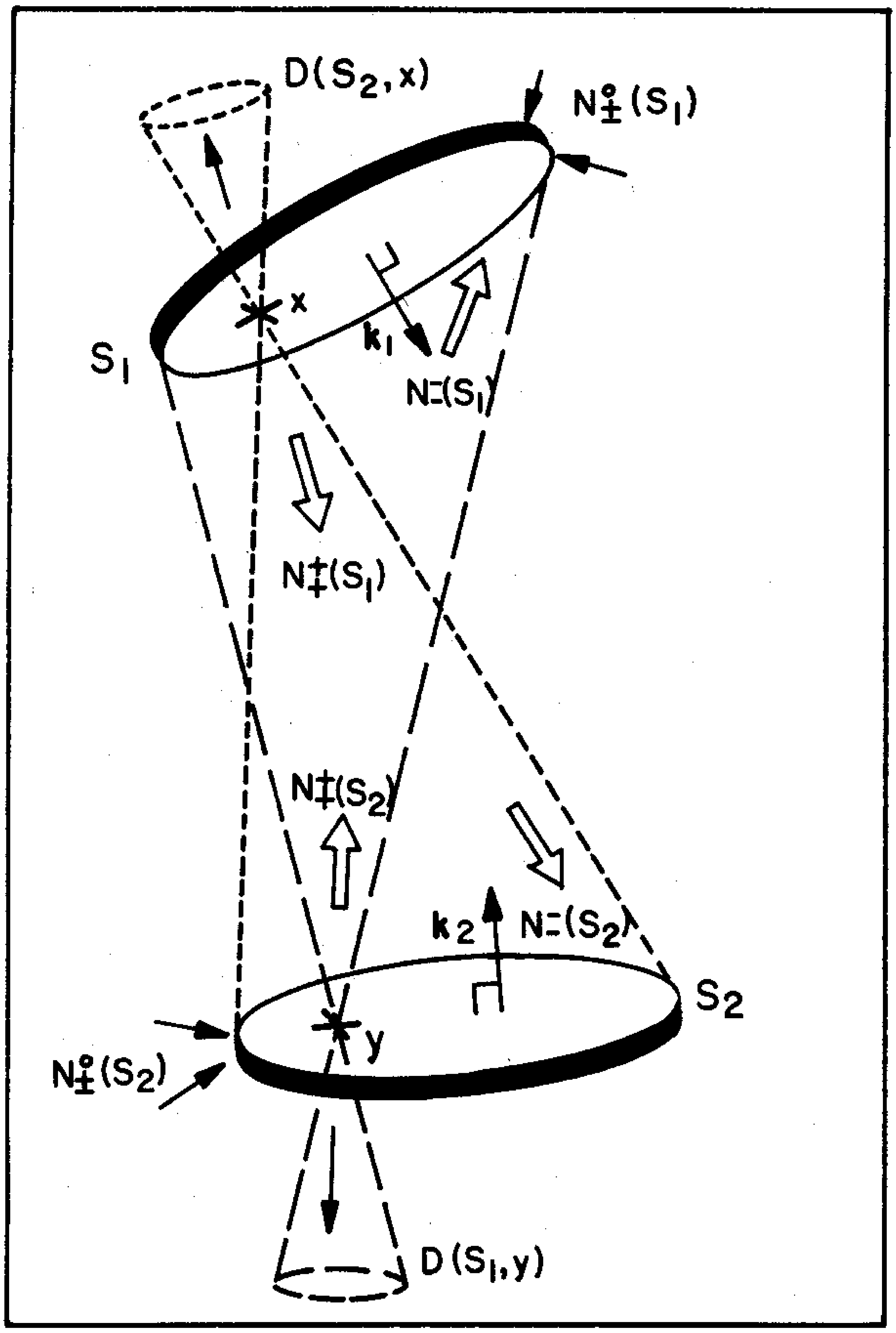

FIG. 3.13 Two interacting finite plane surfaces.

inward $\Xi_-(S_i)$ hemispheres on S_i, $i = 1$ or 2. Thus $\Xi_+(S_i)$ consists of all unit vectors ξ such that $\xi \cdot \mathbf{k}_i > 0$ and $\Xi_-(S_i)$ consists of all unit vectors ξ such that $\xi \cdot \mathbf{k}_i < 0$. This convention of fixing outward and inward hemispheres of *interacting* surfaces is to be distinguished from the corresponding convention for *collecting* surfaces used in Sec. 2.4. For collecting surfaces it is sometimes more convenient to refer the directions of incident flux to a unit *inward* normal. For a surface which is explicitly considered to interact with another, the outward unit normal is occasionally a more convenient reference direction to use. We do not intend, however, to permanently fix such conventions. Rather we shall choose between the conventions as a given situation favors one or the other. With the direction coordinate frames anchored to S_1 and S_2 in the above manner we now require that for every y in S_2 the set $D(S_1,y)$ of all directions from points of S_1 to y to lie in $\Xi_-(S_2)$. Conversely, we require for every x in S_1

the set $D(S_2,x)$ of all directions from points of S_2 to x to lie in $\Xi_-(S_1)$. See Fig. 3.13. These conditions amount to the simple requirement that each surface lie above the other's horizon. This is not an essential restriction; it serves only to shorten the number of cases considered in the analysis below. We require that the reflectance and transmittance functions of S_1 and S_2 be known and that the space between S_1 and S_2 be a vacuum. The two surfaces are irradiated by incident radiance distributions $N^o_\pm(S_i)$, i = 1 or 2. It is then required to formulate and solve the interreflection problem associated with S_1 and S_2 under these conditions. In particular we require the response (surface) radiance distributions $N_\pm(S_i)$ of S_i, i = 1 or 2. Thus $N_+(S_1)$, e.g., is a function which assigns to each x in S_1 and ξ in $\Xi_+(S_1)$ a surface radiance $N_+(x,\xi)$. As usual when the need arises to distinguish between field and surface radiances for S_1, the appropriate superscripts "-" or "+" respectively, will be appended to "N". Thus "$N^-_\pm(S_i)$" will denote field (incident) radiance distributions over S_i, i = 1 or 2, and "$N^+_\pm(S_i)$" will denote the surface (response) radiance distributions of S_i.

We isolate surface S_1 and enumerate the sets of incident radiometric functions on S_1:

A_1: All field radiance distributions like $N^o_-(S_1)$

A_2: All field radiance distributions like $N^o_+(S_1)$

A_3: All field radiance distributions like $N^-_-(S_1)$

The set of response radiometric functions for S_1 are:

B_1: All surface radiance distributions like $N^+_+(S_1)$

B_2: All surface radiance distributions like $N^+_-(S_1)$

Thus, in the case of surface S_1, m = 3, n = 2, and the six interaction operators s_{ij} supplied by the interaction principle are in the form of reflectance and transmittance integral operators as follows:

$$s_{11} \text{ -- } r^o_-(S_1)$$

$$s_{12} \text{ -- } t^o_-(S_1)$$

$$s_{21} \text{ -- } t^o_+(S_1)$$

$$s_{22} \text{ -- } r^o_+(S_1)$$

$$s_{31} \text{ -- } r_-(S_1)$$

$$s_{32} \text{ -- } t_-(S_1)$$

The six operators $r^o_-(S_1),\ldots,t_-(S_1)$, are instances of definitions (10) and (11) of Sec. 3.3. They are handled using the techniques illustrated in Example 5. Then, according to the interaction principle the response radiance distributions are given by:

$$N_+^+(S_1) = N_-^0(S_1)r_-^0(S_1) + N_+^0(S_1)t_+^0(S_1) + N_-^-(S_1)r_-(S_1) \quad (89)$$

$$N_-^+(S_1) = N_-^0(S_1)t_-^0(S_1) + N_+^0(S_1)r_+^0(S_1) + N_-^-(S_1)t_-(S_1). \quad (90)$$

By repeating this process of application of the interaction principle to surface S_2, we arrive at the analogous pair of statements:

$$N_+^+(S_2) = N_-^0(S_2)r_-^0(S_2) + N_+^0(S_2)t_+^0(S_2) + N_-^-(S_2)r_-(S_2) \quad (91)$$

$$N_-^+(S_2) = N_-^0(S_2)t_-^0(S_2) + N_+^0(S_2)r_+^0(S_2) + N_-^-(S_2)t_-(S_2). \quad (92)$$

An inspection of (89) and (91) shows that these equations are identical in structure; similarly for (90) and (92). The present choice of coordinate frames has rendered the formulation completely symmetrical with respect to S_1 and S_2. It is of interest to note that the domain of integration of the operator $r_-(S_1)$, e.g., may be limited at each x of S_1 to $D(S_2,x)$, and that of $r_-(S_2)$, may be limited to $D(S_1,y)$ at each y of S_2. Similar observations hold for $t_-(S_i)$, i = 1 or 2.

The solution procedure of the system (89)-(92) will not be exhibited; it is similar in all essential respects to that for the system (76)-(79). Those who wish to solve (91)-(92) explicitly should observe that the present counterparts to (80), (81) are given by the symmetric pair of auxiliary equations:

$$N_+^+(S_2) = N_-^-(S_1) \quad (93)$$

$$N_+^+(S_1) = N_-^-(S_2) \quad (94)$$

where the domain of the distributions are suitably restricted. Thus, e.g., (93) is understood to state that

$$N_+^+(y,\xi) = N_-^-(x,\xi) \quad (95)$$

for every x in S_1 and y in S_2 such that

$$\xi = (x-y)/|x-y| \quad .$$

By allowing S_1 and S_2 to be mutual point sources as in the preliminary example of Sec. 3.1, and by setting $N_\pm^0(S_i) = 0$, i = 1 or 2, the reader may easily show that (89) can be reduced to (2) of Sec. 3.1 and that (91) can be reduced to (3) of Sec. 3.1. In this reduction, observe that D_{12} in Fig. 3.2 is now replaced by $-D(S_2,x)$, and D_{21} by $-D(S_1,y)$. Of even more interest is the fact that the present formulation contains as a limiting case all the preceding examples on infinite parallel planes (set S_1 and S_2 parallel, and let them become arbitrarily large).

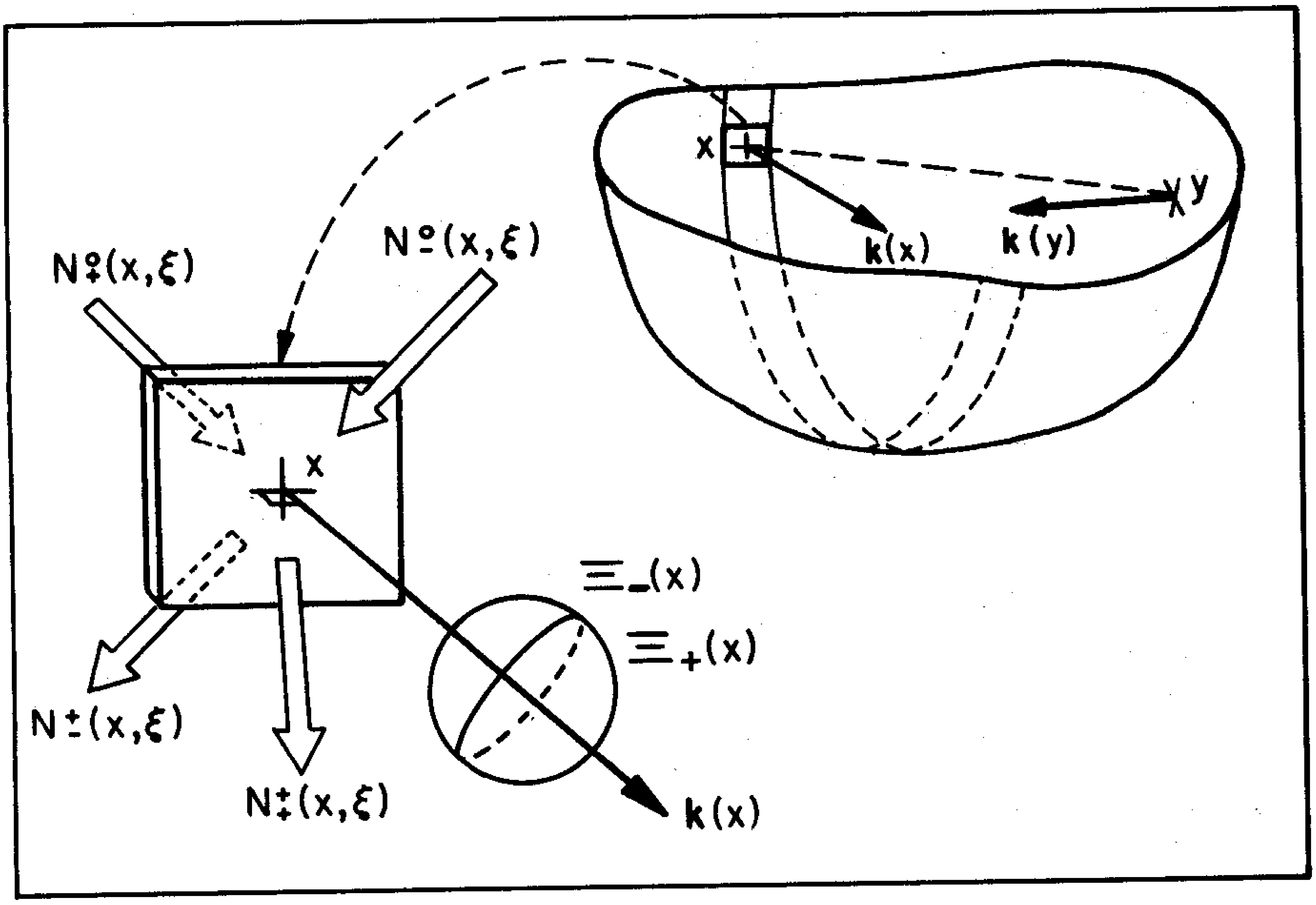

FIG. 3.14 Geometric conventions for radiometry on open concave surfaces.

<< Table of Contents << Title Page >> Next Page >>

3.5 Applications to Curved Surfaces

The distinguishing feature of curved surfaces for radiometry in general and the interaction principle in particular is the fact that such surfaces, unlike plane surfaces, may interact radiometrically with themselves. In this section we illustrate the application of the interaction principle to curved surfaces with this feature of self-interaction particularly in mind.

Example 1: Open Concave Surfaces

As a first illustration, consider a smooth open concave surface S in an optical medium X which is otherwise a vacuum. S is of finite extent and, as depicted in Fig. 3.14, has the general appearance of a dish or bowl. Each point x on S is visible to every other point y on S. At each point x of S we erect a unit outward normal **k**(x) which automatically determines the *outward* hemisphere: $\Xi_+(x)$; and *inward* hemisphere: $\Xi_-(x)$, at x. Instead of going into complete analytical specifications of the sense of "outward", we let Fig. 3.14 help fix the sense which is intended: the angles between **k**(x) and the directions to every other y in S from x are less than 90°. Here "outward" direction at x, as usual, means "away from S" in the immediate vicinity of x along some specified direction. By traveling in an outward direction from a plane surface, one is carried ever farther from the plane.