3.14 Derivation of the Volume Scattering Function

We come now to the derivation of the second inherent optical property of a general optical medium X. This property is summarized in the form of the volume scattering function which quantitatively describes at each point x of X the fraction of incident radiance $N(x,\xi')$ in the direction ξ' scattered in the generally different direction ξ, without change in wavelength. The volume scattering function together with the volume attenuation function derived in Sec. 3.11 are the two basic inherent optical properties for optical media used in radiative transfer theory.

Figure 3.32 depicts an arbitrary right-cylindrical subset C of the optical medium X. The axis of the cylinder is a straight line segment with initial point x, terminal point z, direction ξ, and length $r(\xi)$. Our present goal is to derive the volume scattering function from the appropriate interaction operator for C as supplied by the interaction principle.

Let a and b be the two bases of C and c be the flanking surface of C. Let $N_-(c)$ be a member of the set of incident radiance distributions on c. We assume C to be a one-parameter optical medium with the customary geometrical conventions. We are interested in the response radiances $N_-(b)$ emerging from the base b and such that $N_-(b)$ is of the same wavelength as $N_-(c)$. The interaction principle supplies an operator $\mathcal{S}(C;c,b)$ such that:

$$N_-(b) = N_-(c)\,\mathcal{S}(C;c,b) \quad . \tag{1}$$

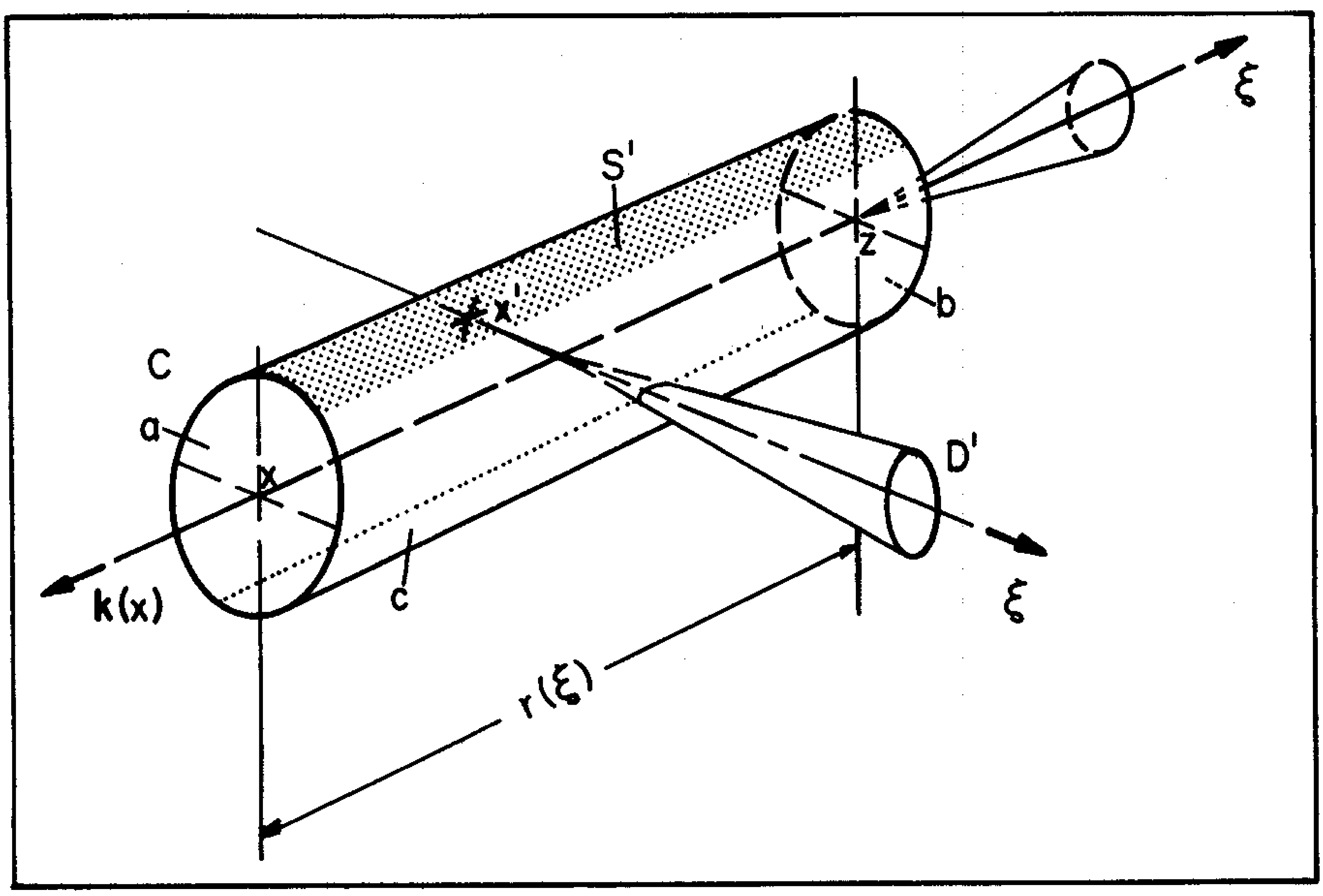

FIG. 3.32 Detail for the volume scattering function derivation.

The linear dimensions of C are now chosen so that the variation in size and shape of the radiance distributions from point to point over the volume of C is very small. This can always be done in natural light fields. The mathematical regularity property of $N_-(c)$ reflecting this natural phenomenon is that $N_-(c)$ be continuous and nearly constant with location over C. Furthermore we restrict the radiance distribution to be of uniform value over a fixed small solid angle D' containing a given direction ξ' distinct from ξ. Thus $N_-(c)$ is constant valued in D' and has zero value for directions outside D'. Again this is a feasible restriction, being operationally realizable in practice using appropriate optical shielding of C from appropriate parts of Ξ. The reason for these restrictions on $N_-(c)$ stems from the highly specialized geometrical structure of the volume scattering function which we are trying to characterize.

When these conditions are applied to (1) that equation takes on a considerably simpler form. Thus, evaluating $N_-(b)$ at z in the direction ξ we have from (1):

$$N(z,\xi) = \int_c \int_{\Xi_-(x')} N(x',\xi'')S_b(C;x',\xi'';x,\xi)\; d\Omega(\xi'')dA(x')$$

$$= N(S'',D') \int_{S'} \int_{D'} S_b(C;x',\xi'';x,\xi)\; d\Omega(\xi'')dA(x') \qquad (2)$$

Here S' is the part of the boundary of C which is bathed in the present radiance distribution. More precisely, S' is the set of all points x' of the flank c of C such that $\xi'\cdot\mathbf{k}(x')\leq 0$, where $\mathbf{k}(x')$ is the unit outward normal to c at x', and ξ' is the fixed direction of incidence of the radiance distribution $N_-(c)$ (see shaded region in Fig. 3.32). "N(S",D')" denotes the constant value of the radiance distribution $N_-(c)$ over S' and D'. S" is the projection of S' on a plane normal to ξ'. By (1), (3) and (4) of Sec. 3.8, $N(z,\xi)$ can be written as:

$$N(z,\xi)= N(S'',D')S(C;S',D';x,\xi)A(S')\Omega(D') \qquad (3)$$

and we are essentially back where we started from--except for one important difference: the radiance $N(z,\xi)$ is now clearly seen (by (1) of Sec. 3.12) to be the incipient path radiance for the path $\mathscr{P}_{r(\xi)}(x,\xi)$ under the present special lighting conditions. Hence by the agreement leading to (2) of Sec. 3.12 we are motivated to consider the ratio $N(z,\xi)/r(\xi)$, which by (3) is of the form:

$$\frac{N(z,\xi)}{r(\xi)} = N(S'',D')\left[S(C;S',D';x,\xi)\;\frac{A(S')}{r(\xi)}\right]\Omega(D') \qquad (4)$$

We have now reached the threshold of the definition of the volume scattering function. It remains only to write:

$$"\sigma(x;\xi';\xi)" \quad \text{for} \quad \lim_{\begin{cases} C\to \mathscr{P}_{r(\xi)}(x,\xi) \\ r(\xi)\to 0 \\ D'\to\{\xi'\} \end{cases}} S(C;S',D';x,\xi)\ \frac{A(S')}{r(\xi)} \ . \tag{5}$$

The function σ which assigns to each point x and distinct pair of directions ξ',ξ the non negative number σ(x;ξ';ξ) defined in (5) is called the *volume scattering function.* Since σ was derived from the operator $\mathscr{S}(C;c,b)$ with the kernel function S(C;·,·,·,·) supplied by the interaction principle, it follows (as in the case of α in Sec. 3.11) that σ is an inherent optical property of X. The dimensions of σ are $Lr^{-1}\Omega^{-1}$. This radial length and solid angle are associated with the direction ξ. The verbal interpretation of σ(x;ξ';ξ) is obtained directly from (4) as follows: σ(x;ξ';ξ) *is the scattered radiance generated, without change in wavelength, per unit length in the direction* ξ *by unit irradiance incident at* x *in the direction* ξ'.

An operational definition of σ, suitable for experimental use in natural optical media, is given in Sec. 13.6. An alternate approach to the volume scattering function is given in Example 1 of Sec. 3.17. The approach to (5) via empirical operations is given in Sec. 18 of Ref. [251].

Regularity Properties of σ

We have gone as far as we can in the present approach to the definition of σ: the value of σ at each x,ξ',ξ has been defined by means of (5). In order to go on and use σ in the mathematical theory of radiative transfer we must assume regularity properties of σ. In other words we must make explicit certain continuity properties of σ in order to use the calculus. Therefore σ will be assumed a piecewise continuous function of each of the arguments x,ξ', and ξ. Furthermore, we assume that for every point y in X there is a sphere $X_a(y)$ of center y and radius a such that for every connected subset C of $X_a(y)$:

$$S(C;x',\xi';x,\xi) = \sigma(x';\xi';\xi)\ \frac{r(\xi)}{A(S')} + o\big(r(\xi)\big) \tag{6}$$

for every pair of distinct directions ξ',ξ, and every pair of points x',x on the boundary of C. Here "o(r(ξ))" is the value of a function o(·) which has the property that $\lim_{x\to 0} o(x)/x=0$. r(ξ) is the length of the shortest path through C with initial point x and in the direction -ξ. S' is that part of the boundary Y of C consisting of points x such that $\xi'\cdot\mathbf{k}(x)\le 0$ where $\mathbf{k}(x)$ is the unit outward normal to Y at x. S' has a very simple geometric interpretation: shine a parallel beam of light on all of C along the direction ξ'. Then S' is that part of Y that is lit by the beam.

The Integral Representation of the Path Function

The final main step toward the derivation of the integral equation for radiance over a path $\mathscr{P}_r(x,\xi)$ can now be taken. This is the establishment of the integral representation of the path function N_* over $\mathscr{P}_r(x,\xi)$ using the volume scattering function σ along $\mathscr{P}_r(x,\xi)$.

Let $X_a(y)$ be a sphere of radius a about a point y on $\mathscr{P}_r(x,\xi)$. Let the boundary of $X_a(y)$ be b. Then:

$$N_-(b)(x,\xi) = \int_b \int_{\Xi_-(x')} N(x',\xi')S(X_a(y);x',\xi';x,\xi)\, d\Omega(\xi')dA(x') \tag{7}$$

Let the radius of $X_a(y)$ be such that (6) holds and $N_-(c)$ is independent of position over $X_a(y)$. Then (7) becomes:

$$N_-(b)(x,\xi) =$$

$$= \int_b \int_{\Xi_-(x')} N(y,\xi')\left[\sigma(y;\xi';\xi)\,\frac{r(\xi)}{A(S')} + o\big(r(\xi)\big)\right] d\Omega(\xi')dA(x')$$

$$= \int_\Xi N(y,\xi')\left[\sigma(y;\xi';\xi)r(\xi) + A(S')o\big(r(\xi)\big)\right] d\Omega(\xi') \quad .$$

Hence:

$$N_-(b)(x,\xi) =$$

$$= r(\xi)\int_\Xi N(y,\xi')\sigma(y;\xi';\xi)\, d\Omega(\xi') + o\big(r(\xi)\big)\int_\Xi N(y,\xi')A(S')\, d\Omega(\xi') \; .$$

The radiance $N_-(b)(x,\xi)$ is thus seen to be nearly the path radiance for the path through $X_a(y)$ emerging at x in the direction y, and of length $r(\xi)$ inside the spherical region $X_a(y)$. Hence $N_-(b)(x,\xi)/r(\xi)$ is nearly the value of the path function at x in the direction ξ. By letting the radius a of $X_a(y)$ go to zero we have:

$$\lim_{a\to 0} \frac{N_-(b)(x,\xi)}{r(\xi)} = N_*(y,\xi) = \int_\Xi N(y,\xi')\sigma(y;\xi';\xi)\, d\Omega(\xi') \; .$$

Summarizing: For every point x in an optical medium, the relation between the path function $N_*(x,\cdot)$ at x, the radiance distribution $N(x,\cdot)$ at x and the volume scattering function $\sigma(x;\cdot,\cdot)$ is:

$$N_*(x,\xi) = \int_{\Xi} N(x,\xi')\sigma(x;\xi';\xi)\, d\Omega(\xi') \tag{8}$$

<< Table of Contents << Title Page >> Next Page >>

3.15 The Equation of Transfer for Radiance

All the pieces of the integral equation of transfer for radiance have now been carved out of the interaction principle. It remains only to assemble them into the desired statement.

Let $\mathscr{P}_r(x,\xi)$ be a path in an optical medium X. Let C be a right cylinder with $\mathscr{P}_r(x,\xi)$ as axis, and let C be a one-parameter optical medium with boundary Y composed of upper and lower parameter surfaces a and b, and flank c, as in Fig. 3.29. Let the incident radiance distribution over the outer surface Y of C be $N_-(Y)$. Then there is an interaction operator $\mathscr{J}(C;Y,b)$ such that:

$$N_-(b) = N_-(Y)\,\mathscr{J}(C;Y,b) \quad .$$

This is the interaction equation governing the response radiance of C over the base b. With this as a starting point, it was eventually reduced, as shown in Sec. 3.13, to the statement:

$$N_r(z,\xi) = N_r^o(z,\xi) + N_r^*(z,\xi)$$

where $N_r(z,\xi)$ is the apparent radiance of the field at z in the direction ξ. From (6) of Sec. 3.13 this can be written in the form:

$$N_r(z,\xi) = N_o(x,\xi)T_r(x,\xi) + \int_0^r N_*(x',\xi)T_{r-r'}(x',\xi)\, dr' \quad .$$

From (8) of Sec. 3.14 this becomes:

$$N_r(z,\xi) = N_o(x,\xi)T_r(x,\xi) + \int_0^r \left[\int_{\Xi} N(x',\xi')\sigma(x';\xi';\xi)\, d\Omega(\xi')\right] T_{r-r'}(x',\xi)\, dr' \tag{1}$$

which is the requisite *integral equation of transfer for radiance*. The subscripts "r" and "o" may be dropped wherever possible when it is convenient to divest (1) of all explicit ties with the path $\mathscr{P}_r(x,\xi)$. The result in such a case is: