

Example 8: Several Interacting Separate Media

The quantitative evaluation of the radiometric interaction of separate clouds in the atmosphere, or separate portions of lakes, oceans, or other natural hydrosols forms an interesting and difficult radiative transfer problem. The methods of discrete space theory have been used to develop a systematic means of computing the \mathcal{V} -operators of such irregular types of media (see, e.g., Chapter X, Ref. [251]). For actual computations, the latter formulation may use the estimates of the \mathcal{V} -operators as supplied by the methods of Chapter X of Ref. [251].

3.10 Derivation of the Beam Transmittance Function

In this and the following four sections the interaction principle is used to derive the basic inherent optical properties and the integral equation of transfer for radiance in general optical media. In the present section the beam transmittance function is derived which in turn will yield the first of the inherent optical properties in Sec. 3.11, namely the volume attenuation function.

Let X be a general source-free optical medium. Consider a point x in X and a direction ξ at x . These together determine a natural path in X --a path for a light ray through x with direction ξ . By specifying a length r , a path segment $\mathcal{P}_r(x, \xi)$ is determined within X . For simplicity of exposition we shall introduce the beam transmittance concept in a medium X in which the index of refraction is constant. Then $\mathcal{P}_r(x, \xi)$ is a sensed, straight line segment in X with initial point x , terminal point $x+r\xi$ and length r . We shall write " z " for $x+r\xi$. The requisite steps for the definition in a completely general medium with variable index of refraction will be clear from the following derivation.

Our goal is to define the beam transmittance function--the well-known function of radiative transfer theory which assigns to an initial radiance $N_0(x, \xi)$ at point x of $\mathcal{P}_r(x, \xi)$ the residual radiance $N_r^0(z, \xi)$ at z after traversing $\mathcal{P}_r(x, \xi)$. This residual radiance $N_r^0(z, \xi)$ may be described as consisting of photons which have travelled the entire length of $\mathcal{P}_r(x, \xi)$ without having undergone scattering or absorption. Were we not in possession of the interaction principle, we would simply define the beam transmittance of $\mathcal{P}_r(x, \xi)$ as the ratio $N_r^0(x, \xi)/N_0(z, \xi)$. Such a definition is quite acceptable and implicitly assumes the terms "scattering" and "absorption" as understood, using their meanings as grounded in other departments of physics such as electromagnetic theory, and thus requiring no further elucidation. However, we choose not to use the words "scattering" and "absorption" at this stage of the exposition. Instead we ask: can the basic idea of "beam transmittance" be communicated without presupposing the physical notions of scattering and absorption? If this is possible then the theory of radiative transfer, at least in this area, is kept self-contained and freed from unnecessarily using

in the present discussion and with the concept of the \mathcal{L} -operator at our disposal, we shall use it to define the beam transmittance function geometrically.

Let C be a right cylinder in X containing $\mathcal{P}_r(x, \xi)$ as its axis. This cylinder is depicted schematically in Fig. 3.29. The plane base of C at x is denoted by "a", that at z by "b". Let $k(x)$ ($= -\xi$) be the unit outward normal to C over base a. Then C is an instance of a one-parameter optical medium with distance along the direction $-k(x)$ as the parameter of the space (cf. Example 2, Sec. 3.9 and part (e) of Fig. 3.25). Consider C isolated. Since C is convex we shall not need to black-convexify it. Let $N_-(a)$ be an incident radiance distribution over a circular conical set D of directions with axis ξ and on base a. Let $N_-(a)$ be the only source of radiant flux in C . Since C is isolated, radiant flux leaving C is not considered to enter it again nor is the radiant flux in X considered to enter it. In short "isolated" means, as usual, that a subset is *conceptually* excised from its master set and placed in a dark vacuum for a controlled radiometric study.

We now direct attention to the radiance distribution

$$N_-(a) \mathcal{T}(a, y, b)$$

where y is an intermediate plane cross section of C , as in Fig. 3.29, and $\mathcal{T}(a, y, b)$ is the complete transmittance operator for C (cf. (11) of Sec. 3.9). This radiance distribution is thought of as that part of $N_-(a)$ transmitted from a to y . Hence:

$$(N_-(a) \mathcal{T}(a, y, b)) \mathcal{T}(y, b, b)$$

is that part of $N_-(a)$ transmitted from a to b and emerging over base b. By (12) of Sec. 3.9 (with $z = b$) we have:

$$N_-(a) \mathcal{T}(a, y, b) \mathcal{T}(y, b, b) = N_-(a) \mathcal{T}(a, b, b) \quad (1)$$

or, alternatively:

$$N_-(a) \mathcal{T}(a, y, b) T(y, b) = N_-(a) T(a, b) \quad .$$

The crucial step in the definition of the beam transmittance can now be taken. We select the value of $N_-(a) \mathcal{T}(a, b, b)$ at z for the direction ξ and let $a \rightarrow \{x\}$, which is tantamount to letting $C \rightarrow \mathcal{P}_r(x, \xi)$. We then go on to write:

$$"N_r^0(z, \xi)" \quad \text{for} \quad \lim_{C \rightarrow \mathcal{P}_r(x, \xi)} [N_-(a) T(a, b)](z, \xi) \quad (2)$$

where the notation indicates that $N_-(a) T(a, b)$ is evaluated at z in the direction ξ during the limit process. Let us also write:

$$"N_0(x, \xi)" \quad \text{for} \quad N_-(a)(x, \xi) \quad ,$$

i.e., $N_0(x, \xi)$ is the value of $N_-(a)$ at x and ξ . Then we shall write:

$$\text{"}T_r(x, \xi)\text{" for } \lim_{C \rightarrow \mathcal{P}_r(x, \xi)} [T(a, b)](z, \xi) \quad (3)$$

evaluated for a unit radiance function $N_-(a)$ and at z in the direction ξ . By (2) the definitional identity:

$$N_r^0(z, \xi) = N_0(x, \xi) T_r(x, \xi) \quad (4)$$

readily follows, where $z = x + r\xi$. For every x, ξ, r , the quantity $T_r(x, \xi)$ is a dimensionless, non negative real number associated with the path $\mathcal{P}_r(x, \xi)$, and is called the *beam transmittance* of $\mathcal{P}_r(x, \xi)$. The radiance $N_r^0(z, \xi)$ is the *directly transmitted* (or *residual*, or *reduced* or *unattenuated*) radiance.

The compatibility of the limit in (3) with classical theory can only be fully made after a sufficient amount of the theory has been developed. Those readers wishing an immediate indication of the compatibility may consult statements Ia and Ib of Sec. 23 in Ref. [251]. Statement Ia shows at any rate that the limit of $T(a, b)$ need not go to zero as $a \rightarrow \{x\}$ and $D \rightarrow \{\xi\}$ for short paths. Furthermore statement Ib states that $R(a, b)$ does go to zero under these conditions. The rigorous proof of the existence of the limit in (3) requires the specific postulation of very mild regularity properties of the underlying radiative process. We shall not digress here to establish such fine points. Interested readers are referred to Chapter III of Ref. [251] for a complete discussion of what regularity properties are needed in this matter.

Next, by means of (42) of Sec. 3.7 we see that the statement:

$$\lim_{C \rightarrow \mathcal{P}_r(x, \xi)} \mathcal{T}(a, y, b) = T(a, y) \quad (5a)$$

follows from:

$$\lim_{C \rightarrow \mathcal{P}_r(x, \xi)} R(y, b) = 0 \quad (5b)$$

for every y , $a \leq y \leq b$. Statement (5b), in turn, follows from the S-continuity of the function $S(X; \cdot, \cdot, \cdot, \cdot)$. The S-continuity of $S(X; \cdot, \cdot, \cdot, \cdot)$ follows from statement Ib of Sec. 23 of Ref. [251]. From (1), (3) and the preceding observation (5a) we have for every x, ξ, s and t such that $s + t = r$:

$$T_r(x, \xi) = T_s(x, \xi) T_t(x + s\xi, \xi) \quad (6)$$

or

$$T_{s+t} = T_s T_t$$

for short when x and ξ are understood. Furthermore, from the fact that

$$\mathcal{T}(a, a, a) = I \quad ,$$

we have for every x and ξ :

$$\boxed{T_0(x, \xi) = 1} \quad , \quad (7)$$

or simply:

$$T_0 = 1 \quad ,$$

for short. In other words the beam transmittance of a path of length zero, is unity. Finally, from (15), (17) of Sec. 3.8 it follows that for every x, ξ, r :

$$\boxed{T_r(x, \xi) \leq 1} \quad (8)$$

or

$$T_r \leq 1 \quad ,$$

for short. Property (6) is called the *multiplicative (or semigroup) property*; property (7) is the *identity property*, and (8) is the *contraction property* of beam transmittance. These properties can also be based on properly presented physical arguments (see, e.g., Sec. 16 of Ref. [251]). However, it is of some interest to see that they can be made to follow formally from the interaction principle and (in the case of (8)) the law of conservation of radiant energy.

The intuitive meaning of $T_r(x, \xi)$ for a path $\mathcal{P}_r(x, \xi)$ in an optical medium can now be discerned. In everyday words, the radiance $N_-(a) \mathcal{T}(a, b, b)$ ($= N_-(a) T(a, b)$) transmitted through a finite cylinder C arrives at base b after having possibly been scattered several times. However, as C approaches $\mathcal{P}_r(x, \xi)$, so that base a approaches the set $\{x\}$ consisting of the single point x , there is progressively less "elbow" room for the flux in transmission to scatter about within C on its way to b . In the limit as C goes to $\mathcal{P}_r(x, \xi)$, any radiant flux that travels from x to z along $\mathcal{P}_r(x, \xi)$ has the "straight and narrow path" to follow. If it happens that a photon in transit from point x to point z is scattered exactly forward or backward a certain number of times, this photon is considered to constitute part of the residual radiance at z , for it is impossible at present to operationally distinguish, in the time-averaged residual radiance, between a photon which has travelled along $\mathcal{P}_r(x, \xi)$ without scattering and one which has travelled along $\mathcal{P}_r(x, \xi)$ and which has been scattered. Hence it will not be inconsistent to speak of $N_r^0(z, \xi)$ as radiance consisting of those photons originally

constituting $N_0(x, \xi)$ which have not been absorbed or scattered from $\mathcal{P}_r(x, \xi)$ as they travelled from x to $x+r\xi = z$. This is the *phenomenological interpretation* of $N_r^0(x, \xi)$, and we shall adopt it in the present work.

The generalization of the foregoing results to the case of non-constant index of refraction is effected by repeating all steps with $(N_-(a)/n^2)$ instead of $N_-(a)$, and building a tube of natural paths around $\mathcal{P}_r(x, \xi)$ using cross section a as a base from which the paths begin with normal incidence. The motivation for using the quotient $N_-(a)/n^2$ rests in the n^2 -law for radiance in (4) of Sec. 2.6.

This discussion is concluded with the observation that the beam transmittance $T_r(x, \xi)$ associated with the path $\mathcal{P}_r(x, \xi)$ in X is independent of the radiance distribution in X . This may be seen by returning to (3) and recalling that the standard transmittance operator $R(a, b)$ is an integral operator whose kernel function S_b is derived from an interaction operator obtained via the interaction principle. A re-examination of the conclusion of the interaction principle will show that an interaction operator is independent of the members of its domain sets A_i and range sets B_j , in other words interaction operators do not depend on the radiance distributions (i.e., the light fields) in X .

Inherent and Apparent Optical Properties

The foregoing observation is of considerable importance in establishing the basic optical properties of a natural optical medium X . *An optical property P of an optical medium X (P in the form of a number, function, or operator) which is independent of the light fields (in the form of radiance distributions) in X will be called an inherent optical property of X ; otherwise, P is an apparent optical property.* Hence, the beam transmittance function T_r which assigns to the path $\mathcal{P}_r(x, \xi)$ in X the beam transmittance $T_r(x, \xi)$, being independent of the radiance distribution in X , is an inherent optical property of X . We shall return to the systematic study of inherent and apparent optical properties in Chapter 9.

3.11 Derivation of the Volume Attenuation Function

The volume attenuation function is a measure of how much radiance a light beam loses per unit length of travel under the joint action of scattering and absorption processes. In this section we shall develop the concept of the volume attenuation function with the beam transmittance function as a starting point.

Let $\mathcal{P}_r(x, \xi)$ be a natural path in an optical medium X with associated beam transmittance $T_r(x, \xi)$. If a parcel of radiant flux of unit initial radiance traverses $\mathcal{P}_r(x, \xi)$, then on the one hand $T_r(x, \xi)$ is the amount of radiance transmitted over $\mathcal{P}_r(x, \xi)$, and on the other hand:

$$1 - T_r(x, \xi)$$