may be developed. The preceding approach is specifically designed to play up the deep group-theoretic similarities of the quantum and phenomenological levels of radiative transfer theory. Unquestionably, the simplest connection between quantum mechanics and radiative transfer theory--the connection that would require a minimum of re-doing of existing constructions, is that which would derive the interaction principle (Sec. 3.2) from the tenets of quantum mechanics with a specific representation of the interaction operator in terms of the quantum properties of matter. Then all the constructions of [251] and the present work would stand ready-made for use without any further effort. In this way one can go on to solve important remaining problems of radiative transfer theory (Sec. 141 of [251]) with a minimum of duplication of effort. For further observations on the similarity of the structures of radiative transfer equations and quantum dynamical equations, see the closing remarks of Sec. 8.2.

3.8 Interaction Operators for General Spaces

The third step in the present sequence of constructions of the main concepts of radiative transfer theory will be taken in this section. We shall develop the concept of the interaction operator for a general three-dimensional optical medium. This development therefore augments the store of operators constructed in Sec. 3.3 for surfaces and in Sec. 3.6 for plane-parallel media. As in the latter case, the present constructions will utilize the space-level interpretation of the interaction principle.

The subsets of Euclidean three-space we shall consider in this section are those that are connected--i.e., in one piece--and we may envision them as members of an ensemble of interacting connected sets. The interaction equations will be stateable for the ensemble once the interaction operator for each connected component of the ensemble is known. The connected spaces we consider may be of finite or infinite extent and fall conveniently for radiometric purposes into two main classes: Those that have convex surfaces and those that have non convex (concave) surfaces.

Geometrical Conventions

Figure 3.22 depicts a general connected optical medium X and a point x on the boundary Y of X. Let "$\mathbf{k}(x)$" denote the unit outward normal to Y at x. Then "$\Xi_+(x)$" will denote, as usual, the set of all directions in Ξ such that $\xi\cdot\mathbf{k}(x)>0$, and "$\Xi_-(x)$" will denote the set of all directions ξ in Ξ such that $\xi\cdot\mathbf{k}(x)<0$. The directions in $\Xi_\pm(x)$ are called the *outward* (+) or *inward* (-) directions at x. Radiance distributions $N(x,\cdot)$ at points x of the boundary Y are split, as usual, into two parts: the *outward radiance distribution* $N_+(x,\cdot)$ and the *inward radiance distribution* $N_-(x,\cdot)$. If "a" denotes a part of Y, then "$N_+(a)$" and "$N_-(a)$" denote the outward and inward radiance distributions of Y restricted to part a. The part a can vary from a set {x} consisting of one point x of Y, up to

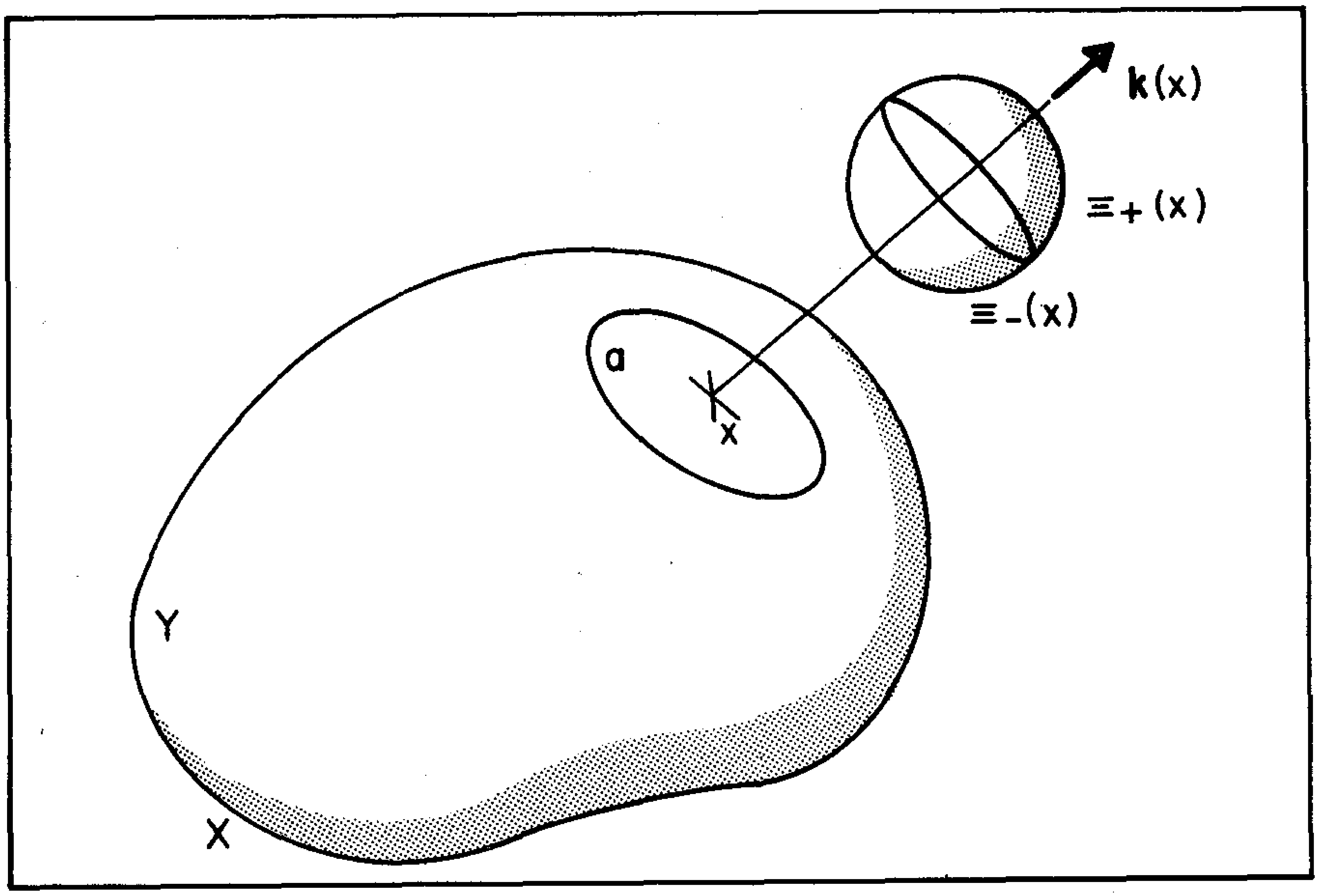

FIG. 3.22 Direction convention for interaction operators on general spaces.

Y itself.

The geometrical conventions for the empirical quantities D',D,S',S, established in Sec. 3.3, will also hold below.

Another geometric convention we shall require is that based on the process of *convexification* of a concave optical medium. This process will allow in many instances both convex and concave media to be treated alike during a given discussion. Let any of parts (a), (b), (c) of Fig. 3.23 represent an optical medium which has a concave boundary. This, it will be recalled, means that some points of the boundary can be joined by straight lines lying partially outside the surface Y. To be specific, we have pictured *solid* subsets X for the present discussion. It should be noted that all that we say below can be applied, *mutatis mutandis* to surfaces also. Now imagine a rubber sheet to be neatly applied all around X, enclosing X like a tight-fitting cocoon. On those parts of X where its surface is convex, the rubber sheet will cling and follow the contours of the original surface. On those parts of X where the surface is concave, the rubber sheet will soar as a plane surface across the concave hollow and will thereby establish a smooth convex surface enclosing X, of minimal possible area. Thus the step-like concavity of X in (b) of Fig. 3.23 will be ideally bridged by the rubber coating as sketched by the dashed lines in the figure, and the hollows and holes of (a) and (b) of the figure will be enclosed likewise. The net result will be a new region X' containing X with the rubber sheet as a *convex* boundary of the newly encased volume X'.

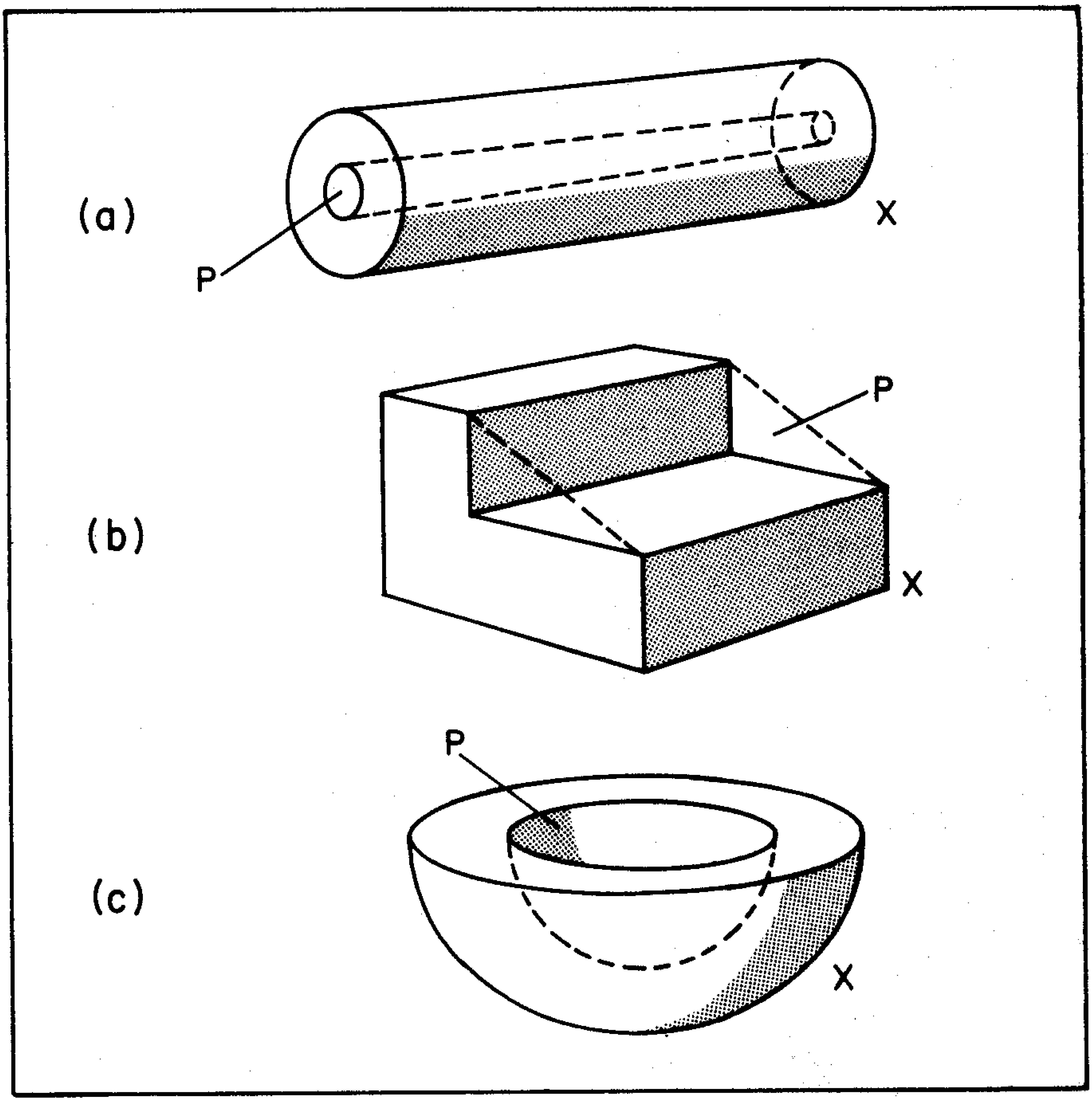

FIG. 3.23 Illustrating the convexification of concave media.

The new surface so formed is called the *convex hull* of X. In short, X' is the smallest convex solid containing X. So far we have engaged in pure geometry.

Next we introduce a radiometric element into the discussion. We consider all the regions which comprise the difference P between X and its convex hull X', including any holes inside X. For example, the triangular prism region P in Fig. 3.23 is one such region, and the hole in (a) and the hemisphere in (b) are further examples of the difference P between X' and X. It is found that certain theoretical considerations of X are facilitated by considering all such regions like P as filled either (a) by a hypothetical vacuum of unit transmittance and zero reflectance, or (b) by its antithesis: a hypothetical black material of zero transmittance and zero reflectance. In the case (a), we use X' and say that X has been *white convexified* and in case (b) we use X' and say that

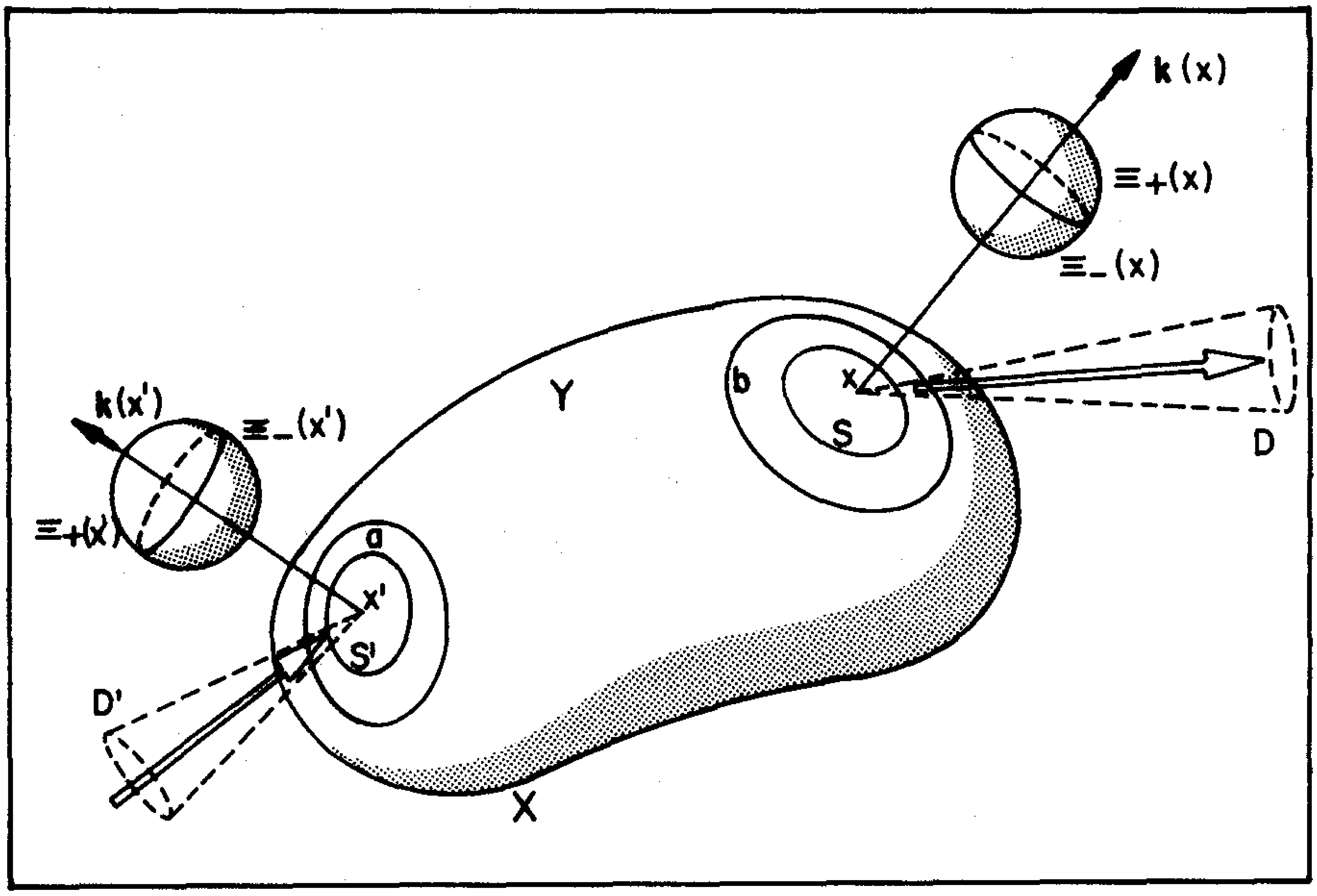

FIG. 3.24 Details for defining empirical scattering functions on arbitrary optical media.

X has been *black convexified.* It is obvious that if X is convex to begin with (and hence also with no holes), then either its black or white convexification results in X once again.

It is perhaps needless to add that a convexified X (either way) is still a *conceptual* object which can be considered irradiated or probed at will at any point of its surface or interior. However, the definitions of convexified media have an operationally meaningful cast which, if the necessity ever arose, could quite possibly be realized in many instances.

The Empirical Scattering Functions

The empirical scattering functions will now be established for a general optical medium X. The medium X may be convex or concave. If X has a concave boundary Y then we shall consider X to have been either white or black convexified. The present discussion is independent of the particular choice of these convexifications and hence we need not distinguish between them.

Consider two parts a and b of the boundary Y. Let S' be a small patch of part a around point x', and S be a small patch of b around point x, as in Fig. 3.24. Thus the present geometric situation is similar--as far as the present general geometry will allow--to Fig. 3.18. Let an amount N(S',D') of radiance be incident over S' and within the narrow conical solid angle D' which lies wholly inside $\Xi_-(x')$. This is the only source of irradiation either in or on X. (Again "S'", as in Sec. 3.6, should be replaced in "N(S',D')" by the name of the projection of S' on a plane normal to the axis of D'. However, brief notation wins out over logical notation once again.) Let N(S',D';S,D) be the resultant radiance of S within the conical solid angle D which lies wholly inside $\Xi_+(x)$. Then let us write:

$$\text{"}S(X;S',D;S,D)\text{"} \quad \text{for} \quad \frac{N(S',D';S,D)}{N(S',D')A(S')\Omega(D')} \tag{1}$$

The non negative valued function S(X;·,·,·,·) is the *standard* (empirical) *scattering function* for X.

Occasionally it is convenient to know if X has been white or black convexified, and when it is necessary to explicitly note this fact in the symbol for the standard empirical scattering function we shall write:

$$\text{"}S_w(X;S',D';S,D)\text{"} \quad \text{for} \quad S(X;S',D';S,D) \tag{2}$$

if X has been white convexified and:

$$\text{"}S_b(X;S',D';S,D)\text{"} \quad \text{for} \quad S(X;S',D';S,D) \tag{3}$$

if X has been black convexified.

At about this point in the corresponding developments of Secs. 3.3 and 3.6, it was customary to observe that the counterparts to S(X;·,·,·,·) obeyed D and S additivity and continuity properties. The observance of this procedure is now well established and, therefore, in order not to repeat unnecessarily, these facts need only be alluded to here with an observation that these properties are stated in detail in Sec. 18 of Ref. [251]. Of course, while we are currently giving slight attention to these properties, this does not in any way mitigate their supreme importance in allowing the rigorous deduction from the interaction principle of the standard $\mathcal{U}$ and $\mathcal{U}$ operators below, and hence, ultimately, all of radiative transfer theory on discrete or continuous optical media. At any rate, the formal establishment of all these functions in 3.3, 3.6 and the present function, starting from the interaction principle, will be discussed in detail in Sec. 3.16. In particular, it will be shown in that section that each of the various S'-additivity and D'-additivity properties will take its formal place as an appropriate property of the interaction measure, and the various D' and S' continuity properties will be formulated as the so-called *AC property* of the interaction measure.

The Theoretical Scattering Functions

Let us write:

$$\text{"}S(X;S',D';x,\xi)\text{"} \quad \text{for} \quad \lim_{\substack{S\to\{x\}\\ D\to\{\xi\}}} S(X;S',D';S,D) \tag{4}$$

and

$$\text{"}S(X;x',\xi';x,\xi)\text{"} \quad \text{for} \quad \lim_{\substack{S'\to\{x'\}\\ D'\to\{\xi'\}}} S(X;S',D';x,\xi) \ . \tag{5}$$

These limits exist by virtue of the various D and S additivity and continuity properties of the empirical scattering function. If black or white convexification is to be explicitly noted, then "b" and "w" subscripts are inherited, appropriately, from (2),(3). We go on to write:

$$\text{"}\mathcal{V}(X;a,b)\text{"} \quad \text{for} \quad \int_a \int_{\Xi_-(x')} [\ \ ]S_b(X;x',\xi';x,\xi)\ d\Omega(\xi')dA(x') \tag{6}$$

where a and b are parts of the original surface Y of X and x is in b, and ξ is in $\Xi_+(x)$. Further, we write:

$$\text{"}\mathcal{U}(X;a,b)\text{"} \quad \text{for} \quad \int_a \int_{\Xi_-(x')} [\ \ ]S_w(X;x',\xi';x,\xi)\ d\Omega(\xi')dA(x') \tag{7}$$

where a and b are parts of the (original) surface Y of X and x is in b and ξ is in $\Xi_+(x)$. $\mathcal{V}(X;a,b)$ (or "$\mathcal{V}$" for short when X,a,b are understood) is the *standard* $\mathcal{V}$*-operator for* X *over* a *and* b. $\mathcal{U}(X;a,b)$ (or "$\mathcal{U}$" for short) is the *standard* $\mathcal{U}$ *-operator for* X *over* a *and* b.

To see the relative roles played by $\mathcal{V}$ and $\mathcal{U}$ we observe that $\mathcal{V}$ is to a black convexified X as $\mathcal{U}$ is to a white convexified X. The theoretical connections between $\mathcal{V}$ and $\mathcal{U}$ for a given concave space X have been given in Sec. 25 of Ref. [251]. It suffices to say that this connection is intricate and its applications have not yet been completely explored. Of the two, the standard $\mathcal{V}$-operator is by far the more useful in the immediate generalizations of classical radiative transfer theory, especially in the theory of one-parameter carrier and general spaces (Examples 4,5 in Sec. 3.9). The operators $\mathcal{V}$ promise to help organize and systematize the theory on the more general spaces which have little or no symmetry or regular structure.

It will be instructive for the reader to give simple verbal proofs, based on the appropriate definitions, of the following statements:

(a) For every X, a, b, if X is convex and a and b are parts of the boundary of X, then $\mathcal{V}(X;a,b) = \mathcal{U}(X;a,b)$.

(b) For every X,a,b, if X is concave and a and b are parts of the boundary of X, then $|N_-(a)\ \mathcal{V}(X;a,b)| \le |N_-(a)\ \mathcal{U}(X;a,b)|$.

In statement (b) above, we have used the definition of radiometric norm (Example 5, Sec. 3.4) extended to curved surfaces

Y. Thus, we write in general:

$$\text{"}|N_{\pm}(Y)|\text{"} \quad \text{for} \quad \frac{1}{A(Y)} \int_{Y} \int_{\Xi_{\pm}(x)} N(x,\xi)\, d\Omega(\xi)\, dA(x) . \qquad (8)$$

Furthermore, we have written in statement (b):

$$\text{"}N_{-}(a)\mathscr{V}(X;a,b)\text{"} \quad \text{for}$$

$$\int_{a} \int_{\Xi_{-}(x')} N_{-}(x',\xi')S_{b}(X;x',\xi';x,\xi)\, d\Omega(\xi')dA(x') \quad .$$

A similar definition holds for the term $N_{-}(a)\mathscr{U}(X;a,b)$.

Since we have defined the radiometric norm for radiance distributions over surfaces Y bounding general optical media X, it is natural to try to extend the definition of the norm of a reflectance operator, as given in Sec. 3.4, to a more general object such as the $\mathscr{V}$-operator for a medium X. The requisite sequence of definitions for the norm of $\mathscr{V}(X;a,b)$ is patterned closely after (44)-(49) of Sec. 3.4, and proceeds as follows. First we agree that if X has a boundary Y of finite area A(Y) then we normalize all radiometric norms of radiance distributions $N_{\pm}(a)$, defined over parts a of Y, with respect to A(Y) rather than with respect to A(a). Thus on a fixed finite boundary surface Y of an optical medium X we agree to write:

$$\text{"}|N_{\pm}(a)|_{Y}\text{"} \quad \text{for} \quad \frac{1}{A(Y)} \int_{a} \int_{\Xi_{\pm}(x)} N(x,\xi)\, d\Omega(\xi)\, dA(x) \ . \qquad (9)$$

If A(Y) if infinite, then, as in Sec. 3.4, we employ a limit process to define the norm. In practice, when working with a fixed boundary Y, then "Y" may be dropped from the norm notation, for brevity.

Next we write:

$$\text{"}S_{b}(X;x',\xi';x)\text{"} \quad \text{for} \quad \int_{\Xi_{+}(x)} S_{b}(X;x',\xi';x,\xi)\, d\Omega(\xi) \qquad (10)$$

where x' and x are in Y and ξ' is in $\Xi_{-}(x')$. We have chosen to work with S_b simply to be specific. All that follows below holds also for S_w. Further, we agree to write:

$$\text{"}\beta(X,N;x',x)\text{"} \quad \text{for} \quad \frac{\int_{\Xi_{-}(x')} N(x',\xi')S_{b}(X;x',\xi';x)\, d\Omega(\xi')}{\int_{\Xi_{-}(x')} N(x',\xi')\, d\Omega(\xi')} \qquad (11)$$

Next, we write:

$$"\beta(X,N;x)" \quad \text{for} \quad \frac{\int_a \beta(X,N;x',x)\left[\int_{\Xi_-(x')} N(x',\xi')\, d\Omega(\xi')\right] dA(x')}{\int_a\int_{\Xi_-(x')} N(x',\xi')\, d\Omega(\xi')\, dA(x')} \tag{12}$$

And finally:

$$"\beta(X,N)" \quad \text{for} \quad \int_b \beta(X,N;x)\, dA(x) \quad . \tag{13}$$

The motivation for this sequence of definitions is made clear by computing the norm $|N_-(a)\mathscr{S}(X;a,b)|_Y$. Thus:

$$|N_-(a)\mathscr{S}(X;a,b)|_Y =$$

$$= \frac{1}{A(Y)}\int_b\int_{\Xi_+(x)} N_-(a)\mathscr{S}(X;a,b)\, d\Omega(\xi)\, dA(x)$$

$$= \frac{1}{A(Y)}\int_b\int_{\Xi_+(x)}\left[\int_a\int_{\Xi_-(x')} N(x',\xi')S_b(X;x',\xi';x,\xi)\,d\Omega(\xi')\,dA(x')\right]d\Omega(\xi)\,dA(x)$$

$$= \frac{1}{A(Y)}\int_b\int_a\left[\int_{\Xi_-(x')} N(x',\xi')S_b(X;x',x';x)\, d\Omega(\xi')\right] dA(x')dA(x)$$

$$= \frac{1}{A(Y)}\int_b\int_a\left[\beta(X,N;x',x)\int_{\Xi_-(x')} N(x',\xi')\, d\Omega(\xi')\right] dA(x')dA(x)$$

$$= \frac{1}{A(Y)}\int_b \beta(X,N;x)\left[\int_a\int_{\Xi_-(x')} N(x',\xi')\, d\Omega(\xi')dA(x')\right] dA(x)$$

$$= \beta(X,N)\ |N_-(a)|_Y \tag{14}$$

As in the case of the norm of the surface reflectance operators $r_\pm(a)$, $t_\pm(a)$ (Sec. 3.4) it can be shown with the help of the energy conservation principle that:

$$0 \le \beta(X,N) \le 1 \tag{15}$$

for every X,a,b on X, and every radiance function N. For a given X, we write

$$"\beta(X)" \quad \text{for} \quad \max_N \beta(X,N) \quad , \tag{16}$$

where the maximum operation is taken over the set of all radiance functions on Y. Then the conclusion in (14) implies:

$$|N_-(a)\mathcal{J}(X;a,b)|_Y \le \beta(X)|N_-(a)|_Y \quad . \qquad (17)$$

We say that $\mathcal{J}(X;a,b)$ is *norm contracting* if:

$$0 < \beta(X) < 1 \quad . \qquad (18)$$

Variations of the Basic Theme

The operators $\mathcal{J}$ or $\mathcal{U}$ can be used as a basis for further definitions of operators which work with radiometric quantities other than radiance. Thus, following the patterns established in Secs. 3.3, 3.6, we could redesign $\mathcal{J}$ so as to map radiance into radiant emittance, or irradiance into radiance, etc. These brief comments will suffice to make the reader aware of the potential variations he himself may wring from $\mathcal{J}$ and $\mathcal{U}$ as the occasion may arise.

It should be noted in conclusion that the operators $\mathcal{J}(X;a,b)$ and $\mathcal{U}(X;a,b)$ serve the capacities of both reflectance and transmittance operators depending on the relative disposition of parts a and b over the boundary of X. Thus we agree to call $\mathcal{J}(X;a,b)$ or $\mathcal{U}(X;a,b)$ a *reflectance operator* whenever a = b, and call it a *transmittance operator* whenever a and b are disjoint, i.e., have no points in common. This convention attains its greatest conceptual utility when X is very irregular and no simple directional conventions are possible, such as are available in the case of plane-parallel media. Observe, that if X is a plane-parallel medium X(a,b), then our present convention essentially reduces to that established earlier for a plane-parallel medium X(a,b) with upper boundary a and lower boundary b. (See, e.g., (8)-(11) of Sec. 3.6).

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3.9 Applications to General Spaces

The applications of the interaction principle will now be extended to general optical media. We will begin with some relatively simple but important extensions of the principles of invariance to curvilinear media such as spherical, cylindrical and toroidal media. Then the abstract versions of these media--one-parameter carrier spaces--are considered, and finally the illustrations culminate in the principles of invariance for completely arbitrary media which are not represented explicitly as one-parameter media. Throughout this section, the proceedings may best be viewed once again from the two vantage points defined and discussed in the introduction to Sec. 3.7. In regard to these vantage points, Sections 3.4-3.8 and the present section begin to illustrate the efficacy of the interaction principle, not only as a theoretical tool, but as one which shows promise in fostering novel methods of numerical computations in radiative transfer problems.