where ∇ is the usual gradient operator of vector analysis.

The steady state equation of transfer for polarized radiance has the same Gestalt as (3); thus,

$$\frac{dN(z,\xi)}{dr} = -\alpha(z,\xi)N(z,\xi) + \int_{\Xi} N(z,\xi')p(z;\xi';\xi)\, d\Omega(\xi') \qquad (7)$$

where $N(z,\xi)$ is as defined in Sec. 2.10 and $p(z;\xi';\xi)$ is the standard observable volume scattering matrix--the polarized counterpart to $\sigma(z;\xi';\xi)$. For a complete definition of $p(z;\xi';\xi)$ and related concepts, see Sec. 112 of Ref. [251]. Helpful background techniques for the derivation of (7) from the interaction principle are contained in Sec. 113, 114, and 126 of Ref. [251]. In particular, the work in Sec. 126 of Ref. [251] may serve as a prototype for the requisite derivation of (7). What is required for the general derivation is the consistent elevation of all concepts of the prototype derivation from the scalar to the vector level. The extension of (4) and (7) to the case of internal sources is accomplished by appending suitable source terms to each. For example, one may append $N_\eta(x,\xi,t)$ to (4), and $N_\eta(x,\xi)$ to (7). A discussion of the relative consistency of (3) and (7) when the light field in a given medium is unpolarized is given in Sec. 13.11. In addition, the problem of the fidelity of (3) in the context of polarized light fields is raised and discussed in Sec. 13.11.

3.16 On the Integral Structure of the Interaction Operators

In this section we discharge a series of obligations which have been accumulating ever since Sec. 3.3. These concern the assertions that the interaction principle formally implies the existence of the various integral operators for reflectances and transmittances of surfaces, plane-parallel media, and the scattering properties for general optical media. Our purpose here is to cite and apply the appropriate mathematical theorems which, under suitable regularity conditions, yield the requisite integral operators arising under the use of the interaction principle along with their physically interesting kernels. By methodically applying these theorems to the various geometric settings encountered in Sec. 3.3, 3.6, 3.8 and 3.9, and many other settings, a veritable cornucopia of classical and novel integral operator formulas for radiative transfer phenomena is tapped and brought into formal existence by means of the interaction principle. The following discussion, while mainly mathematical in flavor, is written principally with the physicist in mind. Emphasis will constantly be on the physical or geometrical meanings of the terms discussed. As a result, mathematical rigor will not be of primary concern.

The Mathematical Prerequisites

In our everyday activities we frequently perform certain operations to determine various weights and measures of objects; measures of area, direction, length, volume, mass, and so on. When an attempt is made to formulate a mathematical representation of these operations and to summarize their salient characteristics, one arrives at the logical concepts of *linear functional* and *measure*.

Here is a simple example of a linear functional. Consider a straight flat sidewalk S one block long. A child's wagon loaded with objects is pulled over S from point a to b. As a result work is done in transporting these objects over S from a to b. The more objects in the wagon, the more force generally that must be exerted to push it from a to b. Suppose f(s) is the force parallel to the sidewalk that must be applied to a loaded wagon at point s on the sidewalk to keep it moving at constant speed from a to b. Then f is a force function defined on the sidewalk S. To f we can now assign the amount of work W(f) required to push its associated loaded wagon over S. For another wagon load, there is another function g and yet another amount W(g) of work done in transporting *that* wagonload over S. Hence W is a function which assigns to f or g a number, the work done in transporting over S the wagonload associated with f or g. W is called a *functional* because it acts on functions f, g and not numbers (as f and g do).

Now for typical sidewalks and most ordinary objects and wagons, the functional W is *linear* in the sense that:

$$\text{(i)} \quad W(f + g) = W(f) + W(g)$$

and

$$\text{(ii)} \quad W(cf) = cW(f)$$

where c is a real number. An instance of (i) arises, e.g., when we stack f's load of objects on top of g's load of objects and push the assembled load, whose force function then is f + g, along S. An instance of (ii) arises when several copies of an object with force function f are stacked on top of each other and transported along S. (In this case c would be an integer.)

Here is an example of a measure. Consider once again the sidewalk S described above. Every two points s, s' on the centerline of S determine a segment [s,s'] of S between them which has length say l([s,s']). If [s,s'] and [t,t'] are two such separate segments along S, then $l([s,s'] \cup [t,t']) = l([s,s']) + l([t,t'])$, where "$[s,s'] \cup [t,t']$" denotes the set of points on S consisting of the points of [s,s'] or [t,t']. Here l is an example of a measure--in this case a length measure: it assigns a number l([s,s']) or "l(s,s')", for short, to the interval [s,s'] of points--the *length* of the interval [s,s'].

Here is an example of a linear functional closer to the present subject of radiative transfer theory. Let S be the same sidewalk as above, and let H be the function which assigns to each point s of S, the irradiance on S at s. To this function H there is associated an amount P(H) of radiant flux falling on S. By simultaneous irradiation of S by skylight and street lamplight, etc., various irradiance functions H_1, H_2, etc. can be defined over S, and with resultant amounts $P(H_1)$, $P(H_2)$, etc. of radiant flux falling on S. P is therefore a linear functional in the sense that:

$$\text{(iii)} \quad P(H_1 + H_2) = P(H_1) + P(H_2)$$

$$\text{(iv)} \quad P(cH) = cP(H) \quad .$$

The pertinent measure in this example is area measure A of parts of S: and with the property that $A(S_1 \cup S_2) = A(S_1) + A(S_2)$ for two separate parts S_1 and S_2 of S.

Now the substance of the first of the two prerequisite theorems of present interest concerns the representation of linear functionals, such as W and P above, by means of an *integral with respect to a certain measure*. Thus, in the case of W, the theorem states that there is a measure μ over the sidewalk centerline such that for every force function f,

$$W(f) = \int_a^b f(s)\, d\mu(s) \quad . \tag{1}$$

We know from elementary physical theory that μ is simply the length measure illustrated above; so

$$W(f) = \int_a^b f(s)\, dl(s) \quad . \tag{2}$$

In the radiometric example, the theorem asserts that for every irradiance function H on S, there is a measure μ over the sidewalk such that:

$$P(H) = \int_S H(s)\, d\mu(s) \quad . \tag{3}$$

From (6) of Sec. 2.4, we know that μ must be area measure A; so

$$P(H) = \int_s H(s)\, dA(s) \quad . \tag{4}$$

The general version of the special cases (2), (4) may be stated after the following preliminaries are covered: Let S be a closed bounded subset in Euclidean n-space X_n, over which a family F(S) of real-valued continuous functions is defined. This setting is of sufficient generality to serve all our present purposes. Observe that F(S) is a vector space, (e.g., the sum of two functions in F(S) is again in F(S); the product of a member of F(S) by a real number is again in F(S)). By examples (i)-(iv) above we know what a linear functional L is. A *positive* linear functional L on F(S) is a linear functional on F(S) such that $L(f) \geq 0$ if f is a non negative valued member of F(S) (both examples W and P above are examples of positive linear functionals). Then the requisite theorem (which is a general form of the Riesz representation theorem) goes as follows:

> *Theorem A. If* L *is a positive linear functional on* F(S), *then there exists a (Borel) measure* μ *on* S *such that for every* f *in* F(S)

$$\boxed{L(f) = \int_S f(x)\, d\mu(x)} \quad .$$

A complete general development of this theorem may be found, e.g., in Sec. 56 of Ref. [103].

The second theorem we shall need concerns measures which are absolutely continuous with respect to other measures. A measure μ is absolutely continuous with respect to a measure ν on a space X if $\mu(E) = 0$ whenever $\nu(E) = 0$ for every measurable subset E of X. This ostensibly forbidding-sounding description hides a very simple idea which may be illustrated as follows. To each subset X of ordinary Euclidean three-space assign the radiant energy content U(X) of that subset, as, e.g., we did in (14) of Sec. 2.7. Now it is obvious from the relation (14) of Sec. 2.7 that $U(X) = 0$ whenever $V(X) = 0$. That is, the radiant energy content of a set X of zero volume is zero. Using the present terminology we say that the radiant energy measure U is *absolutely continuous* with respect to the volume measure V. Other common examples may be found: mass measure, heat measure, etc., are absolutely continuous with respect to volume measure. Now the next theorem we have in mind says that for the case of U and V, for example, there is a function f on X such that:

$$U(X) = \int_X f(x)\, dV(x) \quad . \qquad (5)$$

In other words, the theorem guarantees the existence of an energy density function f which when integrated over X gives the radiant energy content U(X) of X. In the other two cases cited we have the existence guaranteed of the mass density functions and heat density functions. Another way of writing f above is as:

$$\frac{dU}{dV}$$

pointing up the nature of f as a volume derivative of energy. We could then write the preceding integral as:

$$U(X) = \int_X \frac{dU}{dV}\, dV(x) \quad .$$

The function f above is a special instance of the general concept of a *Radon-Nikodym derivative* of one measure μ with respect to another ν. This derivative of μ exists whenever μ is absolutely continuous with respect to ν. The general statement is as follows:

> *Theorem B. Let* S *be a subset of Euclidean n-space* X_n *and let* ν *be a finite valued measure on* S. *Let* μ *be a finite valued measure on* S *which is absolutely continuous with respect to* ν. *Then there exists a finite valued function* f *on* S *such that*

$$\boxed{\mu(E) = \int_E f(x)\ d\nu(x)}$$

> *for every subset* E *of* S.

The wording of this theorem, whose full version may be found in Sec. 31 of Ref. [103], has been deliberately simplified--references to fixed measure spaces and fixed families of measurable sets and functions have been suppressed and are to be implicitly understood. We are concerned here with only the essential conceptual content of Theorems A and B, what mathematical things they yield up for use, and their pertinence to the physical radiative transfer context. In the context of Theorem B we shall write

$$\text{"}\frac{d\mu}{d\nu}\text{"} \quad \text{for} \quad f \ . \tag{6}$$

The final theorem we shall need has been anticipated by the integral representation of U(X) above. Its statement goes as follows:

> *Theorem C. Let* S *be a subset of Euclidean n-space* X_n. *If* μ *and* ν *are finite valued measures on* S *and* μ *is absolutely continuous with respect to* ν *and* g *is a function on* S *such that* $\int_E g\ d\nu$ *is defined for every subset* E *of* S, *then:*

$$\boxed{\int_E g\ d\mu = \int_E \left[g\ \frac{d\mu}{d\nu}\right]\ d\nu}$$

> *for every subset* E *of* S.

Again the wording of this theorem has been mercifully simplified so that one is encouraged to follow its physical applications below. Its unexpurgated and generalized version may be found in Sec. 32 of Ref. [103].

Interaction Operators for Surfaces

The preceding mathematical theorems will now be applied to the case of reflectance and transmittance operators for surfaces. Let us return to Fig. 3.3 of Sec. 3.3. We are interested in particular in the interaction properties of the surface Y depicted in part (a) of that figure. If a is any subset of Y, (e.g., a could be S of the figure) then the interaction method yields an operator $r_-(a)$ such that:

$$N_+(a) = N_-(a)r_-(a)$$

for the incident downward radiance distribution $N_-(a)$ on a and the reflected upward radiance distribution $N_+(a)$. Recall that $N_-(a)$ is a function which assigns to each point x on a and direction ξ in $\Xi_-(x)$ the incident (field) radiance $N_-(a)(x,\xi)$, called "$N_-(x,\xi)$" for short. The set $\Xi_-(x)$ is depicted in (b) of Fig. 3.3.

According to the interaction principle, $r_-(a)$ is a positive linear functional which, for a fixed choice x in a, and ξ in $\Xi_+(x)$, assigns to each $N_-(a)$ in the set of incident radiance functions on a the reflected radiance $N_+(a)(x,\xi)$ or "$N_+(x,\xi)$" for short. More specifically, the set S in Theorem A is the set $\Xi_-(x)$. F(S) is now the set of all incident radiance distributions $N_-(x,\cdot)$ at x on a. Hence, by Theorem A, there is a measure μ, depending on the current fixed choice of x and ξ, such that:

$$N_+(x,\xi) = \int_{\Xi_-(x)} N_-(x,\xi')\, d\mu(x;\xi';\xi) \quad . \tag{7}$$

Here we have written the "μ" in Theorem A with sufficient notational paraphernalia so as to completely identify and keep track of it. The variable ξ' is like the x in the theorem. The variables x,ξ remind us that we have momentarily limited the values of $N_+(a)$ to x and ξ in a and $\Xi_+(x)$, respectively.

Now the measure $\mu(x;\cdot;\xi)$ just obtained from Theorem A is defined on $\Xi_-(x)$. That is, it assigns to subsets D' of $\Xi_-(x)$ a number whose geometric and physical significance becomes clear by letting $N_-(x,\cdot)$ be uniform valued with value 1 over subsets of $\Xi_-(x)$. For example, if D' is a subset of $\Xi_-(x)$ over which $N_-(x,\cdot)$ has value 1 and has value 0 outside D', then from (7):

$$\begin{aligned} N_+(x,\xi) &= \int_{\Xi_-(x')} N_-(x,\xi')\, d\mu(x;\xi';\xi) \\ &= \int_{D'} d\mu(x;\xi';\xi) \\ &= \mu(x;D';\xi) \end{aligned}$$

The construction of $\mu(x;D';\xi)$ in the present case is such (according to the proof of Theorem A) that for every D', if $\Omega(D') = 0$, then $\mu(x;D';\xi) = 0$ under all natural physical conditions. This means that a unit radiance distribution incident on surface a through solid angles of zero measure will induce zero radiance $N_+(x,\xi)$. Hence $\mu(x;\cdot;\xi)$ is to be absolutely continuous with respect to the solid angle measure Ω.

We are now ready to use Theorem B. The subset S is the same set just used in Theorem A. The measure ν is now solid angle measure Ω and μ is $\mu(x;\cdot;\xi)$. Hence Theorem B says that there is a finite valued function f--in this case call it "$r_-(x;\cdot;\xi)$", such that:

$$\mu(x;D';\xi) = \int_{D'} r_-(x;\xi';\xi)\, d\Omega(\xi')$$

for every subset D' of $\Xi_-(x)$. In other words:

$$r_-(x;\xi';\xi) = \frac{d\mu(x;\xi';\xi)}{d\Omega} \quad .$$

Theorem C completes the derivation when we observe that g is now to be $N_-(x,\cdot)$, μ is now $\mu(x;\cdot;\xi)$, and ν is Ω. We therefore have from (7):

$$N_+(x,\xi) = \int_{\Xi_-(x)} N_-(x,\xi')\, d\mu(x;\xi';\xi)$$

$$= \int_{\Xi_-(x)} N_-(x,\xi') r_-(x;\xi';\xi)\, d\Omega(\xi') \quad . \qquad (8)$$

Since x and ξ were arbitrary, (8) holds for every x in a and ξ in $\Xi_+(x)$, and the deduction of the form:

$$\int_{\Xi_-(x)} [\ ] r_-(x;\xi';\xi)\, d\Omega(\xi')$$

from the interaction principle is complete. The integral representations of the remaining three operators $r_+(a)$, $t_\pm(a)$ introduced in Sec. 3.3 are obtained similarly.

Interaction Operators for General Media

We go on now to consider a general optical medium X, bypassing the operators for plane-parallel media as being merely a special case of the present setting. Our goal is to derive the integral form of the linear operator $\mathscr{S}(X;a,b)$ in (6) of Sec. 3.8.

The interaction method yields a linear operator $\mathscr{S}(X;a,b)$ such that

$$N_+(b) = N_-(b)\mathscr{S}(X;a,b) \tag{9}$$

with the geometric conventions as defined in Sec. 3.8. The radiance distribution $N_-(b)$ is one of a family $F(S)$ of incident radiance distributions on S, where S is the set $a \times \Xi_-$ consisting of all pairs of points (x,ξ) with x in a and ξ in $\Xi_-(x)$. $F(S)$ is an instance of an incident set A_1 in the interaction principle and $N_+(b)$ is a member of the set B_1 of response functions. Hence $m = n = 1$ for the interaction principle yielding (9).

The interaction principle implies that $\mathscr{S}(X;a,b)$ induces a positive linear functional over $F(S)$ in the following way. We choose a *fixed* point x on b and *fixed* direction ξ in $\Xi_+(x)$ and consider the value $N_+(x,\xi)$ of $N_+(b)$ at (x,ξ). Theorem A then allows us to write:

$$\begin{aligned} N_+(x,\xi) &= \left(N_-(a)\mathscr{S}(X;a,b)\right)(x,\xi) \\ &= \int_S N_-(x',\xi')\, d\mu(X;x',\xi';x,\xi) \end{aligned} \tag{10}$$

where now $\mu(X;\cdot;x,\xi)$ is the measure on $a \times \Xi_-$, denoted by "S" in (10), whose existence is asserted in Theorem A. Again we have lavishly embellished μ of Theorem A with identifying variables: X,x,ξ. Further (x',ξ') in (10) acts like x in the theorem.

A glance at (10) shows that $\mu(X;\cdot;x,\xi)$ is a measure on S ($= a \times \Xi_-$) and so the integral is a double integral over S. The geometric measure ν over S is the product of the area measure A over a and solid angle measure Ω over Ξ. If we let $N_-(a)$ be of uniform unit value over a subset S' of S and zero outside S', then (10) implies:

$$\begin{aligned} N_+(x,\xi) &= \int_{S'} d\mu(X;x',\xi';x,\xi) \\ &= \mu(X;S';x,\xi) \quad . \end{aligned} \tag{11}$$

As in the case of the reflectance operator for surfaces, we require, for obvious physical reasons, the radiance $N_+(x,\xi)$ to be zero when the measure ν of a subset S' of S is zero. That is, we require $\mu(X;S';x,\xi) = 0$ whenever $\nu(S') = 0$ and we shall assume that this is true.

Now we have:

$$\begin{aligned} \nu(S) &= \int_S d\nu = \int_S d(\Omega \times A) \\ &= \int_a \int_{\Xi_-(x')} d\Omega(\xi')dA(x') \quad . \end{aligned} \tag{12}$$

Recall that a general subset S' of S is a collection of ordered pairs (x',ξ') such that x' is in a subset a' of a and ξ' is in $\Xi_-(x')$; hence (12) is a special case of:

$$\nu(S') = \int_{a'} \int_{\Xi_-(x')} d\Omega(\xi')dA(x') \quad . \qquad (13)$$

We now return to Theorem B which asserts that there is in this case a finite valued function f on $a \times \Xi_-$--call it $S(X;\cdot,\cdot;x,\xi)$--such that:

$$\mu(X;S';x,\xi) = \int_{a'} \int_{\Xi_-(x')} S(X;x',\xi';x,\xi)\ d\nu(x',\xi')$$

for every subset S' of S. In other words:

$$S(X;x',\xi';x,\xi) = \frac{d\mu(X;x',\xi';x,\xi)}{d\nu}$$

$$= \frac{d\mu(X;x',\xi';x,\xi)}{d(\Omega \times A)} \quad .$$

Theorem C allows us to complete the derivation when we observe that g is to be $N_-(a)$, μ is now $\mu(X;\cdot;x,\xi)$ and ν is $\Omega \times A$, the Cartesian product of the solid angle and area measures. We therefore have from (10):

$$N_+(x,\xi) = \int_S N_-(x',\xi')\ d\mu(X;x',\xi';x,\xi)$$

$$= \int_a \int_{\Xi_-(x')} N_-(x',\xi')S(X;x',\xi';x,\xi)\ d\Omega(\xi')dA(x') \quad . \qquad (14)$$

Since x and ξ were arbitrary, (14) holds for every x and ξ in b and $\Xi_+(x)$, respectively, and the deduction of $\mathscr{S}(X;a,b)$ in its integral operator form:

$$\int_a \int_{\Xi_-(x')} [\ ]S(X;x',\xi';x,\xi)\ d\Omega(S')dA(x') \quad ,$$

from the interaction principle, is complete.

Interaction Measures and Kernels

The features common to the two discussions just completed will now be summarized so as to extract the salient steps that must be generally taken in deducing from the interaction principle the requisite integral operator describing a given radiative transfer interaction.

Suppose a particular discussion using the interaction method has reached the stage where the interaction principle yields for a subset S of an optical medium X the operator equation:

$$b = as \tag{15}$$

where a and b are elements of the incident and response classes A and B of radiometric functions, respectively (cf. Sec. 3.2). The functions a and b are quite general and may be either number-valued or vector-valued, or matrix-valued, etc., with domains of space, directional, frequency, time variables, singly or in combination. Let C be the domain of a. For the present discussion we shall view the class A explicitly as a set A(C) of continuous non negative valued functions on C. Similarly B is viewed explicitly as a set B(D) of non negative valued functions on some set D. (For example, in the case of the surface reflectance operators the medium X was three-space, S was a surface a, C was $\Xi_-(x)$, and D was $\Xi_+(x)$ for a fixed point x on surface a. A(C) was the set of all radiance distributions of the form $N_-(x,\cdot)$, and B(D) was the set of all radiance distributions of the form $N_+(x,\cdot)$.) Finally, the subset C is generally assumed to have some pertinent measure ν. (For example in the preceding discussion of the surface reflectance operators ν was Ω the solid angle measure on $\Xi_-(x)$.)

With these preliminaries established, the general method proceeds by selecting an arbitrary fixed point y in D. With this fixed y, we see that, the interaction operator s of (15) becomes a positive linear functional s(y). That is, if:

$$b_1(y) = a_1 s(y) \geq 0$$

and:

$$b_2(y) = a_2 s(y) \geq 0$$

then:

$$\alpha b_1(y) + \beta b_2(y) = (\alpha a_1 + \beta a_2) s(y) \geq 0 \quad ,$$

where α and β are non negative numbers, and where the $b_i(y)$ are images of the a_i under s(y), i = 1,2.

By Theorem A, there is a measure $\mu(S,\cdot,y)$ depending on the subset S of X and the point y in D such that:

$$s = \int_C [\ ]\, d\mu(S,\cdot,y) \tag{16}$$

so that for every y in D, (15) may be represented as:

$$b(y) = \int_C a(x)\, d\mu(S,x,y) \quad . \tag{17}$$

Let us say that $\mu(S,\cdot,y)$ has the *AC property (with respect to ν)* whenever ν and $\mu(S,\cdot,y)$ are such that if: $\nu(E) = 0$ then $\mu(S,E,y) = 0$ for every subset E of C. This property, it should be noted, is not asserted to hold universally. We view it as a *regularity property* whose validity must either be postulated (as an axiom, say) or demonstrated in each situation. Thus the properties of each operator must be suitably stated so that the AC property holds (see remarks on the *Stages of the Interaction Method* in Sec. 3.18). The AC property of $\mu(S,\cdot,y)$ is the abstract version of all the S' and D' continuity statements made in Secs. 3.3, 3.6, and 3.8. The additive property of the measure $\mu(S,\cdot,y)$ is the abstract version of the S' and D' additivity properties stated in these sections. The initials "AC" stand for "absolute continuity".

The next step in the general method is to postulate (or verify) the AC property of $\mu(S,\cdot,y)$ so that we may go on formally to Theorem B which asserts that there exists a function $K(S,\cdot,y)$ on C such that:

$$\mu(S,E,y) = \int_E K(S,x,y)\, d\nu(x) \quad , \qquad (18)$$

for every subset E of C. For example, $S(X;x',\xi';x,\xi)$ defined in the preceding example on general media is the special case of $K(S,x,y)$ for a general optical medium X. In the case of $S(X;x',\xi';x,\xi)$, "x" (in (18)) plays the role of "(x',ξ')", and "y" plays the role of "(x,ξ)", and of course "S" (in (18)) plays the role of X. Hence we have:

$$K(S,x,y) = \frac{d\mu(S,x,y)}{d\nu(x)} \quad . \qquad (19)$$

We call $K(S,\cdot,y)$ the *interaction kernel* for the subset S of X and $\mu(S,\cdot,y)$ the *interaction measure* for S.

An application of Theorem C then completes the general method by allowing us to write:

$$b(y) = \int_C a(x) \frac{d\mu(S,x,y)}{d\nu(x)}\, d\nu(x) \quad .$$

That is, for every y in D, (15) may now be written:

$$b(y) = \int_C a(x)K(S,x,y)\, d\nu(x) \qquad (20)$$

so that:

$$s = \int_C [\ \ ]K(S,x,y)\, d\nu(x) \quad . \qquad (21)$$

Equation (21) is the requisite integral representation of the interaction operator s, associated with the subset S of the optical medium X.