



FIG. 3.18 Setting for empirical reflectances and transmittances of plane-parallel media.

when integrated over in the manner usual for two-sided surfaces. However, by slicing S in a manner designed to render it two-sided, as done above, the same integral considered above and taken over the same point set can have non zero net radiant flux \bar{P} across its extent and such that \bar{P} has valid physical significance.

3.6 Reflectance and Transmittance Operators for Plane-Parallel Media

In this section we continue the sequence of constructions, begun in Sec. 3.3, of the basic concepts used in radiative transfer theory. We shall use the interaction principle to develop the reflectance and transmittance operators for plane-parallel media which will subsequently be used (Sec. 3.7) in the formulation of interaction equations for such media. The development of such operators will be carried on within the space-level interpretation of the interaction principle (cf. Sec. 3.2). This is in contrast to the constructions leading to the r and t functions of Sec. 3.3; they were carried out in the surface-level interpretation of the principle.

Geometrical Conventions

The geometrical conventions for plane-parallel media are depicted in Fig. 3.18. First of all a *plane-parallel optical medium* is a subset of euclidean space X consisting of

all points between and including two infinite parallel planes a and b. The plane-parallel medium is imagined to contain matter which can scatter, absorb or emit radiant flux. In a terrestrially-based reference frame (Sec. 2.4) planes a and b are customarily parallel to and coextensive with the xy coordinate plane, as shown in Fig. 3.18. The sets of downward \bar{E}_- and upward \bar{E}_+ directions are available for use in describing flows of radiant flux within the medium, and we shall use once more the narrow conical circular type of cones D' and D of Sec. 3.3 to establish the empirical reflectances and transmittances of plane-parallel media.

The Empirical Reflectances and Transmittances

Let planes a and b define a plane-parallel medium X, as in Fig. 3.18. The medium is irradiated in the vicinity of point x on one of its boundary planes. No other irradiation falls on X and no sources of radiant flux are within X. Fig. 3.18 shows x on plane a. The point x can also be on plane b. The empirical radiance of the incident flux at x is $N(S',D')$ where S' is the projection on plane a and along the axis of D' , of the collecting surface of a radiance meter, and D' is the small solid angle of directions of the incident flux. D' lies wholly within \bar{E}_- if x is on a, or wholly within \bar{E}_+ if x lies on b. For notational convenience we have used the regions S' and S lying in the boundary planes, instead of their projected counterparts on the planes normal to the axis of the radiance meter (shown dotted in Fig. 3.18). With this irradiation fixed, a radiance meter is directed at various points y on the boundary of X. Let $N(S',D';S,D)$ be the resultant measured surface radiance of X emergent through direction set D in the vicinity of point y on the boundary of X. Let S be the small patch of surface on the boundary of X about y defined as the projection on plane a or b along the axis of D, of the collecting surface of the radiance meter. (Recall the convention for measuring surface radiance in Sec. 2.6.) The cone D lies completely in \bar{E}_+ or \bar{E}_- as the case may be. For a fixed plane-parallel medium X and every such pair S',D' of incident variables and every such pair S,D of response variables, let us write:

$$"S(X;S',D';S,D)" \quad \text{for} \quad \frac{N(S',D';S,D)}{N(S',D')A(S')\Omega(D')} \quad . \quad (1)$$

In this way we can generate a table of values $S(X;S',D';S,D)$ of the function $S(\cdot;\cdot,\cdot;\cdot,\cdot)$, and this table of values would be representative of the reflecting or transmitting properties of the medium X. As it stands the value $S(X;S',D';S,D)$ does not tell whether the scattered radiant flux has been reflected or transmitted by X. To help distinguish between these processes we keep in mind on which hemisphere of \bar{E} , the direction sets D' and D lie. If D' and D lie in the same hemisphere, then the value is a transmittance; if D' and D lie in opposite hemispheres, then the value is a reflectance. Thus part (a) of Fig. 3.18 depicts a transmittance arrangement, and part (b) depicts a reflectance arrangement.

By systematically considering the four general possibilities we arrive at the following set of definitions for a plane-parallel medium X defined by planes a and b . We write:

$$\begin{aligned} "R_+(X;S',D';S,D)" & \text{ for } S(X;S',D';S,D) \text{ if } S' \subset b \\ & \qquad \qquad \qquad D' \subset \Xi_+ \\ & \qquad \qquad \qquad \text{and } S \subset b \\ & \qquad \qquad \qquad D \subset \Xi_- \end{aligned} \quad (2)$$

$$\begin{aligned} "R_-(X;S',D';S,D)" & \text{ for } S(X;S',D';S,D) \text{ if } S' \subset a \\ & \qquad \qquad \qquad D' \subset \Xi_- \\ & \qquad \qquad \qquad \text{and } S \subset a \\ & \qquad \qquad \qquad D \subset \Xi_+ \end{aligned} \quad (3)$$

$$\begin{aligned} "T_+(X;S',D';S,D)" & \text{ for } S(X;S',D';S,D) \text{ if } S' \subset b \\ & \qquad \qquad \qquad D' \subset \Xi_+ \\ & \qquad \qquad \qquad \text{and } S \subset a \\ & \qquad \qquad \qquad D \subset \Xi_+ \end{aligned} \quad (4)$$

$$\begin{aligned} "T_-(X;S',D';S,D)" & \text{ for } S(X;S',D';S,D) \text{ if } S' \subset a \\ & \qquad \qquad \qquad D' \subset \Xi_- \\ & \qquad \qquad \qquad \text{and } S \subset b \\ & \qquad \qquad \qquad D \subset \Xi_- \end{aligned} \quad (5)$$

The preceding four definitions are designed to run parallel to (2)-(5) of Sec. 3.3. As in the earlier case, these definitions could have been based directly on the interaction principle with the same operators resulting. (The interested reader should consult the discussion following (5) of Sec. 3.3.) However, in this introductory discussion, we wish to keep the intuitive and operational aspects of the concepts foremost.

Once a table of reflectances and transmittances is made for a given plane-parallel medium, the table can be used to compute the responses to incident flux. Suppose, e.g., that X is irradiated over S' in a by radiance $N(S',D')$ where the reflected radiance $N(S,D)$ over S in a is sought. Then

$$N(S,D) = N(S',D')R_-(X;S',D';S,D)\Omega(D')A(S') \quad .$$

The essential properties of the empirical radiance $N(S;D';S,D)$, which is the response to $N(S',D')$ of the plane-parallel medium X , is its S' and D' additivities and its S' and D' continuities. These properties are analogous to (i) and (ii) of Sec. 3.1 enunciated for surfaces. These properties are sufficiently important to the theory of plane-parallel media (and media in general) that they will be stated below: In each case D, D' are circular conical solid angles with central directions ξ, ξ' , respectively, etc.

- (i) (*D'-additivity*). If a and b are the plane boundaries of a plane-parallel medium X and either a or b is irradiated in turn by radiances $N(S', D_1')$ and $N(S', D_2')$ with $N(S', D_1'; S, D)$ and $N(S', D_2'; S, D)$ as the respective observed response radiances, then $N(S', D_1'; S, D) + N(S', D_2'; S, D)$ is the observed radiance of S (on a or b) under simultaneous irradiation.

Furthermore:

- (ii) (*D'-continuity*). Let the geometric setting be as defined in (i). If $\Omega(D') = 0$ and $\xi \neq \xi'$ then $N(S', D'; S, D) = 0$.

A similar pair of statements can be made about *S'-additivity* and *S'-continuity* by varying the areas over which the flux is incident, keeping the solid angles fixed. The reason behind all this attention to additivity and continuity rests in the fact that from these properties--which are simply intuitive manifestations of the linearity of radiative processes in the domain of radiative transfer theory--we form an empirical justification of the mathematical model of radiometry by means of which we can rigorously deduce the existence of the classical integral operators for the reflectances and transmittances of plane-parallel slabs of scattering material. The details of the derivation of the appropriate integral operators for the plane-parallel setting are quite close to those for the surface operators discussed in Sec. 3.3. In order to avoid excessive repetition, it will be shown in Sec. 3.16 how all these integral operators for surfaces, slabs and general media can be deduced from the interaction principle.

The Theoretical Reflectances and Transmittances

We now use the definitions (2)-(5) as a basis for the definition of the theoretical reflectances and transmittances of a plane-parallel medium. As was pointed out in the preceding paragraph, the existences of theoretical reflectance and transmittance functions and their associated integral operators all follow mechanically from the interaction principle and will be discussed later. However, it is instructive to show how these functions come about by means of every day limit operations applied to the empirical response functions. This we now do. In (2)-(5), we let $S' \rightarrow \{x'\}$, $S \rightarrow \{x\}$, $D' \rightarrow \{\xi'\}$, $D \rightarrow \{\xi\}$ in any order desired.

We choose first to write:

$$"S(X; S', D'; x, \xi)" \quad \text{for} \quad \lim_{\substack{S \rightarrow \{x\} \\ D \rightarrow \{\xi\}}} S(X; S', D'; S, D) . \quad (6)$$

This limit exists by virtue of the S and D additivity and continuity properties of radiant flux established in Sec. 2.3. Next we write:

$$"S(X;x',\xi';x,\xi)" \quad \text{for} \quad \lim_{\substack{S' \rightarrow \{x'\} \\ D' \rightarrow \{\xi'\}}} S(X;x',\xi';x,\xi) \quad (7)$$

This limit exists by virtue of the S' and D' additivity and properties of the response radiance just cited above. One may envision the physical significance of the magnitude of limit $S(X;x',\xi';x,\xi)$ as follows: the magnitude represents the radiance of X at x in the direction ξ induced by a unit incident radiance on X at x' in the direction ξ' . The establishment of $S(X;x',\xi';x,\xi)$ in the two stages of limit operations (6) and (7) was prompted by a desire to facilitate the reader's study of related matters in Sec. 3.14 and in reference [251], in particular Chapter III of that reference.

We come now to the definition of the integral operators associated with plane-parallel media. These operators are the three-dimensional counterparts to $r_-(Y)$ and $t_-(Y)$ defined in (10), (11) of Sec. 3.3, and the theoretical counterparts to R_- and T_- in (2)-(5) above. In the present case we identify the plane-parallel medium by the two bounding planes. Whenever possible, we choose to use their depths also as identifying names, thus "a" denotes a plane at depth a, etc. However, occasionally, to avoid confusion, the full name " X_y " will be used for the plane at depth y , $a \leq y \leq b$. In what follows we shall, unless explicitly stated otherwise, always assume that $a \leq b$. We write:

$$"R(a,b)" \quad \text{for} \quad \int_{E_-} \int_{X_a} [] S(X;x',\xi';x,\xi) dA(x') d\Omega(\xi') \quad (8)$$

$$"T(a,b)" \quad \text{for} \quad \int_{E_-} \int_{X_a} [] S(X;x',\xi';x,\xi) dA(x') d\Omega(\xi') \quad (9)$$

if $x \in a$, $\xi \in E_+$ for (8); and $x \in b$, $\xi \in E_-$ for (9);

$$"R(b,a)" \quad \text{for} \quad \int_{E_+} \int_{X_b} [] S(X;x',\xi';x,\xi) dA(x') d\Omega(\xi') \quad (10)$$

$$"T(b,a)" \quad \text{for} \quad \int_{E_+} \int_{X_b} [] S(X;x',\xi';x,\xi) dA(x') d\Omega(\xi') \quad (11)$$

if $x \in b$, $\xi \in E_-$ for (10); and $x \in a$, $\xi \in E_+$ for (11).

By using "a" and "b" as explained above, we obviate the need for the signatures "+" and "-" on "R" or "T" as in (6)-(9) of Sec. 3.3 or (2)-(5) above. The operators $R(a,b)$ and $R(b,a)$ are called the *standard reflectance operators* associated with the plane-parallel medium determined by a,b. The operators $T(a,b)$ and $T(b,a)$ are called the *standard transmittance operators* associated with the plane-parallel medium determined by a,b.

The remaining matters requiring mention in this section

can be presented very much in the manner that the corresponding surface concepts were presented in Sec. 3.3 following (10), (11) of Sec. 3.3. In particular, to denote an incident radiance distribution on the upper plane a of a plane-parallel medium X we write " $N_-(a)$ ". Thus, $N_-(a)$ is a function which assigns to x' in a and ξ' in E_- the radiance $N_-(x', \xi')$. The reflected radiance distribution initiated by $N_-(a)$ has its values given by:

$$\int_{E_-} \int_{X_a} N_-(x', \xi') S(X; x', \xi'; x, \xi) dA(x') d\Omega(\xi') \quad (12)$$

(where x is in a and ξ is in E_+), and which we denote simply as: " $N_-(a)R(a,b)$ ". In a similar way we define $N_-(a)T(a,b)$ as the transmitted radiance distribution emerging from level b and initiated by the downward incident distribution $N_-(a)$. Further, $N_+(b)R(b,a)$ and $N_+(b)T(b,a)$ are the reflected and transmitted radiance distributions initiated by the upward incident distribution $N_+(b)$ on the lower boundary b of X . When necessary (i.e., if ambiguity is to be avoided) we can append the signatures "+" and "-" as superscripts to "N" to denote surface and field radiances, respectively. However, it has been found by experience that the *surface* radiance concept by itself is for the most part adequate to describe without ambiguity theoretical discussions involving plane-parallel media, particularly those centering on the principles of invariance. The main functional relations for the operators (8)-(11) will be developed in Chapter 7.

Variations of the Basic Theme

The operators defined in (8)-(11) may be used as a basis for defining still further operators for radiometric concepts other than radiance. We illustrate this observation for the case of radiant emittance and irradiance.

Let X be a plane-parallel medium defined by planes a and b . Let $N_-(a)$ be an incident downward radiance distribution over boundary a . This gives rise to an irradiance function $H_-(a)$ over a . Thus the value of $H_-(a)$ at each x in a is given by:

$$\int_{E_-} N_-(x, \xi') \xi' \cdot (-k) d\Omega(\xi')$$

The medium X responds to this incident flux with a radiance distribution $N_+(x, \cdot)$ at each point x of a , as given in (12). The associated value of radiant emittance $W_+(a)$ of a at x is:

$$\int_{E_+} N_+(x, \xi) \xi \cdot k d\Omega(\xi)$$

Therefore to every plane-parallel medium defined by two parallel planes a and b we can associate a reflectance for irradiance, namely:

$$\frac{1}{H_-(a)} \int_{\Xi_+} \left[\int_{\Xi_-} \int_{X_a} N_-(x'; \xi') S(X; x'; \xi'; x, \xi) dA(x') d\Omega(\xi') \right] \xi \cdot k d\Omega(\xi) \quad (13)$$

We will rarely mix radiance and irradiance calculations in one discussion so that it will usually be possible to economize on notation and write "R(a,b)" for the reflectance (13). Therefore the numerical product $H_-(a)R(a,b)$ denotes the radiant emittance of X induced by $H_-(a)$ on X, where $H_-(a)$ in turn is associated with $N_-(a)$. Hence in the irradiance context "R(a,b)" will denote a pure number; in the radiance context "R(a,b)" will denote an integral operator. We can make three more definitions of reflectance and transmittance for $H_+(b)$ and $H_-(a)$: $R(b,a)$, $T(b,a)$, and $T(a,b)$, respectively. The full details of all these definitions and the discussion of the properties of the R and T quantities are reserved for Chapter 8.

We could now go on and consider further variations on the theme such as the present plane-parallel counterpart to (18) of Sec. 3.3. However, the point that potential variations are possible seems well made by now, and we shall therefore go on to consider the applications of the interaction principle to plane-parallel media.

3.7 Applications to Plane-Parallel Media

The application of the interaction principle to plane-parallel media, which is the main theme of this section, will perhaps be most interesting from the following two points of view. First, the interaction equations for interacting plane-parallel media will be seen to be identical in form to those for interacting planes illustrated in Sec. 3.4. This point of similarity of the two types of interaction equations for ostensibly dissimilar radiative transfer contexts should encourage a closer examination of the classical modes of solution of the problems associated with these media with the purpose in mind of obtaining a unified means of solution for both types of settings. The natural mode of solution, as we shall see, is one candidate for such a unification.

The second interesting observation that can be made about the illustrations below concerns the ontogenetical foundations of radiative transfer theory, that is, the basic concepts on which the theory rests. Advanced students of radiative transfer theory know that the classical framework of the subject can be made to rest on the equation of transfer for radiance and on the principles of invariance. For some time there was a question as to the primacy of one or the other of these concepts; which was more fundamental: the equation of transfer or the principles of invariance? This question is naturally of interest to those who are concerned with the logical connections between these two tap roots of the subject. The principles of invariance have been developed and made a powerful tool of radiative transfer theory by Chandrasekhar