For given x and ξ this is an elementary differential equation for T_r, with known function α, whose solution is:

$$T_r = \exp\left\{-\int_0^r \alpha \, dr'\right\} \quad ,$$

or in more explicit notation:

$$\boxed{T_r(x,\xi) = \exp\left\{-\int_0^r \alpha\big(x(r'),\xi(r')\big) \, dr'\right\}} \quad . \qquad (3)$$

We have used the identity property for T_r ((7) of Sec. 3.10) to find the integration constant for the particular solution (3) of equation (2). Here $x(r')$ and $\xi(r')$ are the location and direction of a variable point within $\mathcal{P}_r(x,\xi)$, at distance r' from x along the path. If the index of refraction were constant, then $x(r') = x + r'\xi$; $\xi(r') = \xi$ for every r', $0 \le r' \le r$; and (3) would become:

$$T_r(x,\xi) = \exp\left\{-\int_0^r \alpha(x + r'\xi,\xi) \, dr'\right\} \quad , \qquad (4)$$

and (2) would take the form:

$$\frac{dT_r(x,\xi)}{dr} = -\alpha(x + r\xi,\xi) T_r(x,\xi) \quad . \qquad (5)$$

For a discussion of (5) in the case of variable index of refraction, see Sec. 17 of Ref. [251]. In that section there is also an alternative derivation of the function α using empirical radiances and empirical attenuating volumes. An experimental procedure for determining α is given in Sec. 13.4 of this work.

3.12 Derivation of Path Radiance and Path Function

We continue the sequence of derivations, begun in Sec. 3.10, leading to the derivation of the integral equation of transfer for radiance along a path $\mathcal{P}_r(x,\xi)$ in an optical medium X. In this section we give a derivation of two important components of this equation: the path radiance, and the path function associated with $\mathcal{P}_r(x,\xi)$, and conclude with a derivation of an important connection between them.

The Path Radiance

Let $\mathcal{P}_r(x,\xi)$ be a path in an optical medium X such as that depicted in Fig. 3.29. Once again, for simplicity, we assume constant index of refraction over X and no internal sources. Let C be a right cylinder about $\mathscr{P}_r(x,\xi)$. To introduce the concept of the path radiance of $\mathscr{P}_r(x,\xi)$ we conceptually isolate C, think of it as a one-parameter optical medium as in Sec. 3.10, and direct attention to the incident radiance distribution $N_-(c)$ on its flank c. The surface of C consists of three parts: two bases a,b, and flank c. The unit outward normal to c is assumed defined at each point of c. The incident radiance over flank c generates a light field within C by scattering processes within C with response radiance $N_-(b)$ over base b given by:

$$N_-(b) = N_-(c)\,\mathscr{S}(C;c,b)$$

where $\mathscr{S}(C;c,b)$ is the requisite interaction operator supplied by the interaction principle. The physical significance of this equation should be thoroughly understood before proceeding. We have enclosed $\mathscr{P}_r(x,\xi)$ (itself a conceptual object) in a hypothetical cylinder C. The radiance distribution at the base b of C is then the response of C to the incident natural light field all along its flank c. We direct attention now to point z in base b and direction $\xi(= -k(x))$ at z. The value of $N_-(b)$ at z and ξ is the radiance, generated within C, in response to the incident distribution $N_-(c)$. We now let C approach $\mathscr{P}_r(x,\xi)$, and keep an analytic eye on the limit of the value of $N_-(b)$ at x and ξ. We write:

$$\text{"}N_r^*(z,\xi)\text{"} \quad \text{for} \quad \lim_{C\to\mathscr{P}_r(x,\xi)} [N_-(c)\,\mathscr{S}(C;c,b)](z,\xi) \tag{1}$$

evaluated at z and ξ. We call $N_r^*(z,\xi)$ the *path radiance* associated with $\mathscr{P}_r(x,\xi)$. The path radiance $N_r^*(z,\xi)$ is commonly known as the "space light" or "haze of day" or "diffuse radiance" along a path of sight. If one looks along a path of sight under water or in the atmosphere such that the path terminates in some dark region, the veiling light between the observer and the dark region is due to path radiance. What we have just done is to go from the observable case for $N_-(b)$ to the mathematical limit (1). The latter is easier to work with for the same general reason that lines and planes in geometry are easier to work with than narrow rods and thin flat sheets.

The Path Function

We consider now the path radiance of very short paths $\mathscr{P}_r(x,\xi)$. In this way we shall come to the concept of the path function. Part (a) of Fig. 3.30 depicts a path $\mathscr{P}_r(x,\xi)$ with a right cylindrical subset C of X about $\mathscr{P}_r(x,\xi)$ as axis. As in Sec. 3.10, we consider C a one-parameter optical medium with the usual geometrical conventions. We start once again with a conceptual isolation of C and consider the equation for $N_-(b)$. We are interested now not only in the limit of $N_-(b)$ as C approaches $\mathscr{P}_r(x,\xi)$, but also in the limit of $N_-(b)$ as r is allowed to go to 0 while x and ξ are held fixed. We can anticipate the limit by examining the specific integral structure of $\mathscr{S}(C;c,b)$. From (6) of Sec. 3.8:

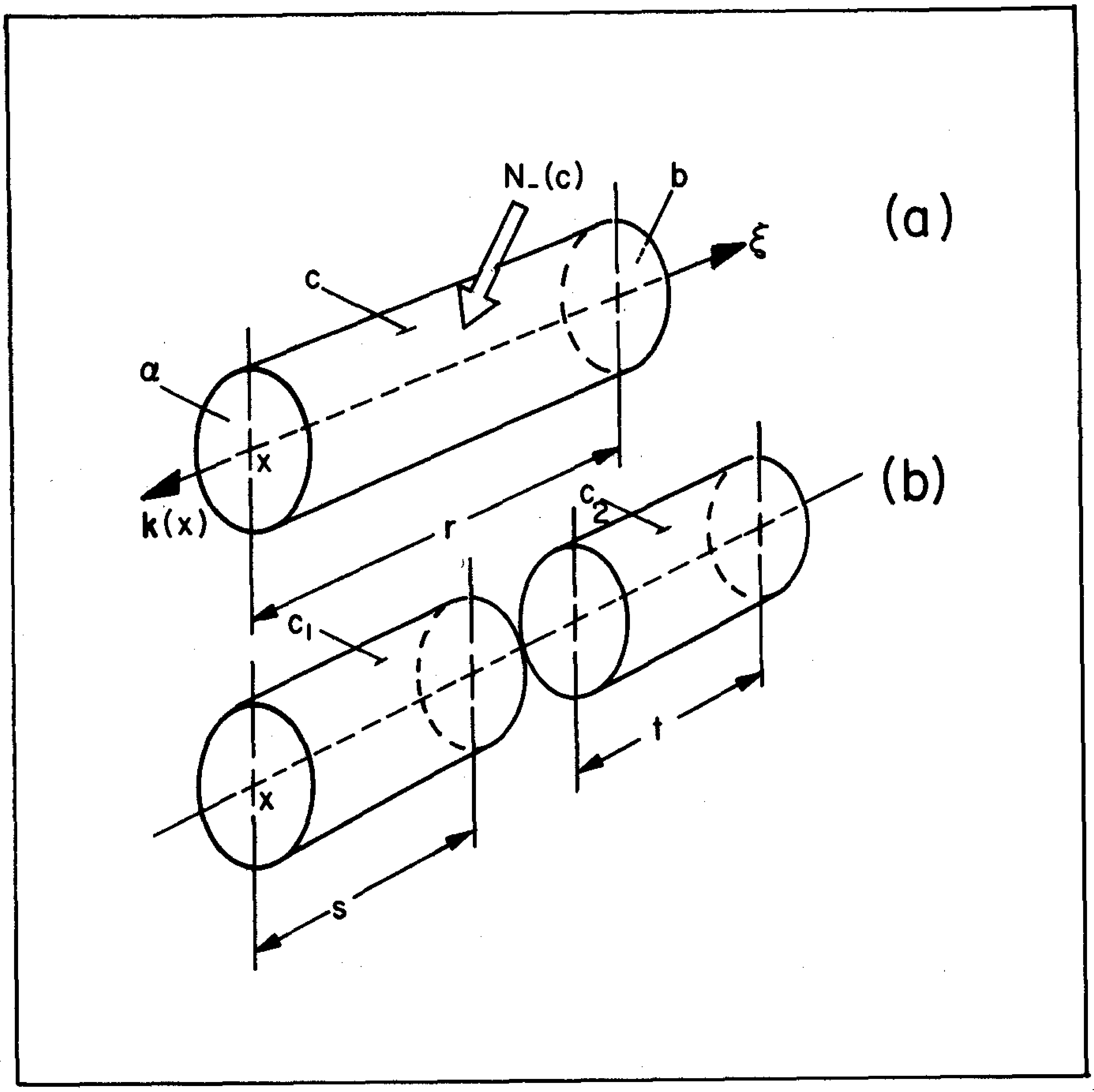

FIG. 3.30 Detail for the path function derivation.

$$N_-(b) = \int_c \int_{\Xi_-(x')} N(x',\xi')S_b(C;x',\xi';x,\xi)\ d\Omega(\xi')dA(x')$$

where $N(x',\xi')$ is the value of $N_-(c)$ at x' on c and ξ' in $\Xi_-(x')$. By the S'-continuity of the empirical funtion S_b given in (1) and (3) of Sec. 3.8, we conclude that $N_-(b)$ goes to zero as r goes to zero. We can say a little more than this: the value $N_r^*(z,\xi)$ eventually goes to zero linearly with r. This can be seen by conceptually slicing C into two parts C_1, C_2 shown in (b) of Fig. 3.30. This partitions the flank c into two parts c_1 and c_2 such that:

$$N_-(b) = N_-(c_1)\mathscr{S}(C_1;c_1,b) + N_-(c_2)\mathscr{S}(C_2;c_2,b)$$

where $c_1 \cup c_2 = c$, and $N_-(c_1)$ is equal to $N_-(c)$ over points of c_1 and zero over points of c_2. $N_-(c_2)$ is defined similarly. Each summand contributes a proportional share to $N_-(b)$. This shows, with the foregoing continuity property, that $N_r^*(z,\xi)$ has additivity and continuity properties with respect to r; and so it becomes plausible that it has an r-derivative.

The foregoing observations lead us to write, for every path $\mathcal{P}_r(x,\xi)$ in X:

$$"N_*(x,\xi)" \quad \text{for} \quad \lim_{r\to 0} \frac{N_r^*(z,\xi)}{r} \quad . \tag{2}$$

From this definition and (1) we have the following useful definitional identity for every path $\mathcal{P}_r(x,\xi)$ in X:

$$N_*(x,\xi) = \lim_{\substack{C\to \mathcal{P}_r(x,\xi) \\ r\to 0}} \frac{[N_-(c)\,\mathcal{J}(C;c,b)](z,\xi)}{r} \tag{3}$$

where the limit is evaluated at x and ξ. The number $N_*(x,\xi)$ is value of the *path function* N_* at x,ξ, on $\mathcal{P}_r(x,\xi)$. The dimensions of $N_*(x,\xi)$ are those of radiance per unit radial length. That is, the dimension of $N_*(x,\xi)$ is $\mathbf{P}^{\pm}\mathbf{A}^{-1}\Omega^{-1}\mathbf{L}_r^{-1} = \mathbf{P}^{\pm}\mathbf{V}^{-1}\Omega^{-1}$ (see Table 3 of Sec. 2.12 and note (h) of that table). Hence $N_*(x,\xi)$ may be thought of in either of two ways: the *radiance per unit length* (in the direction ξ) generated by scattering of an incident radiance distribution at point z into the direction ξ; or the *radiant intensity per unit volume* generated by scattering of an incident radiance distribution at point z into the direction ξ.

To summarize: By imagining a path $\mathcal{P}_r(x,\xi)$ in an optical medium X as the axis of a narrow cylindrical subset C of X, as in Fig. 3.30, and by rigorously relating the incident and response radiances $N_-(c)$ and $N_-(b)$ over C by means of the interaction operator $\mathcal{J}(C;c,b)$, we can define precisely, and on a phenomenological level, the concepts of path radiance $N_r^*(z,\xi)$ and path function N_* associated with the path $\mathcal{P}_r(x,\xi)$.

The Connection Between Path Function and Path Radiance

We now reverse the step taken in (2) and obtain a formula for the path radiance $N_r^*(z,\xi)$ given knowledge of the path function values $N_*(y,\xi)$ at each point y of $\mathcal{P}_r(x,\xi)$ between x and $z = x + r\xi$. The result will be a generally useful relation which also forms an indispensible component of the integral equation of transfer for radiance.

The first step in the derivation is to set up once again the one-parameter optical medium C around an arbitrary path $\mathcal{P}_r(x,\xi)$ in X, as in Fig. 3.29. In this way we have a useful scaffolding around $\mathcal{P}_r(x,\xi)$ with many precise analytical ladders on which to clamber up and down its extent; ladders in the form of the complete reflectance and transmittance operators for the medium, and $\mathcal{J}$-operators for arbitrary portions of the medium. Now a novel aspect of the present light field within C--when C is considered as a one-parameter optical medium--is that radiant flux enters C from X through the flanking surface c of C. This is novel in the sense that normally the invariant imbedding relation formally provides for external incident sources to enter C only on parameter levels a and b. However, by a simple device, to be introduced below, we can rigorously convert the incident radiance distributions on the flank c into internal sources at an arbitrary finite number of parameter levels in C. The problem can then be converted to one which, by Example 3 of Sec. 3.9, we know how to solve in principle. After that it will be a relatively simple matter to establish the requisite relation between path radiance and path function. The relation is established by means of six steps, the first of which begins below.

The conversion of the incident radiance distribution $N_-(c)$ on the flanking surface c of the medium C into internal source radiances is accomplished as follows. Part (a) of Fig. 3.31 depicts the one-parameter medium partitioned into a finite number n of cylindrical segments $C_1, C_2, \ldots, C_n$ by cross section surfaces $a_1, a_2, \ldots, a_{n-1}$, with $a_0 = a$ and $a_n = b$. This partition, in turn, partitions the flanking surface c of C into n pieces $c_1, c_2, \ldots, c_n$. Hence for each i, $i = 1, \ldots, n$, part C_i of C is bounded by parameter surfaces a_{i-1} on top and a_i on bottom (*upward* or *outward*, as usual, is the direction of $\mathbf{k}(\mathbf{x})$). The incident radiance distribution is partitioned correspondingly into n distributions $N_-(c_i)$ such that:

$$N_-(c_i) = N_-(c)$$

over c_i and

$$N_-(c_i) = 0$$

over c outside of c_i. In other words, we have decomposed $N_-(c)$ analytically by writing

$$N_-(c) = \sum_{i=1}^{n} \chi_i N_-(c_i) \tag{4}$$

where χ_i is the characteristic function for part c_i of c such that $\chi_i(y) = 1$ or 0 according as y is, or is not in c_i, respectively. Thus the radiometric effect of $N_-(c)$ on C is equivalent to the radiometric effect of this sum-decomposition of distributions on C. The proof of this is obvious: replace the left side of (4) by the right side of (4) in the interaction equation:

$$N_-(b) = N_-(c)\mathscr{S}(C;c,b) \quad . \tag{5}$$

Thus we have:

$$N_-(b) = \sum_{i=1}^{n} \chi_i N_-(c_i)\mathscr{S}(C;c,b) \tag{6}$$

$$= \sum_{i=1}^{n} N_-(c_i)\mathscr{S}(C;c_i,b) \quad . \tag{7}$$

We now begin the *second step*. The result (7) suggests

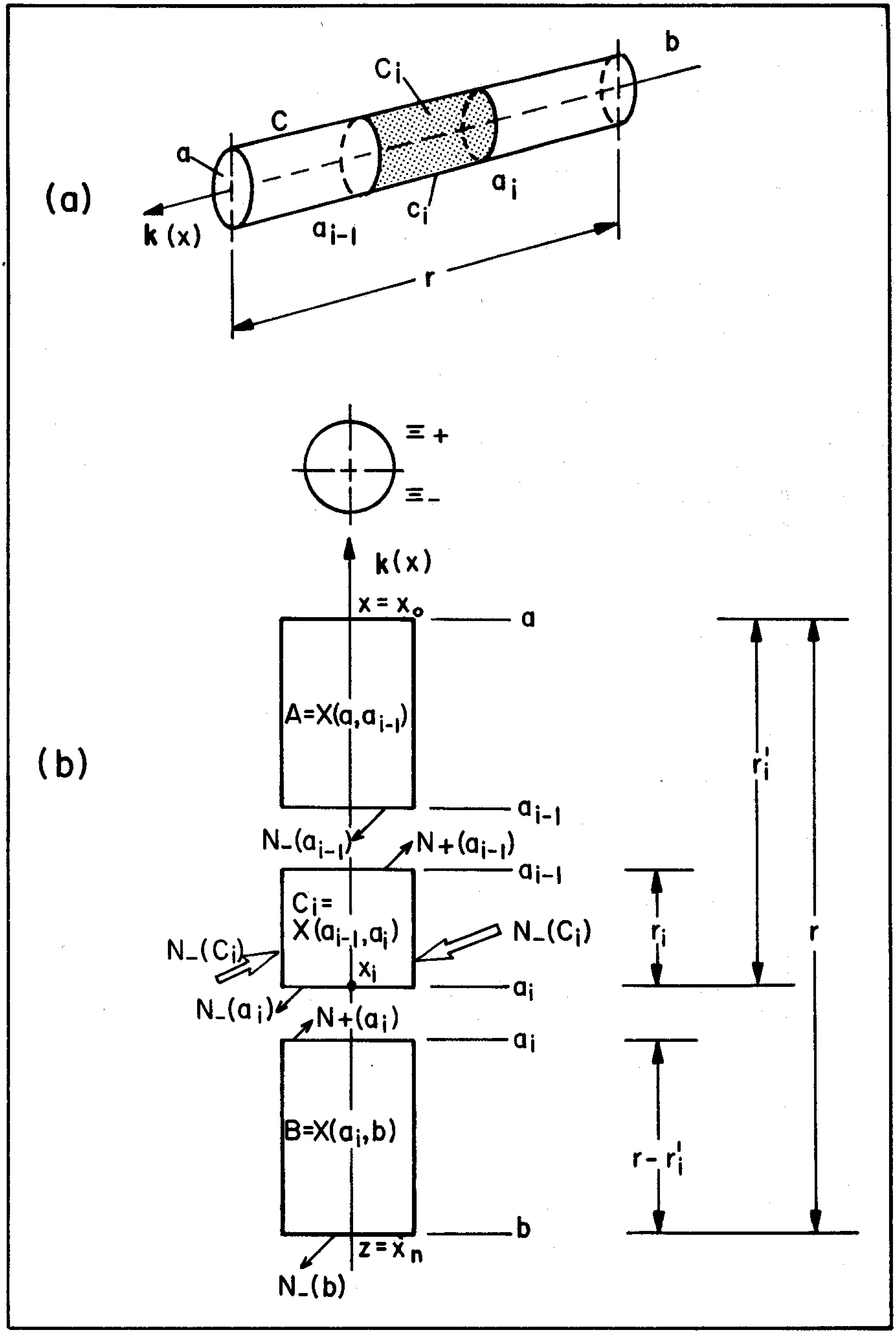

FIG. 3.31 Details for establishing the connection between path function and path radiance.

that the radiance distribution $N_-(b)$ can be thought of as generated by the response of C to the n incident radiance distributions $N_-(c_i)$. We wish to convert the external radiance distribution over c_i to an equivalent set of one or more source radiance distributions inside C. To see what is entailed in such a task, suppose for the moment that $N_-(c_j) = 0$ for every c_j except one, say $N_-(c_i)$. We now apply the interaction method to C_i by supposing C_i is conceptually isolated from C without disturbing the existing light field. Part (b) of Fig. 3.31 will be of help to visualize this conceptual excision of C_i from C. Parts A and B of C outside of C_i are shown in the diagram slightly pulled away from C_i. Next, the sets of incident radiometric functions on C_i are enumerated. These are:

A_1: all incident radiance distributions like $N_-(c_i)$

A_2: all incident radiance distributions like $N_+(a_i)$

A_3: all incident radiance distributions like $N_-(a_{i-1})$.

The sets of response radiance distributions of interest are:

B_1: all response radiance distributions like $N_+(a_{i-1})$

B_2: all response radiance distributions like $N_-(a_i)$.

The members of A_1 are field radiances. All the remaining radiances are surface radiances of their indicated surfaces. In the case of the members of A_2 and A_3, we have used the equality of field and surface radiance at a given point to replace the field radiances. In the present case of the principle, m = 3, n = 2, and the six operators s_{ij} given by the interaction principle are:

$$s_{11} \text{ -- } \mathscr{J}(C_i;c_i,a_{i-1})$$

$$s_{12} \text{ -- } \mathscr{J}(C_i;c_i,a_i)$$

$$s_{21} \text{ -- } T(a_i,a_{i-1})$$

$$s_{22} \text{ -- } R(a_i,a_{i-1})$$

$$s_{31} \text{ -- } R(a_{i-1},a_i)$$

$$s_{32} \text{ -- } T(a_{i-1},a_i)$$

These six operators have been given their standard notation as established earlier in the chapter. The first two are instances of (6) of Sec. 3.8. The remaining four are instances of the standard reflectance and transmittance operators for a general one-parameter medium, as given in Example 2 of Sec. 3.9.

The interaction principle then yields the equations for the response radiance distributions $N_-(a_i)$, $N_+(a_{i-1})$:

$$N_+(a_{i-1}) = N_-(c_i)\mathcal{V}(C_i;c_i,a_{i-1}) + \\ + N_+(a_i)T(a_i,a_{i-1}) + N_-(a_{i-1})R(a_{i-1},a_i) \quad (8)$$

$$N_-(a_i) = N_-(c_i)\mathcal{V}(C_i;c_i,a_i) + \\ + N_+(a_i)R(a_i,a_{i-1}) + N_-(a_{i-1})T(a_{i-1},a_i) \ . \quad (9)$$

The second step in the present derivation continues by comparing the preceding statements with those of the principles of invariance for the medium C under the same partitioning. The partitioning that is presently being used is such that:

C_i corresponds to $X(x,z)$

in principles I and II of Example 2 in Sec. 3.9. Furthermore, equation (8) corresponds to I when we let $y = x$ in I and also let:

y in principle I correspond to a_{i-1} in (8)

z in principle I correspond to a_i in (8) .

This detailed comparison is carried out in order to establish the fact that (8) may be interpreted as an instance of principle I applied to a source-free medium $X(a_{i-1},a_i)$ in C with a source of radiance of magnitude $N_-(c_i)\mathcal{V}(C_i;c_i,a_{i-1})$ at level a_{i-1}. In a similar way we see that (9) is interpretable as an instance of principle II applied to a source-free medium $X(a_{i-1},a_i)$ in C with a downward source (downward in C) of radiance of magnitude $N_-(c_i)\mathcal{V}(C_i;c_i,a_i)$ at level a_i. The conclusion we can now reach is the following: *with the current assumption about* $N_-(c_i)$ *as being the only source on* C *(to which we have momentarily agreed at the outset of this step of the discussion) we can then write the interaction equations for the light field in* C *with the interpretation that there are two sources of radiant flux in* C*: one upward flux confined to level* a_{i-1} *with magnitude* $N_-(c_i)\mathcal{V}(C_i;c_i,a_{i-1})$*, and one downward flux confined to level* a_i *with magnitude* $N_-(c_i)\mathcal{V}(C_i;c_i,a_i)$. This is the crucial observation needed in order to carry the derivation to completion, and since it appears somewhat complex, we state its message in still another way: Suppose a radiance meter were placed inside parts A or B of the cylinder C; for definiteness, say the meter was in B. What we have inferred from (9) is the prediction that the meter's reading would be unchanged were we to replace the external incident radiance distribution $N_-(c_i)$ by an internal source radiance distribution over parameter surface a_i equal to $N_-(c_i)\mathcal{V}(C_i;c_i,a_i)$ and an internal source radiance distribution over parameter surface a_{i-1} equal to $N_-(c_i)\mathcal{V}(C_i;c_i,a_{i-1})$.

The *third step* in the derivation consists in applying to the n-1 other partition pieces of C what we have just learned from consideration of the case of C_i. The net result is that the response radiance distribution $N_-(b)$ (the incipient path radiance) may be considered to be generated within C by n-1 internal source radiances $N^o_\pm(a_i)$ at i = 1,...,n-1, where we have written:

$$"N^o_+(a_{i-1})" \quad \text{for} \quad N_-(c_i)\mathscr{S}(C_i;c_i,a_{i-1})$$

$$"N^o_-(a_i)" \quad \text{for} \quad N_-(c_i)\mathscr{S}(C_i;c_i,a_i)$$

and one external radiance distribution $N^o_-(a_n)$, where we have written:

$$"N^o_-(a_n)" \quad \text{for} \quad N_-(c_n)\mathscr{S}(C_n;c_n,a_n) \ .$$

The *fourth step* in the derivation is the use of (38) of Sec. 3.9 to represent $N_-(b)$. For, by the conclusion of the third step, we have reduced the problem of representing $N_-(b)$ as generated by $N_-(c)$ to the problem of representing $N_-(b)$ as generated by the internal sources at the n-1 interval levels a_i of C. Toward this end, in (38) of Sec. 3.9, let y = b, $s_i = a_i$, and let the summation run from 1 to n-1. Then, for the contribution from the n-1 internal sources we have from (38) of Sec. 3.9:

$$\left(N_+(b),N_-(b)\right) = N(b) = \sum_{i=1}^{n-1} N^o(a_i)\Psi(a_i,b) \quad .$$

From this, and the nth source $N^o_-(a_n)$ we have:

$$N_-(b) = \sum_{i=1}^{n-1} N^o_+(a_i)\Psi_{+-}(a_i,b) + \sum_{i=1}^{n-1} N^o_-(a_i)\Psi_{--}(a_i,b) + N^o_-(a_n) \quad (10)$$

wherein $a_i < b$ for i = 1,...,n-1. The latter inequalities show that we may use the representations of the Ψ-operators in (31)-(34) of Sec. 3.9 once we have interchanged "+" with "-", and interchanged "a" with "b" everywhere in those equations. In particular, we have:

$$\Psi_{+-}(a_i,b) = \Psi_{+-}(a_i,a_i)\mathscr{T}(a_i,b,b) \quad (11)$$

$$\Psi_{--}(a_i,b) = \left(I + \Psi_{--}(a_i,a_i)\right)\mathscr{T}(a_i,b,b) \quad . \quad (12)$$

With this observation, (10) becomes:

$$N_-(c)\mathscr{S}(C;c,b) = N_-(b) =$$

$$= \sum_{i=1}^{n-1} N_-(c_{i+1})\mathscr{S}(C_{i+1};c_{i+1},a_i)\Psi_{+-}(a_i,a_i)T(a_i,b) +$$

$$+ \sum_{i=1}^{n-1} N_-(c_i)\mathscr{S}(C_i;c_i,a_i)\left(I + \Psi_{--}(a_i,a_i)\right)T(a_i,b)$$

$$+ N_-(c_n)\mathscr{S}(C_n;c_n,a_n) \quad . \quad (13)$$

The *fifth step* of the derivation consists in applying to (13) the limit process, $C \to \mathcal{P}_r(x,\xi)$. This application is facilitated by noting that:

$$\lim_{C \to \mathcal{P}_r(x,\xi)} \Psi_{+-}(a_i,a_i) = 0$$

$$\lim_{C \to \mathcal{P}_r(x,\xi)} \Psi_{--}(a_i,a_i) = 0$$

which follow from (21) and (23) of Sec. 3.9 and (5b) of Sec. 3.10. Applying the limit process $C \to \mathcal{P}_r(x,\xi)$ to (13) we have, with the aid of definition (3) of Sec. 3.10, and (1):

$$N_r^*(z,\xi) = \sum_{i=1}^{n-1} N_{r_i}^*(x_i,\xi) T_{r-r_i}(x_i,\xi) + N_{r_n}^*(x_n,\xi) \quad .$$

Hence:

$$\boxed{N_r^*(z,\xi) = \sum_{i=1}^{n} N_{r_i}^*(x_i,\xi) T_{r-r_i'}(x_i,\xi)} \qquad (14)$$

or, briefly:

$$N_r^* = \sum_{i=1}^{n} N_{r_i}^* T_{r-r_i'} \quad ,$$

where r_i, r_i' and x_i are defined as in (b) of Fig. 3.31. (Hence $x_0 = x$, $x_n = z$, $r_n' = r$, and $T_{r-r_n'}(x_i,\xi) = 1$.) Equation (14) is a useful exact formula for the path radiance of $\mathcal{P}_r(x,\xi)$ in terms of the path radiance over the component segments $\mathcal{P}_{r_i}(x_{i-1},\xi)$; $i = 1,\ldots,n$.

The *sixth and final step* of the derivation consists of dividing and multiplying the ith summand in (14) by r_i, and letting $n \to \infty$ in such a way that the maximum r_i in the set $\{r_1,\ldots,r_n\}$ goes to zero. Thus we first write (14) as:

$$N_r^*(z,\xi) = \sum_{i=1}^{n} \frac{N_{r_i}^*(x_i,\xi)}{r_i} T_{r-r_i'}(x_i,\xi) r_i$$

and apply the limit process $n \to \infty$, max $\{r_i\} \to 0$, which formally yields, by means of (2), the following Riemann integral:

$$\boxed{N_r^*(z,\xi) = \int_0^r N_*(x',\xi) T_{r-r'}(x',\xi)\, dr'} \qquad (15)$$

which is the desired integral representation of the path radiance $N_r^*(z,\xi)$ for the path $\mathcal{P}_r(x,\xi)$ in terms of its path function N_* and beam transmittance function T_r. In the integral we have written:

$$\text{"z"} \quad \text{for} \quad x + r\xi$$

$$\text{"x'"} \quad \text{for} \quad x + r'\xi \quad ,$$

and have derived the formula under the conditions of constant index of refraction over $\mathcal{P}_r(x,\xi)$, and no internal sources over $\mathcal{P}_r(x,\xi)$. This completes the derivation.

The form of (15) is unchanged when the index of refraction is non constant, providing the appropriate form of the beam transmittance for such a case is used (see, e.g., Sec. 16 of Ref. [251]). The case of internal sources is covered by introducing the emission function N_η. (See, e.g., Sec. 19 of Ref. [251] and (1) of Sec. 5.8 below.)

It might be well to explicitly summarize what has transpired between (5) and (15). We started with (5) which is the precise formula for the observable path radiance $N_-(b)$ over path $\mathcal{P}_r(x,\xi)$ in an optical medium X. Here we have used one very convenient feature of the interaction principle, namely that it provides the exact mathematical rendition of all possible radiometric operations with radiance meters and other light measuring devices in natural optical media. Equation (5) summarizes how we can view the $N_-(b)$ as the response to incident flux $N_-(c)$, over the boundary c, of a subset C of X. This formula is then rearranged, using some of the laws of radiative transfer and the properties of the interaction operators we have deduced so far, into the classical representation (15) of the path function integral for path radiance. It should be emphasized that no intuitive observations were used as integral parts of the derivation between (5) and (15); however, they were occasionally invoked merely to make intuitively clear or to motivate the various steps in the derivation. The interaction principle has been stocked at the outset with all the intuition of the phenomenological point of view needed to establish radiative transfer theory. The principle thus provides the formal machinery for recovering all the known intuitions, and perhaps some new ones yet to be generated.

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3.13 Derivation of Apparent-Radiance Equation

We turn next to the task of deducing from the interaction principle the well-known intuitive decomposition of the radiance of a distant object into two parts: the residual radiance transmitted from the object over the path of sight to the observer, and the path radiance generated over the extent of the path of sight.

The setting of Fig. 3.29 will serve for the first stage of the present discussion. We imagine the observer at point z of the path of sight $\mathcal{P}_r(x,\xi)$ in an optical medium X and that he is recording the radiance $N(z,\xi)$. For example the path of sight might begin with point x in a mountainside, or