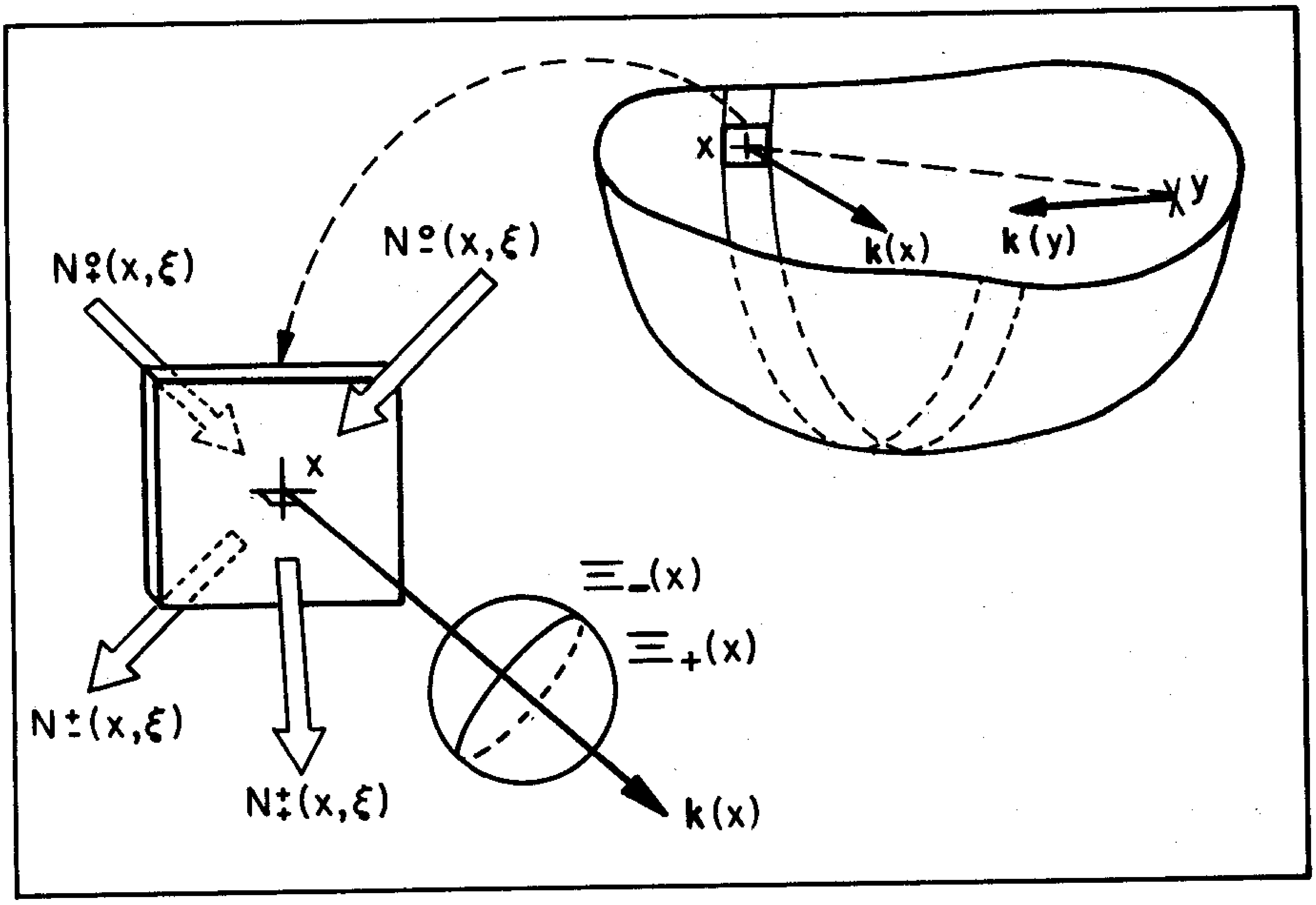

FIG. 3.14 Geometric conventions for radiometry on open concave surfaces.

3.5 Applications to Curved Surfaces

The distinguishing feature of curved surfaces for radiometry in general and the interaction principle in particular is the fact that such surfaces, unlike plane surfaces, may interact radiometrically with themselves. In this section we illustrate the application of the interaction principle to curved surfaces with this feature of self-interaction particularly in mind.

Example 1: Open Concave Surfaces

As a first illustration, consider a smooth open concave surface S in an optical medium X which is otherwise a vacuum. S is of finite extent and, as depicted in Fig. 3.14, has the general appearance of a dish or bowl. Each point x on S is visible to every other point y on S. At each point x of S we erect a unit outward normal $\mathbf{k}(x)$ which automatically determines the *outward* hemisphere: $\Xi_+(x)$; and *inward* hemisphere: $\Xi_-(x)$, at x. Instead of going into complete analytical specifications of the sense of "outward", we let Fig. 3.14 help fix the sense which is intended: the angles between $\mathbf{k}(x)$ and the directions to every other y in S from x are less than 90°. Here "outward" direction at x, as usual, means "away from S" in the immediate vicinity of x along some specified direction. By traveling in an outward direction from a plane surface, one is carried ever farther from the plane.

In the case of a curved surface such as S however, by traveling far enough along some outward directions from x, one can eventually reach S again at a point y and make contact while traveling along an inward direction at y. This elementary observation is a key observation needed in the formulation of the present interreflection problem. We let S be irradiated at each point by steady inward and outward incident radiance distributions $N^o_\pm(S)$ which are conveniently thought of as originating at places other than points on S. Thus the value of $N^o_+(S)$ at x and in the direction ξ in $\Xi_+(x)$ is $N^o_+(x,\xi)$. For example, if S is a portion of the sea surface at an instant of time, then $N^o_-(S)$ is the radiance distribution over that part of the sky visible at the point of interest, and $N^o_+(S)$ is the radiance distribution of that part of the underwater scene visible at the point of interest. These sources initiate and sustain an interreflection process on S where it is now possible for the immediate neighborhoods of every pair of points x and y of S to interact radiometrically. Returning to Fig. 3.14, let "$N^+_+(S)$" denote the resultant response radiance distribution over S. Thus for every x in S and ξ in $\Xi_+(x)$, $N^+_+(x,\xi)$ is the resultant surface radiance of S at x in the direction ξ. As usual, the superscript "+" denotes surface radiance, and the subscript denotes that ξ is in $\Xi_+(x)$. Furthermore let "$N^-_-(S)$" denote the resultant field radiance distribution over S. Thus for every x in S and ξ in $\Xi_-(x)$, $N^-_-(x,\xi)$ is the resultant field radiance at x in the direction ξ. In summary, then, $N^-_-(S)$ plays the role of an incident radiance on S, and $N^+_+(S)$ that of a response radiance of S.

The connection between $N^+_+(S)$ and $N^-_-(S)$ is readily established using the radiance invariance law. We have for every distinct pair x,y of points in S:

$$N^+_+(y,\xi) = N^-_-(x,\xi) \tag{1}$$

whenever x and y are two points whose common line lies in a vacuum, and:

$$\xi = (x-y)/|x-y| \quad .$$

The reader will find it of interest to compare (95) of Sec. 3.4 and (1), and dwell on the points of similarity between the formulations of Example 8 of Sec. 3.4 and those of the present example. In particular he may ask: which of the two problems is more general (in the usual sense that the more general problem yields as a special case the less general problem)?

Equation (1) can be stated in a more useful manner by first letting "D(S,x)" denote the set of all directions ξ from points y in S to the fixed point x. Thus D(S,x) is analogous to the sets $D(S_2,x)$ and $D(S_1,y)$ of Example 8 of Sec. 3.4. Then (1) holds at x in S for every ξ in D(S,x), where $y = x - r\xi$, and $r = |x-y|$. Observe that D(S,x) is part of $\Xi_-(x)$ for every x.

We are now ready to use the interaction principle to formulate the present problem. We isolate S and then enumerate the sets of all incident radiometric functions on S.

A_1: all field radiance distributions like $N^o_-(S)$

A_2: all field radiance distributions like $N^o_+(S)$

A_3: all field radiance distributions like $N^-_-(S)$

Enumerating all the sets of response radiometric functions for S:

B_1: all surface radiance distributions like $N^+_+(S)$

B_2: all surface radiance distributions like $N^+_-(S)$

In the present case m = 3, n = 2, and the interaction principle yields the following six interaction operators s_{ij}:

$$s_{11} \text{ -- } r^o_-(S)$$

$$s_{12} \text{ -- } t^o_-(S)$$

$$s_{21} \text{ -- } t^o_+(S)$$

$$s_{22} \text{ -- } r^o_+(S)$$

$$s_{31} \text{ -- } r_-(S)$$

$$s_{32} \text{ -- } t_-(S)$$

The six operators $r^o_-(S), \ldots, t_-(S)$ are instances of definitions (10), (11) of Sec. 3.3. Then, according to the interaction principle, $N^+_\pm(S)$ are given by:

$$N^+_+(S) = N^o_-(S)r^o_-(S) + N^o_+(S)t^o_+(S) + N^-_-(S)r_-(S) \quad (2)$$

$$N^+_-(S) = N^o_-(S)t^o_-(S) + N^o_+(S)r^o_+(S) + N^-_-(S)t_-(S) \quad (3)$$

This pair of interaction equations and the auxialiary equation:

$$N^+_+(S) = N^-_-(S) \quad (4)$$

form an autonomous system of equations. The latter equation is understood in the sense of (1). The order of solution of the equations is dictated by (2): it must be solved first. Thus using (4) in (2) we have:

$$\boxed{N^+_+(S) = A_+(S) + N^+_+(S)r'_-(S)} \quad (5)$$

where we have written:

$$\text{"}A_+(S)\text{"} \quad \text{for} \quad N^o_-(S)r^o_-(S) + N^o_+(S)t^o_+(S) \quad ,$$

and where we have written:

$$\text{"}r'_-(S)\text{"} \quad \text{for} \quad \int_{D(S,x)} [\ ]' r_-(x;\xi';\xi)\, d\Omega(\xi') \tag{6}$$

The prime (') on the square bracket denotes retardation of the argument of the radiance function on which $r'_-(S)$ operates. Thus, if $N^+_+(S)$ is to be evaluated at x and for ξ in $\Xi_+(x)$, then (5) becomes:

$$N^+_+(x,\xi) = A_+(x,\xi) + \int_{D(S,x)} N^+_+(x-r'\xi',\xi') r_-(x;\xi';\xi)\, d\Omega(\xi')$$

where "$A_+(x,\xi)$" denotes the value of $A_+(S)$ at x and ξ. It is clear that $r'_-(S)$ can operate on any element in the response set B_1. It is particularly to be noted that $A_+(S)$ is an element of B_1. Thus, solving (5), we have:

$$N^+_+(S) = A_+(S)[I - r'_-(S)]^{-1} \tag{7}$$

and the inverse of $[I - r'_-(S)]$ exists provided $r'_-(S)$ is norm contracting (cf. (60) of Sec. 3.4).

The prime on the operator $r'_-(S)$ is adequate to communicate the idea of a retarded argument in (6) and (7) to the general reader whose insight into our intentions fortunately can lighten our expository task. However, if (7) is to be programmed for evaluation on an automatic computer, then another--a more mechanical--expedient must be devised to communicate the idea of retarded arguments of a function. For example we could define a mapping $t_o(S)$ which assigns to every x in S and ξ in $\Xi_-(x)$ the point $x - r\xi$ where r is the distance from x to S along the direction $-\xi$. Knowing the analytic description of S, it is in principle possible to compute this r for each x and ξ in $\Xi_-(x)$ and hence to construct $t_o(S)$. Then "$N^+_+(S)t_o(S)$" will denote the function which assigns to every x in S and ξ in $\Xi_-(x)$ the radiance $N^-_-(x,\xi)$ ($= N^+_+(x-r\xi,\xi)$) where $x - r\xi$ is on S and ξ is in $\Xi_+(x-r\xi)$. With this definition of $t_o(S)$, we can rewrite (4) as:

$$N^-_-(S) = N^+_+(S)t_o(S) \tag{8}$$

without the need of further qualifications as was necessary in qualifying (4) by (1). Then using (8) in (2), the more detailed version of (5) is:

$$N^+_+(S) = A_+(S) + N^+_+(S)t_o(S)r_-(S)$$

from which the more detailed version of (7) follows:

$$\boxed{N^+_+(S) = A_+(S)[I - t_o(S)r_-(S)]^{-1}} \ . \tag{9}$$

It is easy to see that if $r_-(S)$ is norm contracting, then so is $t_o(S)r_-(S)$, where we have written:

$$"t_o(S)r_-(S)" \quad \text{for} \quad \int_{D(S,x)} [\cdot t_o(S)]r_-(x;\xi';\xi)\, d\Omega(\xi') \quad (10)$$

Here any function on which $t^o(S)r_-(S)$ operates automatically has its argument x,ξ (ξ in $\Xi_-(x)$) first retarded to $x-r\xi$, and ξ respectively (ξ now considered in $\Xi_+(x-r\xi)$). With this definition of $t_o(S)$, (9) and (7) are equivalent ways of indicating the computation of the response function $N_+^+(S)$. The computation of $N_-^+(S)$ can be performed using (8), (9) and (3).

In closing we note that one can also view the object $t_o(S)$ as a mapping from response set B into incident set A . This interpretation is based on (8). Such mappings occur naturally in the strictly mechanical formulations of the auxiliary equations arising from step (vi) in the interaction method (cf. Example 2, Sec. 3.4).

Example 2: Closed Concave Surfaces; the Integrating Sphere

In the present example we allow the rim of the surface S in Fig. 3.14 to diminish in diameter while leaving the area of S greater than some fixed constant. Thus S becomes a closed concave surface (as seen from within). It is the purpose of this example to point out that the formulations of Example 1 remain unchanged as the open concave surface becomes a closed concave surface. Indeed, as a review of Example 1 would show there is no essential use made at all of the openness of S as depicted in Fig. 3.14. The only important change to note is that $D(S,x)$ is now exactly $\Xi_-(x)$ for every x in a closed concave surface S. Hence (9) holds also for closed concave surfaces. We shall now illustrate (9) for the most useful case of a closed concave surface: the integrating sphere.

Figure 3.15 depicts a spherical surface S of diameter d enclosing a vacuous region. Incident source radiance is restricted to a general part a of S. For simplicity we let the incident source radiance be represented by $N_+^o(S)$ over part a of S. Hence we will write "$N_+^o(a)$" for $N_+^o(S)$ and set $N_-^o(S)=0$ in $A_+(S)$ of Example 1. $N_+^o(a)$ is of arbitrary directional structure but is independent of location over a with respect to the local direction frame, determined at each point y by $\mathbf{k}(y)$. Then (9) becomes:

$$N_+^+(S) = N_+^o(a)t_+^o(a)\left[I - t_o(S)r_-(S)\right]^{-1} \quad . \quad (11)$$

We next adopt the classical assumption that the inside surface of S is a lambert reflector. In addition we assume a is a lambert transmitter. That is, we are assuming (cf. (17) of Sec. 3.3):

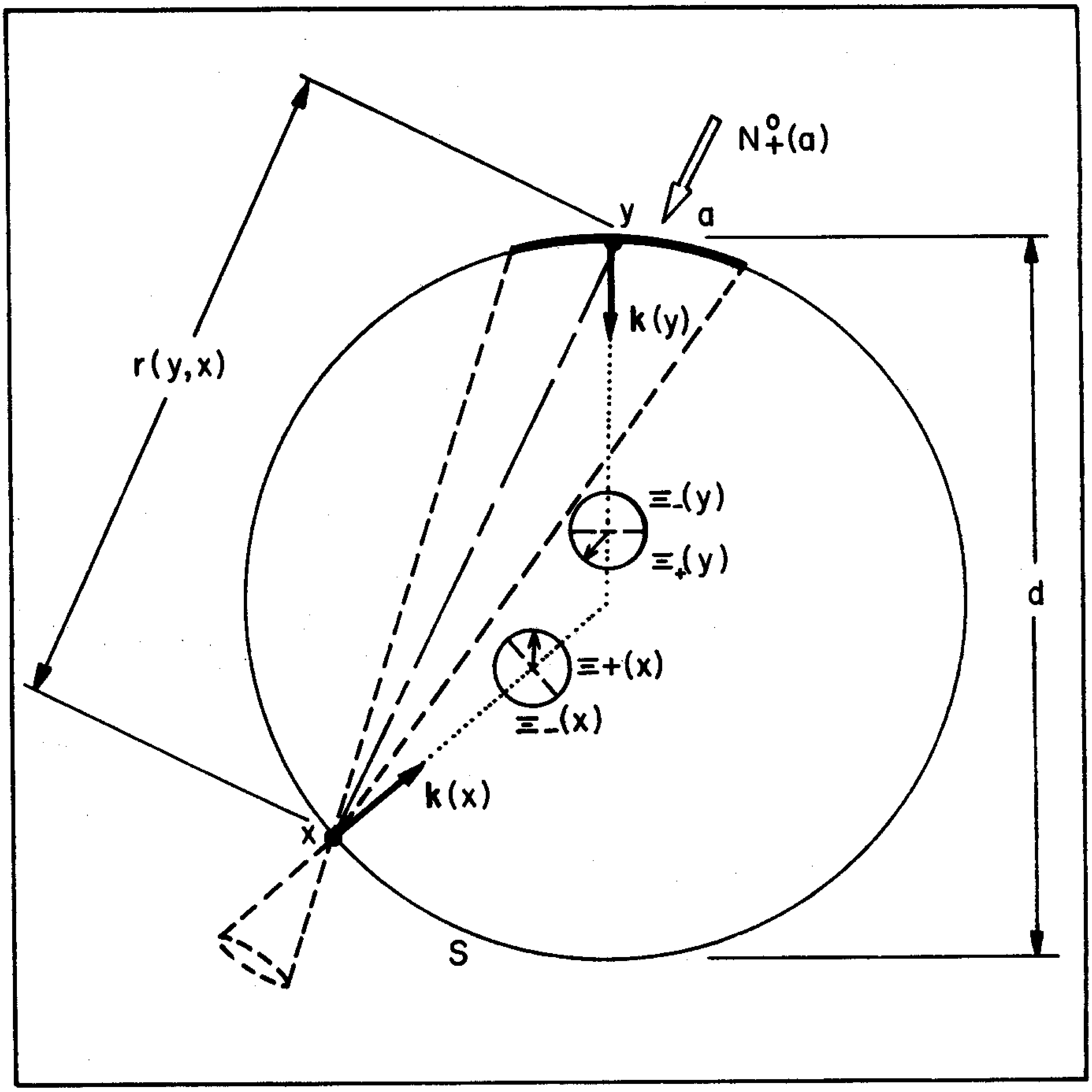

FIG. 3.15 Illustrating the radiometric self-interaction of a closed concave surface. The case of the integrating sphere.

$$r_-(x;\xi';\xi) = \frac{r_-}{\pi}|\xi'\cdot\mathbf{k}(x)|$$

for every x in S, ξ' in $\Xi_-(x)$, and ξ in $\Xi_+(x)$; and with $0 \le r_- < 1$. Further we assume:

$$t_+(x;\xi';\xi) = \frac{t_+}{\pi}\,\xi'\cdot\mathbf{k}(x)$$

for every x in a, ξ' in $\Xi_+(x)$, and ξ in $\Xi_+(x)$. Then:

$$N_+^o(a)\,t_+^o(a) = \frac{t_+}{\pi}\,H_+^o f_a$$

where f_a is a function on S such that $f_a(x) = 1$ if x is in a and $f_a(x) = 0$ if x is not in a. Further, H_+^o is the constant irradiance on a produced by $N_+^o(a)$. Hence $N_+^o(a)t_+^o(a)$ is a uniform radiance distribution over a and is a member of set B_1, (see Example 1) which we shall denote by "N^o".

We now write (11) in the form:

$$N_+^+(s) = N^o\left[I + \sum_{j=1}^{\infty} (t_o(S)r_-(S))^j\right]$$

$$= N^o + \sum_{j=1}^{\infty} N^o(t_o(S)r_-(S))^j \qquad (12)$$

Consider the first term:

$$N^o t_o(S)r_-(S)$$

of the infinite series. We write "N^1" for $N^o t_o(S)r_-(S)$. Then by (10):

$$N^1(x,\xi) = \int_{D(S,x)} N^o(x-r'\xi',\xi')r_-(x;\xi';\xi)\, d\Omega(\xi')$$

for every x in S and ξ in $\Xi_+(x)$. Since the incident source N^o vanishes outside part a of S and is of constant magnitude within a, the domain of integration shrinks from D(S,x) to D(a,x), and

$$N^1(x,\xi) = \frac{t_+}{\pi} H_+^o \int_{D(a,x)} r_-(x;\xi';\xi)\, d\Omega(\xi')$$

$$= \frac{t_+}{\pi} H_+^o \frac{r_-}{\pi} \int_{D(a,x)} |\xi'\cdot\mathbf{k}(x)|\, d\Omega(\xi') \quad .$$

Now by means of an observation following (22) of Sec. 2.11, this integral is readily evaluated:

$$\int_{D(a,x)} |\xi'\cdot\mathbf{k}(x)|\, d\Omega(\xi') = \frac{A(a)}{d^2} \quad .$$

Hence

$$N^1(x,\xi) = \frac{t_+}{\pi} H_+^o\left[r_- \frac{A(a)}{\pi d^2}\right] \quad ,$$

for every x in S and ξ in $\Xi_+(x)$. This result was grouped in the indicated manner to show the effect of the operator $t_o(S)r_-(S)$ on N^o. The effect is to multiply the value of N^o by $[r_- A(a)/\pi d^2] = r_- A(a)/A(S)$.

The uniform surface radiance distribution N^1 is now acted on by $t_o(S)r_-(S)$ to yield the second term of the series.

Observe that N^1 is constant-valued over S, whereas N^o is constant-valued over a and zero over the remainder of S outside a. Thus the second term of the series is:

$$N^1 t_o(S) r_-(S) = \left(N^o t_o(S) r_-(S)\right)\left(t_o(S) r_-(S)\right)$$

$$= N^o \left(t_o(S) r_-(S)\right)^2 \quad .$$

Let us write "N^2" for $N^1 t_o(S) r_-(S)$. Then:

$$N^2(x,\xi) = \int_{D(S,x)} N^1(x - r'\xi', \xi') r(x;\xi';\xi)\, d\Omega(\xi')$$

$$= \frac{t_+}{\pi} H_+^o \left[r_- \frac{A(a)}{A(S)}\right] \frac{r_-}{\pi} \int_{D(S,x)} |\xi' \cdot \mathbf{k}(x)|\, d\Omega(\xi') \quad .$$

With this second iterate of $t_o(S) r_-(S)$, the pattern begins to form. We first note that:

$$\int_{D(S,x)} |\xi' \cdot \mathbf{k}(x) d|\, d\Omega(\xi') = \frac{A(S)}{d^2} = \pi \quad ,$$

for every x in S. Hence:

$$N^2(x,\xi) = \frac{t_+}{\pi} H_+^o \left[r_- \frac{A(a)}{A(S)}\right] r_-$$

$$= N^1(x,\xi) r_- \quad .$$

Thus if we write: for j = 1,2,..., :

"N^j" for $N^{j-1} t_o(S) r_-(S)$,

then for j = 2,3,..., :

$$N^j(x,\xi) = \int_{D(S,x)} N^{j-1}(x - r'\xi', \xi) r_-(x;\xi';\xi)\, d\Omega(\xi')$$

$$= N^{j-1}(x,\xi) \int_{D(S,x)} r_-(x;\xi';\xi)\, d\Omega(\xi')$$

$$= N^{j-1}(x,\xi)\ r_- \quad .$$

Since

$$N^2(x,\xi) = N^1(x,\xi) r_- \quad ,$$

for every x in S and ξ in $\Xi_+(x)$, we then have:

$$N^j(x,\xi) = N^1(x,\xi)(r_-)^{j-1}$$

for j = 2,3,... .

Hence:

$$\sum_{j=1}^{\infty} N^o(t_o(S)r_-(S))^j = N^1 + \sum_{j=2}^{\infty} N^j$$

$$= N^1 + N^1 \sum_{j=2}^{\infty} (r_-)^{j-1}$$

$$= N^1\left(\sum_{j=0}^{\infty} r_-^{\,j}\right)$$

$$= \frac{N^1}{[1-r_-]} \quad .$$

It follows from (12) that

$$N_+^+(S) = N^o + \frac{1}{\pi}\,\frac{\left[r_-t_+H_+^oA(a)\right]/A(S)}{[1-r_-]} \quad .$$

Hence using the explicit values of N^o, we have for every x in S and ξ in $\Xi_+(x)$:

$$N(x,\xi) = \frac{t_+H_+^o}{\pi}\left[f_a(x) + \frac{r_-A(a)/A(S)}{[1-r_-]}\right] \qquad (13)$$

where as noted before $f_a(x) = 1$ if x is in a, and $f_a(x) = 0$ if x is not in a. This formula shows that $N(x,\xi)$ is of uniform directional structure over $\Xi_+(x)$ at each x in S, and is independent of x over a and over the part S-a of S outside of a. However, the radiance distributions over a exceed those in S-a by an amount $t_+H_+^o/\pi$, which is precisely the transmitted radiance through the "window" a of the integrating sphere S. (Observe that no essential use has been made of the sphericity of the surface S. Hence we should expect to extend (13), with only minor changes, to the case of an arbitrary closed surface with lambert properties.)

Example 3: Open and Closed Convex Surfaces

The need to illustrate in great detail the interaction principle for the case of open and closed convex surfaces is obviated by the observation that concavity of surfaces is a relative property, that is, a property relative to the vantage point of the observer. Thus the surface S in Fig. 3.14 is concave relative to an observation point inside the space enclosed by S--i.e., within the bowl of S. On the other hand it appears convex when viewed from below S in the Figure. The interaction equations automatically adjust, without alteration of their general forms, to these two points of view and equations (2), (3), (4) hold also for the convexity interpretation. The only changes in (2)-(4) that might occur are those associated with a reversal of direction of $\mathbf{k}(x)$. *Accordingly, if the user deems to introduce this change, so as to study convex surfaces, then all subscripts "+" and "-" in (2)-(4) and their logical descendants may be interchanged: every occurrence of subscript "+" may be replaced by "-", and conversely.*

Example 4: General Two-Sided Surfaces

In this example we ascend one more rung with respect to the generality of the type of surface considered: we shall apply the interaction principle to general self-interacting one-piece, two-sided surfaces which may be either locally concave or convex, i.e., have alternating hollows or hills. The surfaces may be closed in the sense of enclosing a volume, or open. We assume their reflectance and transmittance properties are known at each point and that they are embedded in a vacuum. As a concrete illustration that may be kept in mind during the following discussion, the instantaneous configuration of a dynamic wind-blown air-water surface will serve well. An application of this example of the air-water surface is made in Sec. 12.10.

Parts (a)-(g) of Figure 3.16 depict some particular instances falling within the scope of the present discussion. An examination of these general surfaces reveals two features which were not present in the cases of concave or convex surfaces considered above. First, for some point x of S there may be points y of S such that x and y are not mutually visible. That is, on the straight line between x and y there lies at least one other point of S. Second, for some points x of S there may be points y of S such that x and y are mutually visible but are on opposite sides of S.

The interaction principle is immediately applicable to general surfaces such as those depicted in Fig. 3.16, and which have the two additional features just described. In order to display the application so that it may be useful in practice and be subject to mechanical manipulation, it is desirable to introduce some preliminary geometric concepts. First we assign, as usual, a unit outward normal $\mathbf{k}(x)$ to S at each x. This fixes $\Xi_+(x)$ at each x and arbitrarily determines the outside and the inside of S. Further, we let "D(S,z)" denote in general the set of all directions from points y of S to a point z. D(S,x) consists of directions ξ in either $\Xi_+(x)$ or $\Xi_-(x)$. Further if ξ is extended to meet S at y, then ξ may be also in either $\Xi_+(y)$ or $\Xi_-(y)$. See, e.g., parts (a) and (e) of Fig. 3.16. It will be necessary to distinguish between such members of D(S,x) which are in $\Xi_+(y)$ or $\Xi_-(y)$.

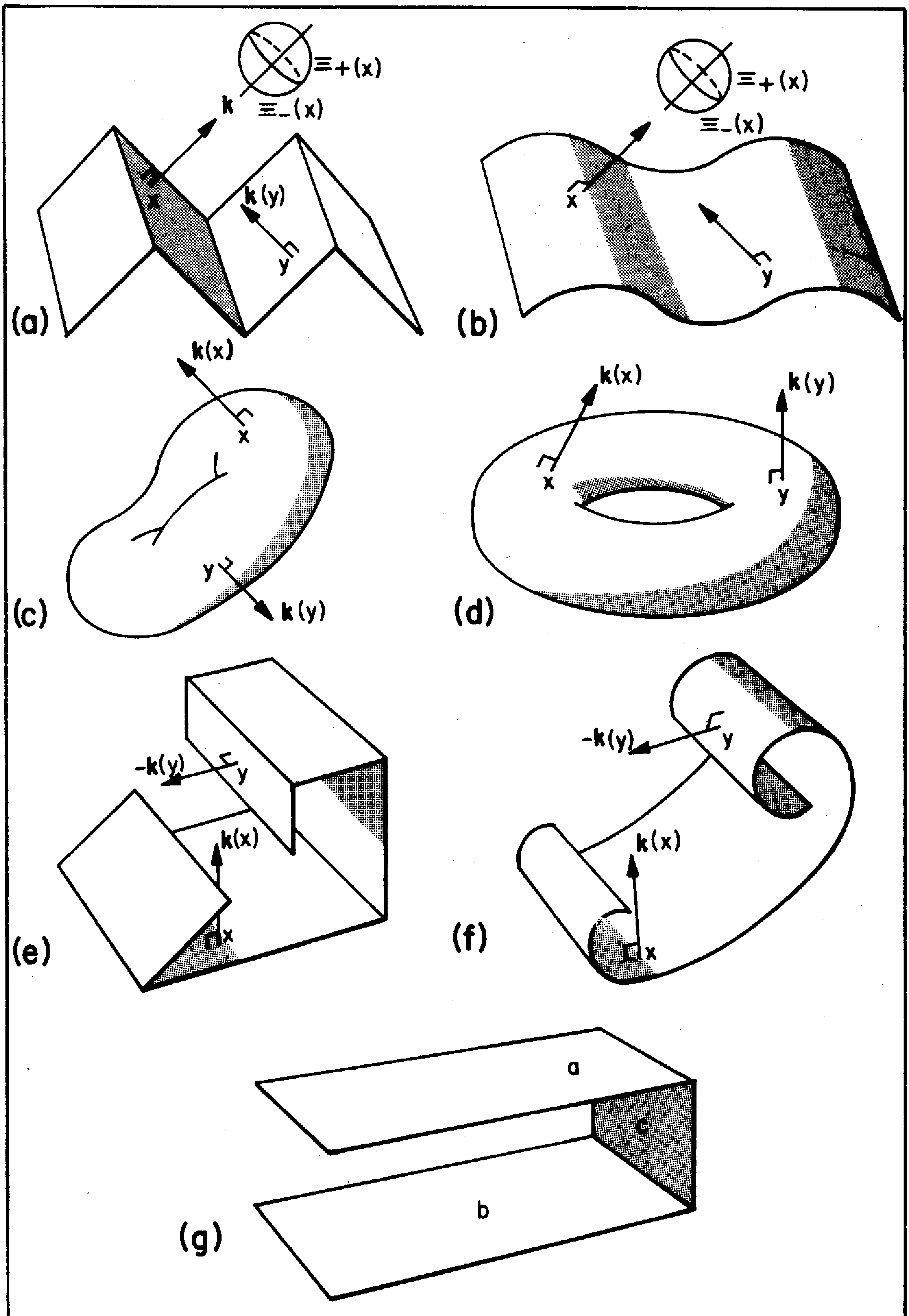

FIG. 3.16 Radiometric self-interaction of two-sided surfaces.

We shall do this in the following way. To begin, for every x in S and ξ in D(S,x) we let "$r_m(x,\xi)$" or simply "r_m" denote the least non negative r such that $x-r\xi$ is a point of S. The geometrical significance of r_m is clear: if at x in S we go along a straight line in direction $-\xi$, then eventually we may reach S, and since some surfaces are corrugated (as in parts (a) or (b) of Fig. 3.16) we may reach S again and again if we continue to travel along the straight line in the direction $-\xi$. The distance r_m is the distance to the first of such meetings with S. Next we let "$t_m(S)$" denote the function which assigns to each x in S and ξ in D(S,x) the point $x-r_m\xi$ on S and the direction ξ. That is, $t_m(x,\xi) = (x-r_m\xi,\xi)$ for every x in S and ξ in D(S,x). Hence $t_m(S)$ is a mapping which is an immediate generalization of the mapping $t_o(S)$ introduced in Example 1. Finally we construct two functions $\chi_+(S)$ and $\chi_-(S)$ with the following properties: for every x in S and ξ in D(S,x), $\chi_\pm(x,\xi) = 1$ if ξ is in $\Xi_\pm(x)$ and $\chi_\pm(x,\xi) = 0$ if ξ is not in $\Xi_\pm(x)$. Finally, for every x in S and ξ not in D(S,x), $\chi_\pm(x,\xi) = 0$. With these geometric preliminaries completed, we can now go on directly to the application of the interaction principle.

Let $N_\pm^o(S)$ be steady incident source radiance distributions on S. These incident distributions generally initiate and sustain a self-interreflection process over S. Let $N_\pm^+(S)$ and $N_\pm^-(S)$ be the resultant surface and field radiance distributions over S.

We first isolate S and then enumerate the incident radiometric functions on S:

A_1: all field radiance distributions like $N_-^o(S)$

A_2: all field radiance distributions like $N_+^o(S)$

A_3: all field radiance distributions like $N_-^-(S)$

A_4: all field radiance distributions like $N_+^-(S)$.

The sets of response functions of S are:

B_1: all surface radiance distributions like $N_+^+(S)$

B_2: all surface radiance distributions like $N_-^+(S)$.

In the present case m = 4, n = 2, and the interaction principle yields the following eight interaction operators s_{ij}:

s_{11} -- $r_-^o(S)$

s_{12} -- $t_-^o(S)$

s_{21} -- $t_+^o(S)$

s_{22} -- $r_-^o(S)$

s_{31} -- $r_-(S)$

s_{32} -- $t_-(S)$

$$s_{41} \text{ -- } t_+(S)$$

$$s_{42} \text{ -- } r_+(S)$$

The preceding eight operators $r_-^o(S),\ldots,r_+(S)$ are instances of definitions (10), (11) of Sec. 3.3. Then, according to the interaction principle, $N_\pm^+(S)$ are given by:

$$N_+^+(S) = N_-^o(S)r_-^o(S) + N_+^o(S)t_+^o(S) + N_-^-(S)r_-(S) + N_+^-(S)t_+(S) \tag{14}$$

$$N_-^+(S) = N_-^o(S)t_-^o(S) + N_+^o(S)r_+^o(S) + N_-^-(S)t_-(S) + N_+^-(S)r_+(S). \tag{15}$$

From the radiance invariance-law and the definitions of the geometric functions $X_\pm(S)$, $t_m(S)$, we have the two auxiliary equations:

$$N_-^-(S) = N_+^+(S)\cdot X_+(S)t_m(S) + N_-^+(S)\cdot X_-(S)t_m(S) \tag{16}$$

$$N_+^-(S) = N_+^+(S)\cdot X_+(S)t_m(S) + N_-^+(S)\cdot X_-(S)t_m(S) \;. \tag{17}$$

These auxiliary equations together with (14) and (15) constitute an autonomous system of integral equations governing the surface radiance distribution $(N_\pm^-(S), N_-^+(S))$ over an arbitrary two-sided surface S. The dots denote multiplication of functions, and the multiplication is done after the operation $t_m(S)$ is applied to $X_\pm(S)$ and $N_\pm^+(S)$.

As an illustration of the use of (16), let x be a point of S and let ξ be in D(S,x). Then the value of $t_m(S)$ at (x,ξ) is $(x-r_m\xi,\xi)$ and this is used in the argument of $X_-(S)$. It follows that one of the two values $X_+(x-r_m\xi,\xi)$ or $X_-(x-r_m\xi,\xi)$ must be 0, and the other 1. Say the former is 0 and the latter is 1. Then the value of $N_-^-(x,\xi)$ of $N_-^-(S)$ at x, is:

$$N_-^-(x,\xi) = N_-^+(x-r_m\xi,\xi) \quad .$$

Hence the downward field radiance $N_-^-(x,\xi)$ comes from the downward surface radiance at $x-r_m\xi$. Thus x and $x-r_m\xi$ are on opposite sides of S, and so S must be curled like that in (e) or (f) of Fig. 3.16. This illustration shows that $X_-(S)t_m(S)$ is to be interpreted as the *composition* of the functions $X_-(S)$ and $t_m(S)$.

The integral operations in (14) and (15) are all generally alike. For the purposes of illustration we take $N_+^-(S)t_+(S)$ as typical. Then, according to (11) of Sec. 3.3 with $Y = S_1$, the value of $N_+^-(S)t_+(S)$ at x in S and ξ in $\Xi_+(x)$ is:

$$\int_{\Xi_+(x)} N_+^-(x,\xi')t_+(x;\xi';\xi)\, d\Omega(\xi')$$

It is clear that we may replace $\Xi_+(x)$ by $\Xi_+(x) \cap D(S,x)$, i.e., by that part of $D(S,x)$ in $\Xi_+(x)$. Furthermore, if we use (17) to replace $N_+^-(x,\xi')$ in the integrand, the integral becomes:

$$\int_{\Xi_+(x)\,\cap\,D(S,x)} \Big[N_+^+(x-r_m\xi',\xi')\cdot\chi_+(x-r_m\xi',\xi') + N_-^+(x-r_m\xi',\xi')\cdot\chi_-(x-r_m\xi',\xi')\Big]\, t_+(x;\xi';\xi)\; d\Omega(\xi')$$

The reader should now examine the set (14)-(17) with the purpose in mind of noting that the set contains as special cases the convex and concave examples above. Plane surfaces are also covered; for then $D(S,x)$ has zero solid angle measure for every x on S and the last two terms in (14), (15) vanish by virtue of (16), (17) and the definitions of $\chi_\pm$. The preceding illustration bears this out in part. Furthermore, by invoking a certain amount of geometric-radiometric trickery, the set (14)-(17) can also yield, in the limit, the cases of a set of finite or infinite parallel planes. For example, S may have the three part configuration as in part (g) of Fig. 3.16, with parts a and b parallel planes and part c having zero reflectance and unit transmittance. This would yield the case of parallel finite planes. These observations will make plausible the assertion that the system (14)-(17) actually constitutes the interaction equations for a general collection of two-sided surfaces S, where S is either in one piece or several distinct pieces, and of concave, convex, or mixed curvature. It is not intended, however, that the set (14)-(17) itself always be reduced to each case as it arises. We have exhibited the preceding interaction equations mainly to show the scope of the interaction method and the mechanical ease with which it can formulate radiometric interaction problems. It is desirable, rather, especially for students of the subject, that each specific instance of an interaction problem be derived anew from the principle and that simplifications be made and auxiliary equations invoked which are motivated by the particular features of the individual case.

Example 5: General One-Sided Surfaces

In this example we apply the interaction method to the formulation of the interreflection light field over one-sided surfaces. Before going into the details it is of interest to observe that one-sided surfaces are mathematical objects which arose originally in critical studies of the classical surface integration theorems of Stokes and Gauss. One-sided surfaces were used principally as counterexamples to show the limitations of the classical forms of these theorems. It is because of this predominantly negative role played by one-sided surfaces in the early training of physics and mathematics students, and because of the spectacular and intuitively bizarre claims made for these surfaces, that a student eventually carries away with him the general impression that one-sided surfaces are conceptual beasts which are inferior to their more applicable two-sided cousins, and are better left alone. This impression is, for the most part, defensible since the classical surface integration theorems in the usual physical applications of mathematics pertain only and implicitly to two-sided surfaces. The implicitness of the two-sided condition is eventually forgotten, and the ingrained avoidance of one-sided surfaces prevents their use in the application of the usual theorems. However, with a little additional effort the one-sided surfaces can occasionally be brought into physical discussions and their physical properties compared--usually with deeper resultant insight--with the corresponding properties of two-sided surfaces. In this example we shall perform this service for the radiative transfer context. We shall briefly consider the interaction principle applied to the most notorious of one-sided surfaces, the Möbius Strip. What we shall find out in this application will be typical of the radiometric properties of one-sided surfaces in general, and no more dramatic than the simple but useful insight that it takes exactly one half the number of equations to formulate the radiometric interaction equations for one-sided surfaces as it does for two-sided surfaces. Hence the four general equations of Example 4 will be reduced to two for the most general one-sided surface.

To help fix ideas we shall adopt as the prototype of one-sided surfaces the Möbius strip depicted in Fig. 3.17. The Möbius strip S is shown in plan view in part (b) of Fig. 3.17 and its mode of generation is shown in perspective in (a) of Fig. 3.17. To generate S, one can imagine first of all a circle C_o of radius r_o in a plane. Then a line segment L of length 2a is placed so that its center is on C_o and so that the line segment, extended, goes through the center of C_o. If L is moved around C_o with this orientation maintained, and remaining in the plane of C_o, L will sweep out a circular annulus of radii r_o+a and r_o-a. To generate the Möbius strip S itself, instead of keeping L in the plane of C_o, now, keeping L perpendicular to C_o, let L rotate with its center always on C_o, and at a uniform angular speed so that as L makes one circuit of C_o, it will turn 180° about its point of contact with C_o. The equations for this Mobius strip are given in parametric form using cylindrical coordinates as shown in (b) of Fig. 3.17:

$$r = r_o + \rho \cos (\phi/2)$$

$$\phi = \phi$$

$$z = \rho \sin (\phi/2)$$

$$-a \le \rho \le a, \quad 0 < a < r_o, \quad 0 \le \phi \le 4\pi \qquad (18)$$

where ρ and ϕ are parameters for the surface and r, ϕ, z are the usual cylindrical coordinate variables. The surface (the set of points in space) is then completely specified once we give the magnitudes a and r_o.

The parameters ρ and ϕ can be used to locate a point on S just as latitude and longitude are used to locate a point

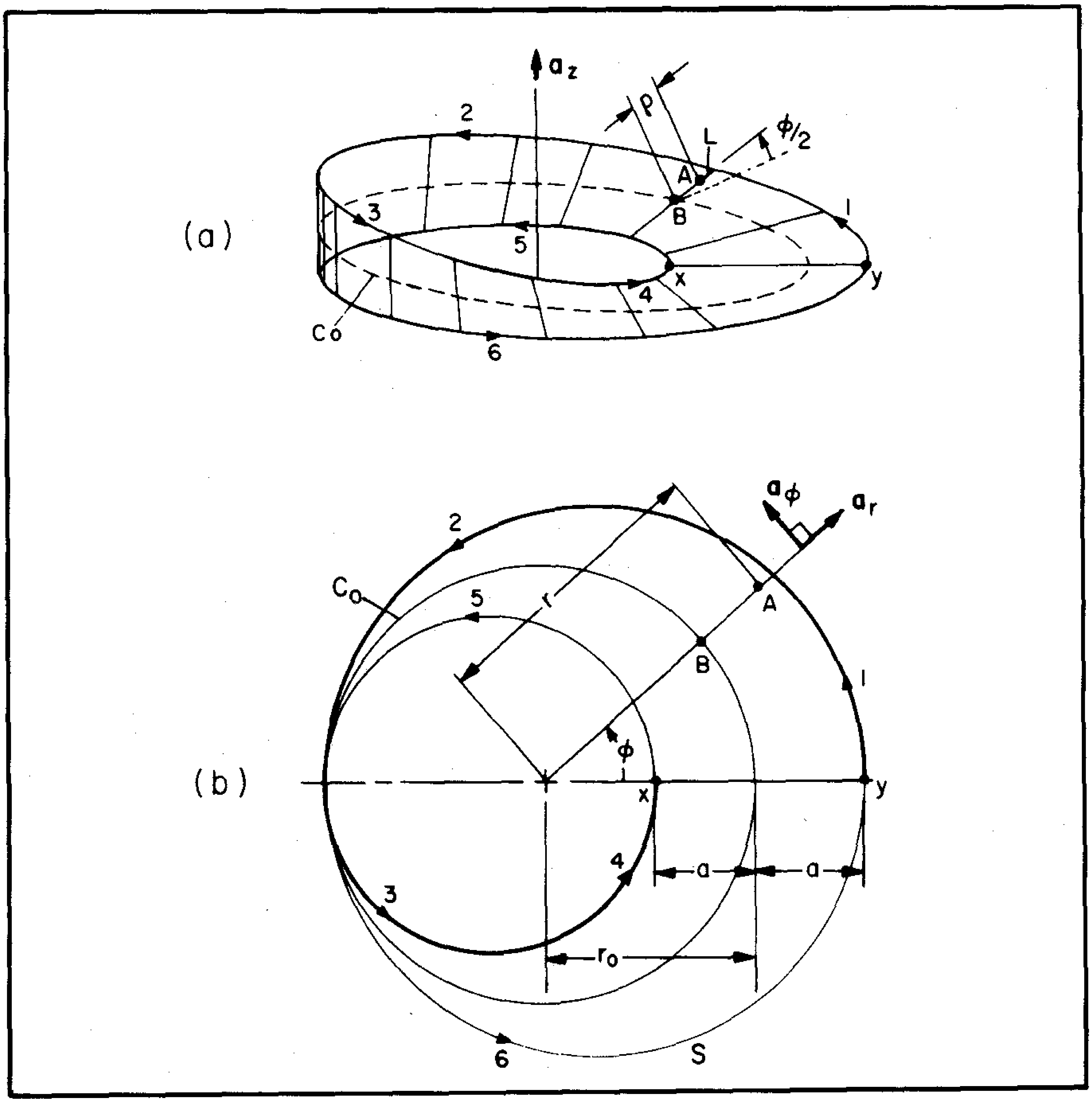

FIG. 3.17 Radiometric self-interaction of one-sided surfaces. The case of the Möbius strip.

on the surface of the earth. For example, given $\rho = a, \phi = 90°$; one finds the corresponding point on S by walking on S along C_0 in the direction of increasing ϕ until a point on C_0 is reached which is on the 90° radial line through the origin. Then, still standing on S and still facing in the direction of increasing ϕ, one's extended right arm points in the direction of increasing ρ; and one's extended left arm points in the direction of decreasing ρ. In the present case one goes a units of distance along S to the right to get to (a,90°). The coordinates at this point in the cylindrical reference frame are obtained by setting $\phi = 90°$ and $\rho = a$ in the system (18) of equations:

$$r = r_o + a\sqrt{2}/2$$

$$\phi = 90°$$

$$z = a\sqrt{2}/2$$

Now, from a radiometric point of view any realization of the surface S will serve as a perfectly good example of a passive reflector and transmitter of radiant flux. By placing a realization of S say, in the form of a translucent matte white plastic strip out in sunlight, one can observe the play of light over its surface and see the interreflection effects where flux from one part augments the natural external flux falling on another part. Hence, the one-sidedness of S does not at all interfere with the inherent interactions of S with light. The one-sidedness of S enters the picture when an unsuspecting human observer wishes to unambiguously fix the unit outward **k**(x) at some point x of S preparatory to constructing the sets $\Xi_+(x)$ and $\Xi_-(x)$ of outward and inward directions relative to S at x. The observer soon sees that a certain property of "two-sided" surfaces is now lacking--a property which is so deeply ingrained in our geometric intuition about surfaces that it is virtually taken as universal and as possessed by every surface. This property of two-sided surfaces may be phrased as follows: *if one travels a closed curve* C *on a two-sided surface* S, *constructing* $\Xi_+(x)$ *at each* x *of* C *from knowledge of the surface coordinates* ρ *and* ϕ *according to some fixed rule, then when the journey along* C *is brought to a close, the last set* $\Xi_+(x)$ *constructed coincides with the set* $\Xi_+(x)$ *constructed at the outset of the journey.* This property is not possessed by the Möbius Strip S. Thus suppose we agree that, as we stand at any x on S, the direction of **k**(x) will be from one's foot to one's head. If one walks from a point x_0 around a small closed path in the neighborhood of a point x, then as one's feet return to x_0 his head will be in the same spot as when he started out. However, if he walks from x_0 around a larger circuit--say all around circle C_0, then when he returns to x_0, he will be relatively upside down from his original position. By making the circuit around C_0 once more, he will regain his original orientation. Hence for the present surface S, he can return, after traversing certain closed paths, to his starting point, but with an orientation opposite to that with which he started out. A surface S is *two-sided* (or *orientable*) if the preceding italicized statement holds for *every* C on S; otherwise a surface is *one-sided* (or *non orientable*).

The salient effect of one-sided surfaces on radiative transfer theorizing is that such surfaces obviate the necessity of considering *both* the upper $\Xi_+(x)$ and the lower hemispheres $\Xi_-(x)$ of the unit sphere Ξ. It is found that the upper hemisphere $\Xi_+(x)$, e.g., alone suffices to describe the radiometric interactions of such a surface. This may be seen as follows. First, by means of some elementary vector analysis, we see that a unit "outward" normal **k**(x) at point x on the Möbius Strip is given by the equation:

$$\mathbf{k}(x) = \frac{1}{1+\frac{\rho^2}{2}}\left[-a_r(x)\sin\left(\frac{\phi}{2}\right) - \frac{\rho}{2}\,a_\phi(x) + a_z(x)\cos\left(\frac{\phi}{2}\right)\right] \qquad (19)$$

in which we have written:

$$\text{"x"} \quad \text{for} \quad (r,\phi,z) \quad ,$$

and r, ϕ, and z are as given in (18) for every choice of parameters ρ and ϕ. Further, the unit vectors $\mathbf{a}_r(x)$, $\mathbf{a}_\phi(x)$, $\mathbf{a}_z(x)$ of the cylindrical coordinate system point along the directions of increasing r, ϕ, and z in that system, as usual. Hence $\mathbf{k}(x)$ is uniquely determined by each choice of ρ and ϕ, $-a \le \rho \le a$, $0 \le \phi \le 2\pi$. We can denote this fact also by writing:

$$\text{"}\mathbf{k}(\rho,\phi)\text{"} \quad \text{for} \quad \mathbf{k}(x) \quad .$$

Now the heart of the one-sidedness of S resides in the following fact: for every ρ,ϕ, by (18), (ρ,ϕ) and $(-\rho,\phi+2\pi)$ determine the same point x on S and:

$$\mathbf{k}(\rho,\phi) = -\mathbf{k}(-\rho,\phi+2\pi) \quad . \tag{20}$$

Next for each $\mathbf{k}(x)$ we define $\Xi_\pm(x)$ as usual, where x corresponds to (ρ,ϕ). To point up this dependence of $\Xi_\pm(x)$ on (ρ,ϕ), we shall write "$\Xi_\pm(\rho,\phi)$" for $\Xi_\pm(x)$. Then we see that:

$$\boxed{\Xi_-(\rho,\phi) = \Xi_+(-\rho,\phi+2\pi)} \quad . \tag{21}$$

This is what we wished to show: the lower hemisphere corresponding to (ρ,ϕ) is equal to the upper hemisphere corresponding to $(-\rho,\phi+2\pi)$. Hence a downward direction at (ρ,ϕ) may be thought of as an upward direction at $(-\rho,\phi+2\pi)$.

It should be recalled that the numbers ρ and ϕ are coordinates one uses while maneuvering about the surface S. If one starts at (ρ,ϕ) where $\rho = 0$ and $\phi = 0$ and walks to (ρ,ϕ) in the manner explained above, and then starts all over again and walks to $(-\rho,\phi+2\pi)$, his feet will come to the same point in space but the final positions of his head will be diametrically opposed. We shall call the number pairs (ρ,ϕ), $(-\rho,\phi+2\pi)$ *conjugate coordinates*. If a $\rho\phi$-coordinate system were painted on a realization of the Möbius Strip S, then the conjugate coordinates (ρ,ϕ) and $(-\rho,\phi+2\pi)$ could be thought of as specifying two "different" points on S. These points could be imagined different in the sense that if S were sufficiently large and opaque then two people standing on points with conjugate coordinates would be hidden from each other. However, from a construction point of view, i.e., viewing S as an assembly of points, the point located by (ρ,ϕ), once placed into position, obviated the need of placing that corresponding to $(-\rho,\phi+2\pi)$.

We can now apply these accumulated findings to the problem of formulating the radiometric interaction equations for a general one-sided surface S. If (ρ,ϕ) are the coordinates of a point on S, with ρ and ϕ variables drawn from some suitable range of numbers such as that suggested above, we let "$\mathbf{k}(\rho,\phi)$" denote the unit outward normal to S at the point with coordinates (ρ,ϕ). To each point on S with coordinates

(ρ,ϕ) there exist conjugate coordinates (ρ',ϕ') such that (ρ,ϕ) and (ρ',ϕ') determine the same point on S but:

$$\mathbf{k}(\rho,\phi) = -\mathbf{k}(\rho',\phi') \quad . \tag{22}$$

Let "Φ" denote the function which assigns to each coordinate (ρ,ϕ) its conjugate coordinate (ρ',ϕ'):

$$(\rho',\phi') = \Phi(\rho,\phi) \quad .$$

In the case of the Möbius Strip above, $\rho' = -\rho$ and $\phi' = \phi+2\pi$. Then (22) can also be written:

$$-\mathbf{k}(\rho,\phi) = \mathbf{k}\big(\Phi(\rho,\phi)\big) \quad . \tag{23}$$

Further

$$\Xi_-(\rho,\phi) = \Xi_+\big(\Phi(\rho,\phi)\big) \tag{24}$$

which may be abbreviated to

$$\Xi_- = \Xi_+\Phi \quad . \tag{25}$$

Still further, if $N_-^+(\rho,\phi,\xi)$ is the downward surface radiance of S at (ρ,ϕ) in direction ξ in $\Xi_-(\rho,\phi)$, then the radiance may be represented as the upward surface radiance $N_+^+(\rho',\phi',\xi)$ where (ρ,ϕ) and (ρ',ϕ') are conjugate coordinates on S. Hence:

$$\begin{aligned} N_-^+(\rho,\phi,\xi) &= N_+^+(\rho',\phi',\xi) \\ &= N_+^+\big(\Phi(\rho,\phi),\xi\big) \end{aligned}$$

or more briefly:

$$N_-^+(S) = N_+^+(S)\Phi \quad , \tag{26}$$

for the one-sided surface S. Analogously,

$$N_+^-(S) = N_-^-(S)\Phi \quad . \tag{27}$$

Hence on a one-sided surface S, *downward surface* radiance and *upward field* radiances can be transformed away according to (26) and (27) into respective surface and field radiances of the *opposite* polarity. It is therefore sufficient at a point x on S to speak only of *upward surface* radiance $N_+^+(S)$ and *downward field* radiance $N_-^-(S)$. In particular, incident source radiance distributions therefore are limited to downward field radiances $N_-^o(S)$.

With these geometrical preliminaries in mind, we are now ready to apply the interaction principle to a general one-sided surface S' with conjugate mapping Φ. We first isolate S' and then enumerate the incident radiometric functions on S':

A_1: all field radiance distributions like $N_-^o(S')$

A_2: all field radiance distributions like $N_-^-(S')$

The sets of response functions of S' are:

B_1: all surface radiance distributions like $N_+^+(S')$

In the present case m = 2, n = 1, and the interaction principle yields the following two interaction operators:

$$s_{11} \text{ -- } r_-^o(S')$$

$$s_{12} \text{ -- } r_-(S')$$

The two operators $r_-^o(S')$ and $r_-(S')$ are *not* instances of definitions (10), (11) of Sec. 3.3, as has been the case all along in applications to two-sided surfaces. We shall see the specific form of these operators in a moment. For the present we go on to write the interaction equation for a one-sided surface S as:

$$N_+^+(S') = N_-^o(S')r_-^o(S') + N_-^-(S')\ r_-(S') \qquad (28)$$

The auxiliary equation in the present case is :

$$N_-^-(S') = N_+^+(S')t_m(S') \qquad (29)$$

where $t_m(S')$ has exactly the same task in the present one-sided case as it did in the two-sided case of Example 4. Thus (28) and (29) constitute the general radiative transfer equations for a one-sided surface. The formal solution of (28) and (29) is:

$$\boxed{N_+^+(S') = N_-^o(S')r_-(S')\left[I - t_m(S')r_-(S')\right]^{-1}} \qquad (30)$$

The similarity of this solution with (11) above for the integrating sphere is particularly to be noted.

We next establish the respective connections between the operators $r_-^o(S')$, $r_-(S')$, and $t_m(S')$ for a one-sided surface, with their correspondents for a two-sided surface. This can most easily be done by making the present one-sided surface S' a two-sided surface S without changing its radiometric properties. Thus we make any radial cut in S', such as from x to y in Fig. 3.17. Then the equations of system (18) hold but we limit ϕ to the range $0 \leq \phi \leq 2\pi$. The resultant surface S is two-sided in the sense defined above. Clearly, its local and global radiometric properties are the same as those of S'. The salient difference between S and S' is that in the case of S there are no pairs of conjugate coordinates assigned to each point and so there is no mapping which conveniently rids us of $\Xi_-(x)$, etc. It follows that (14)-(17) now apply to S.

Comparing (28) and (14), we have the following functional relations between the reflectance and transmittance operators for S' and S:

$$r_-^o(S') = r_-^o(S) + \Phi t_+^o(S) \tag{31}$$

$$r_-(S') = r_-(S) + \Phi t_+(S) \quad . \tag{32}$$

Comparing (16) and (29) we have:

$$\begin{aligned} t_m(S') &= \chi_+(S)t_m(S) + \chi_-(S)\Phi t_m(S) \\ &= [\chi_+(S) + \chi_-(S)\Phi]\, t_m(S) \quad . \end{aligned}$$

Since we have not changed the definition of $t_m(S)$ in going to the one-sided context, we require:

$$t_m(S') = t_m(S) \quad .$$

Thus in the one-sided context we should have

$$\chi_+(S) + \chi_-(S)\Phi = 1$$

which is indeed the case, by the definitions of $\chi_\pm(S)$ and Φ.

One final set of remarks may be in order concerning the role of Möbius Strips and general one-sided surfaces in radiative transfer matters. As noted in the introductory comments to this example, one-sided surfaces arose in the search for the domains of validity of the classical surface integral formulas. Surface integrals in radiative transfer theory arise, for example, in irradiance calculations as we saw in Examples 8-10 of Sec. 2.11. Hence caution must be exercised in using Stokes' Theorem, for example, in transferring from line to surface integrals when working with one-sided surfaces. Surface integrals also arise in calculations of net flux across surfaces S. Thus,

$$\int_S \mathbf{H}(x)\cdot\mathbf{k}(x)\ dA(x)$$

normally gives the net radiant flux across a two-sided surface S, where $\mathbf{H}(x)$ is the vector irradiance at x. This integral can have positive, negative, or zero values. Now as an indication of a radiometric pathology arising on a one-sided surface, let S be the Möbius Strip defined by (18) and in Fig. 3.17. If "x" denotes the point (r,ϕ,z) then:

$$\int_{\phi=0}^{4\pi}\int_{\rho=-a}^{a} \mathbf{H}(x)\cdot\mathbf{k}(x)\ dA(x) = 0$$

for every irradiance field $\mathbf{H}$ over S. In other words a Möbius Strip S, lets every light field "slip through its fingers"

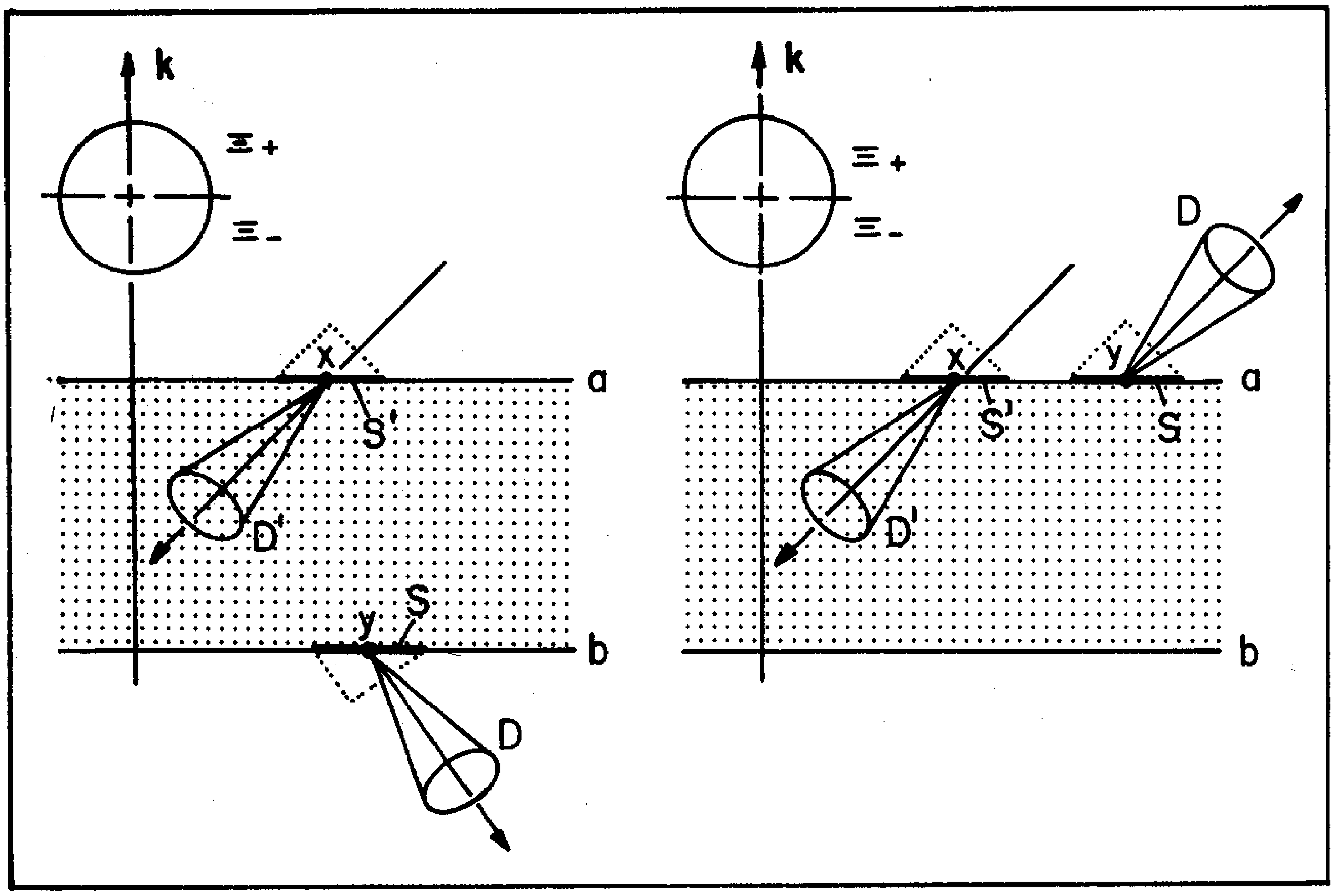

FIG. 3.18 Setting for empirical reflectances and transmittances of plane-parallel media.

when integrated over in the manner usual for two-sided surfaces. However, by slicing S in a manner designed to render it two-sided, as done above, the same integral considered above and taken over the same point set can have non zero net radiant flux $\overline{F}$ across its extent and such that $\overline{F}$ has valid physical significance.

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3.6 Reflectance and Transmittance Operators for Plane-Parallel Media

In this section we continue the sequence of constructions, begun in Sec. 3.3, of the basic concepts used in radiative transfer theory. We shall use the interaction principle to develop the reflectance and transmittance operators for plane-parallel media which will subsequently be used (Sec. 3.7) in the formulation of interaction equations for such media. The development of such operators will be carried on within the space-level interpretation of the interaction principle (cf. Sec. 3.2). This is in contrast to the constructions leading to the r and t functions of Sec. 3.3; they were carried out in the surface-level interpretation of the principle.

Geometrical Conventions

The geometrical conventions for plane-parallel media are depicted in Fig. 3.18. First of all a *plane-parallel optical medium* is a subset of euclidean space X consisting of