$$N_*(x,\xi) = \int_{\Xi} N(x,\xi')\sigma(x;\xi';\xi)\, d\Omega(\xi') \tag{8}$$

3.15 The Equation of Transfer for Radiance

All the pieces of the integral equation of transfer for radiance have now been carved out of the interaction principle. It remains only to assemble them into the desired statement.

Let $\mathscr{P}_r(x,\xi)$ be a path in an optical medium X. Let C be a right cylinder with $\mathscr{P}_r(x,\xi)$ as axis, and let C be a one-parameter optical medium with boundary Y composed of upper and lower parameter surfaces a and b, and flank c, as in Fig. 3.29. Let the incident radiance distribution over the outer surface Y of C be $N_-(Y)$. Then there is an interaction operator $\mathscr{I}(C;Y,b)$ such that:

$$N_-(b) = N_-(Y)\,\mathscr{I}(C;Y,b) \quad .$$

This is the interaction equation governing the response radiance of C over the base b. With this as a starting point, it was eventually reduced, as shown in Sec. 3.13, to the statement:

$$N_r(z,\xi) = N_r^o(z,\xi) + N_r^*(z,\xi)$$

where $N_r(z,\xi)$ is the apparent radiance of the field at z in the direction ξ. From (6) of Sec. 3.13 this can be written in the form:

$$N_r(z,\xi) = N_o(x,\xi)T_r(x,\xi) + \int_0^r N_*(x',\xi)T_{r-r'}(x',\xi)\, dr' \quad .$$

From (8) of Sec. 3.14 this becomes:

$$N_r(z,\xi) = N_o(x,\xi)T_r(x,\xi) + \int_0^r \left[\int_{\Xi} N(x',\xi')\sigma(x';\xi';\xi)\, d\Omega(\xi')\right] T_{r-r'}(x',\xi)\, dr' \tag{1}$$

which is the requisite *integral equation of transfer for radiance*. The subscripts "r" and "o" may be dropped wherever possible when it is convenient to divest (1) of all explicit ties with the path $\mathscr{P}_r(x,\xi)$. The result in such a case is:

$$N(z,\xi) = N(x,\xi)T_r(x,\xi) + \int_0^{r(z,\xi)} \int_{\Xi} N(x',\xi')\sigma(x';\xi';\xi)T_{r-r'}(x',\xi)\, d\Omega(\xi')dr' \ . \tag{2}$$

This is the mathematical form of the integral equation of transfer for the radiance function N defined on $X \times \Xi$. Here $r(z,\xi)$ (abbreviated in (1) as "r") is the length of the shortest distance from z to the boundary of X along the direction $-\xi$ (see Fig. 3.33). Hence r depends on z and ξ and all are connected by the equation: $z = x + r\xi$. The distances r' and r are measured from the boundary point x to the point x' and the point z, respectively, along the direction ξ.

In mathematical discussions requiring the determination of N on $X \times \Xi$, it is assumed that the volume attenuation function α and the volume scattering function σ are known, with appropriate regularity properties, and that $N(x,\xi)$ is given for each incident direction ξ at each point x on the boundary of X. As it stands, Equation (2) is an integral equation for N and applies to source-free optical media of arbitrary geometric shape with arbitrary α and σ and constant index of refraction n. With slight modifications (2) can be made to hold in media with internal sources by suitably including the emission function $N_\eta = N_e + N_s$ (where N_e represents true emission

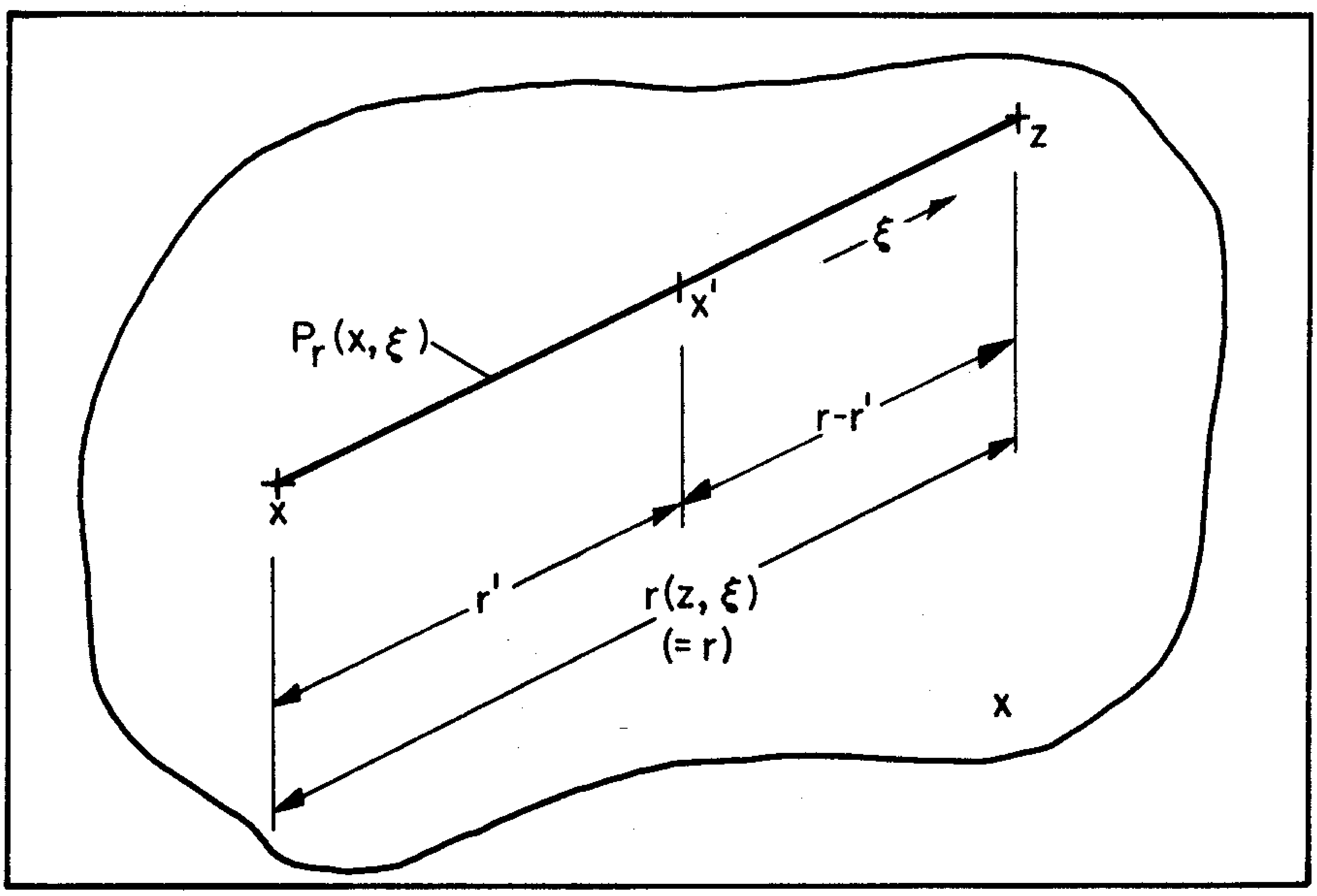

FIG. 3.33 The setting for the equation of transfer for radiance in a general medium X.

radiance and N_s transpectral scattered radiance), and variable index of refraction n. See, e.g., Sec. 21, Ref. [251], in which the boundary radiances and the optical properties can also be changing rapidly with time.

Steady State Equation of Transfer

We conclude the main discussion of this section by deriving from (1) the classical form of the integrodifferential equation of transfer. This equation is designed to describe the rate of change of $N_r(x,\xi)$ along $\mathscr{P}_r(x,\xi)$, with respect to r. Thus, holding x and ξ fixed, we shall let the path length r vary and then compute $dN_r(x,\xi)/dr$. By doing this, we in effect imbed $\mathscr{P}_r(x,\xi)$ in a family $\mathscr{P}$ of paths of the same x and ξ, and observe that the general functional form of (1) is invariant over the members of $\mathscr{P}$. In order to find $dN_r(x,\xi)/dr$, we can apply the operator d/dr to each of the two main terms of (1):

$$\frac{dN_o(x,\xi)T_r(x,\xi)}{dr} = N_o(x,\xi)\,\frac{dT_r(x,\xi)}{dr}$$

$$= N_o(x,\xi)\left(-\alpha(x,\xi)T_r(x,\xi)\right) \quad .$$

The second equality is based on (2) of Sec. 3.11. Next, we find (using (15) of Sec. 3.12 and (5) of Sec. 3.13):

$$\frac{dN_r^*(z,\xi)}{dr} = \frac{d}{dr}\int_0^r N_*(x',\xi)T_{r-r'}(x',\xi)\;dr'$$

$$= \int_0^r N_*(x',\xi)\;\frac{dT_{r-r'}(x',\xi)}{dr}\;dr' + N_*(z,\xi)$$

$$= -\alpha(x,\xi)\int_0^r N_*(x',\xi)T_{r-r'}(x',\xi)dr' + N_*(z,\xi)$$

$$= -\alpha(z,\xi)N_r^*(z,\xi) + N_*(z,\xi) \quad .$$

The second equality is obtained by means of the Leibniz rule of differentiating an integral with respect to a parameter. Collecting these results, we have:

$$\frac{dN_r(z,\xi)}{dr} = -\alpha(x,\xi)\left[N_r^o(z,\xi) + N_r^*(z,\xi)\right] + N_*(z,\xi) \quad .$$

Using (5) of Sec. 3.13, (8) of Sec. 3.14, and dropping "r" from "N_r" we finally obtain:

$$\frac{dN(z,\xi)}{dr} = -\alpha(z,\xi)N(z,\xi) + \int_{\Xi} N(z,\xi')\sigma(z;\xi';\xi)\, d\Omega(\xi') \tag{3}$$

which is the desired *integrodifferential form of the equation of transfer.* The settings in which (3) holds are also those for (1). The generalizations available for (3) are also those for (1).

It will be instructive for the reader to interpret each term of (3) by means of losses and gains of $N(x,\xi)$ due to attenuation and scattering. Thus the term $-\alpha(z,\xi)N(z,\xi)$ summarizes the rate of loss from $N(z,\xi)$ by means of attenuation, and the integral term $N_*(z,\xi)$ summarizes the rate of gain of $N_r(z,\xi)$ by means of scattering. A direct derivation of (3) by means of such loss-gain arguments is made, e.g., in Sec. 21 of Ref. [251]. We have chosen the present route to (3) to add to the accumulating evidence in this chapter that all of classical radiative transfer theory is derivable from the interaction principle. Having finally arrived at (3) in this manner we may essentially rest our case.

Time Dependent and Polarized Equations of Transfer

The chain of arguments in Sec. 3.10 to the present starting from the interaction principle and culminating in the equations of transfer (2) and (3), can now be repeated in all their essential steps but with more general formulations as the end result. For example, two immediate generalizations of (3) are obtained by considering time dependent radiances with time dependent optical properties, and by considering polarized radiance functions. The derivations of these generalizations of (3) will not be recorded here and are left as important (and nontrivial) exercises for interested students of the subject. (See Sec. 127 of [251].) The resultant time dependent equation of transfer is:

$$\frac{1}{v}\frac{DN(z,\xi,t)}{Dt} = -\alpha(z,\xi,t)N(z,\xi,t) + \int_{\Xi} N(z,\xi',t)\sigma(x;\xi';\xi,t)\, d\Omega(\xi') \tag{4}$$

where we have written:

$$\text{"}\frac{D}{Dt}\text{"} \quad \text{for} \quad \frac{\partial}{\partial t} + v\frac{d}{dr} \tag{5}$$

and where D/Dt, as in (4) is the usual (*mobile* or *substantial*) derivative operator along the path $\mathcal{P}_r(x,\xi)$. Furthermore, v is the speed of light at z, and we have written:

$$\text{"}\frac{d}{dr}\text{"} \quad \text{for} \quad \xi\cdot\nabla \quad , \tag{6}$$

where ∇ is the usual gradient operator of vector analysis.

The steady state equation of transfer for polarized radiance has the same Gestalt as (3); thus,

$$\frac{dN(z,\xi)}{dr} = -\alpha(z,\xi)N(z,\xi) + \int_{\Xi} N(z,\xi')p(z;\xi';\xi)\, d\Omega(\xi') \qquad (7)$$

where $N(z,\xi)$ is as defined in Sec. 2.10 and $p(z;\xi';\xi)$ is the standard observable volume scattering matrix--the polarized counterpart to $\sigma(z;\xi';\xi)$. For a complete definition of $p(z;\xi';\xi)$ and related concepts, see Sec. 112 of Ref. [251]. Helpful background techniques for the derivation of (7) from the interaction principle are contained in Sec. 113, 114, and 126 of Ref. [251]. In particular, the work in Sec. 126 of Ref. [251] may serve as a prototype for the requisite derivation of (7). What is required for the general derivation is the consistent elevation of all concepts of the prototype derivation from the scalar to the vector level. The extension of (4) and (7) to the case of internal sources is accomplished by appending suitable source terms to each. For example, one may append $N_\eta(x,\xi,t)$ to (4), and $N_\eta(x,\xi)$ to (7). A discussion of the relative consistency of (3) and (7) when the light field in a given medium is unpolarized is given in Sec. 13.11. In addition, the problem of the fidelity of (3) in the context of polarized light fields is raised and discussed in Sec. 13.11.

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3.16 On the Integral Structure of the Interaction Operators

In this section we discharge a series of obligations which have been accumulating ever since Sec. 3.3. These concern the assertions that the interaction principle formally implies the existence of the various integral operators for reflectances and transmittances of surfaces, plane-parallel media, and the scattering properties for general optical media. Our purpose here is to cite and apply the appropriate mathematical theorems which, under suitable regularity conditions, yield the requisite integral operators arising under the use of the interaction principle along with their physically interesting kernels. By methodically applying these theorems to the various geometric settings encountered in Sec. 3.3, 3.6, 3.8 and 3.9, and many other settings, a veritable cornucopia of classical and novel integral operator formulas for radiative transfer phenomena is tapped and brought into formal existence by means of the interaction principle. The following discussion, while mainly mathematical in flavor, is written principally with the physicist in mind. Emphasis will constantly be on the physical or geometrical meanings of the terms discussed. As a result, mathematical rigor will not be of primary concern.