from which we can immediately read the physical significance of N_q: If $N_r < N_q$ at a point on the path, then $dN_r/dr > 0$, i.e., N_r is increasing at that point. In general, N_r always tends toward the fixed radiance N_q, and $dN_r/dr = 0$ if and only if $N_r = N_q$. Therefore N_q takes on the aspect of an equilibrium value (in an every day sense) toward which the values N_r unceasingly tend. The equilibrium radiance N_q is often observable over long horizontal uniformly lighted paths through a homogeneous natural aerosol or hydrosol.

It should be observed that the derivation of (1) places no conditions on the orientation or the location of the path $\mathcal{P}_r$ in an optical medium. The essential point to observe in the derivation is that (1) follows from (1) of Sec. 3.15 upon assuming only that α, σ and N_* are constant long $\mathcal{P}_r$. This leaves $\mathcal{P}_r$ free to be vertical, inclined, or horizontal, as the case may be. An interesting example of (1) for inclined paths of sight in the atmosphere may be obtained from the results in [71].

4.4 The Classical Canonical Equation

In this section we continue to ascend the ladder of generality and derive still further instances of canonical radiance equations. We still have not reached the most general physical setting in which the canonical equation can hold, but we have reached the point where the full canonical structure of the equation finally emerges, and we turn now to the derivation of that form.

Let $\mathcal{P}_r(x,\xi)$ be an arbitrary line of sight in a homogeneous optical medium X. To fix ideas, let the medium X be a natural hydrosol, and let us adopt the standard coordinate frame for such a setting (Sec. 2.4 and Fig. 2.3). Let $\mathcal{P}_r(x,\xi)$ be positioned as shown in Fig. 4.1.

With the geometry fixed as in Fig. 4.1, we now assume α and σ to be independent of location along the generally inclined path $\mathcal{P}_r(x,\xi)$, and that the light field does not vary over a given horizontal plane, i.e., the light field is stratified. The new feature of the canonical equation appears by assuming that there exists a nonnegative real number K (which is less than α) such that:

$$N_*(z,\xi) = N_*(z_0,\xi)\, e^{-K(z-z_0)} \qquad (1)$$

for every path $\mathcal{P}_r(x,\xi)$ in X. This means that we are hypothesizing an exponential decrease of $N_*(z,\xi)$ with depth z in X. The justification for this assumption rests on both experimental and theoretical grounds. For an experimental justification, see Sec. 1.2; for theoretical justifications see Secs. 1.3, 7.10, 8.5, 8.6 and Sec. 10.7. For the present, we are concerned primarily with the resultant form of (6) of Sec. 3.13 to which this assumption leads us. Thus starting with (6) of Sec. 3.13, we have:

$$N_r(z,\xi) = N_0(x,\xi)T_r(x,\xi) + \int_0^r N_*(x',\xi)T_{r-r'}(x',\xi)\, dr'$$

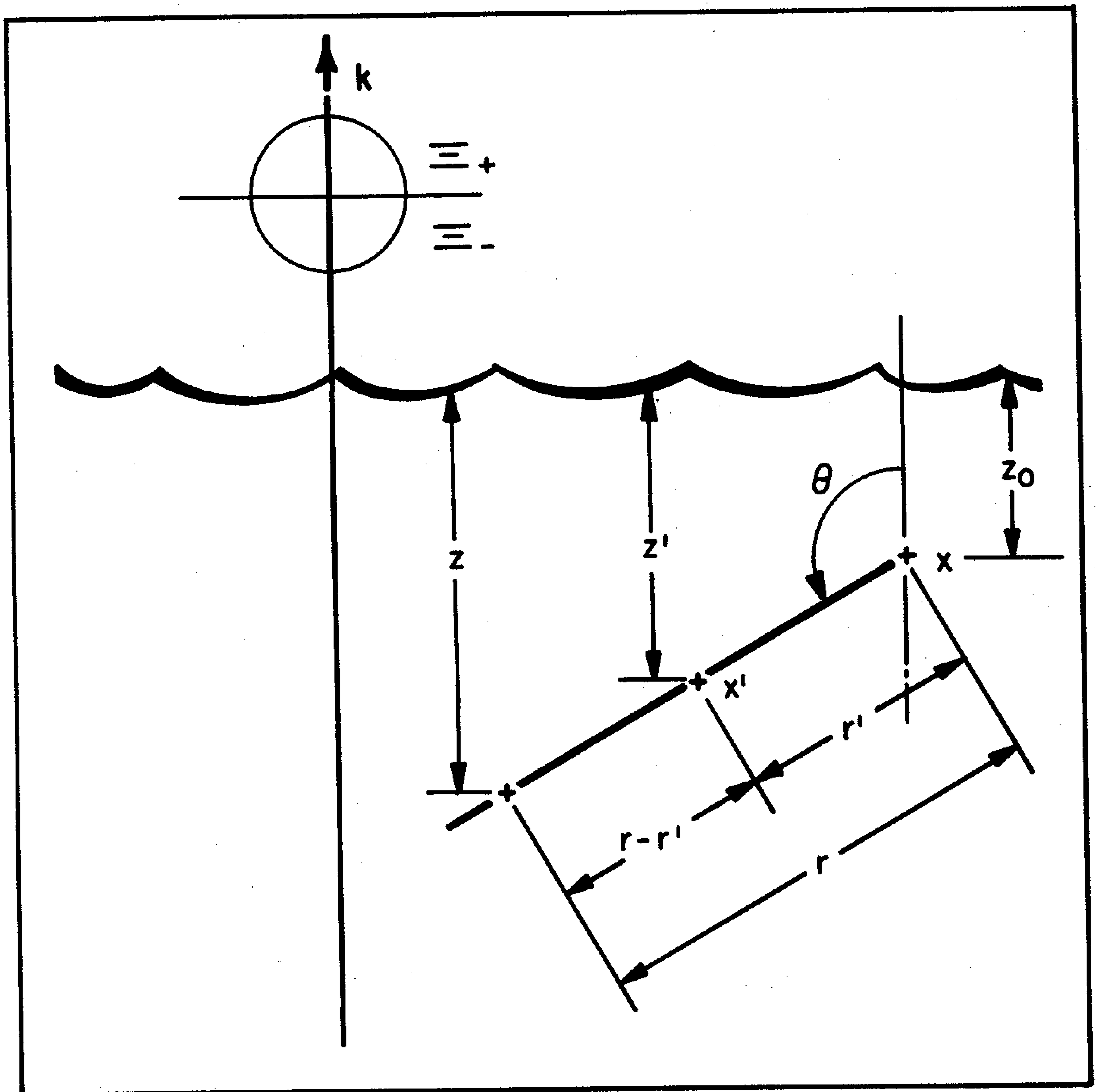

FIG. 4.1 Setting for the derivation of the classical canonical equation for radiance.

Since $N_*(x',\xi)$ depends only on the depth z' of the point x' along $\mathcal{P}_r(x,\xi)$, we may drop references to x and y coordinates and need only relate the variable of integration r' with z' using the relation:

$$z' = z_o - r' \cos\theta$$

so that:

$$dz' = -\cos\theta \, dr' \quad .$$

The equation for $N_r(z,\xi)$ with "z" denoting depth, then becomes:

$$N_r(z,\xi) = N_o(z_o,\xi)e^{-\alpha r} + \int_0^r N_*(z_o,\xi)e^{-K(z'-z_o)}e^{-\alpha(r-r')}\, dr'$$

That is:

$$N_r(z,\xi) = N_o(z_o,\xi)e^{-\alpha r} + N_*(z_o,\xi)\int_0^r \exp\{Kr'\cos\theta - \alpha(r-r')\}dr'$$

$$= N_o(z_o,\xi)e^{-\alpha r} + N_*(z_o,\xi)e^{-\alpha r}\int_0^r \exp\{(\alpha + K\cos\theta)r'\}\, dr' .$$

Hence:

$$N_r(z,\xi) = N_o(z_o,\xi)e^{-\alpha r} + \frac{N_*(z_o,\xi)^{-\alpha r}}{\alpha + K\cos\theta}\left[\exp\{(\alpha + K\cos\theta)r\} - 1\right]$$ Errata

Using (1) once again and the connection between z and r along $\mathcal{P}_r(x,\xi)$, we have:

$$\boxed{N_r(z,\xi) = N_o(z,\xi)e^{-\alpha r} + \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\left[1 - e^{-(\alpha + K\cos\theta)r}\right]} \quad (2)$$ Errata

which is the desired form of the *classical canonical equation for $N_r(z,\xi)$.*

We now make several observations on the structure of (2). First of all, (2) is a proper generalization of equations (1) of Secs. 4.1 and 4.2, and of Koschmieder's equation in Sec. 4.3, reducing to the latter either when $K \neq 0$ and $\theta = \pi/2$, or when $K = 0$ and θ arbitrary. In all real natural hydrosols, $K \neq 0$ so that Koschmieder's equation holds in natural hydrosols only when $\theta = \pi/2$. In the atmosphere on relatively clear days, $K = 0$ (very nearly) over relatively long horizontal or inclined paths, and so Koschmieder's equation holds over relatively extensive regions in the atmosphere (cf., Ref. [71]).

As a second observation, we note that the main use of (2) is to predict the apparent radiance N_r of given objects in natural optical media when α, K and N_o are known or estimable. Furthermore, (2) yields a useful estimate of the path radiance N_r^* generated over a path of sight in an optical medium, that is,

$$\boxed{N_r^*(z,\xi) = \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\left[1 - e^{-(\alpha + K\cos\theta)r}\right]} \quad . \quad (3)$$

If we now write:

$$\text{"}N_q(z,\xi)\text{"} \quad \text{for} \quad N_*(z,\xi)/\alpha(z,\xi) \quad , \qquad (4)$$

which is a straightforward generalization of the equilibrium radiance defined in (2) of 4.3, (3) may then be rendered in the form:

$$N_r^*(z,\xi) = \frac{N_q(z,\xi)}{1 + \frac{K}{\alpha}\cos\theta}\,(1 - e^{-(\alpha + K\cos\theta)r}) \qquad (5)$$

This shows that the equilibrium radiance $N_q(z,\xi)$ is observable only for infinitely long *horizontal* paths in natural hydrosols. For other paths, N_q contributes to the observable quantity N_r^* in the manner shown in (5) but itself is not directly observable.

As a third observation imagine a descent into a deep hydrosol, such as a deep lake or part of the ocean. Let $N_o(z,\xi)$ be the inherent radiance of the air-water boundary for directions ξ in Ξ_-, and $N_o(z_1,\xi)$ be the inherent radiance of the lower boundary of the medium for directions ξ in Ξ_+. Then when the optical distance αr to the boundaries becomes relatively large, $e^{-\alpha r}$ becomes relatively small. Under such conditions $N_r(z,\xi)$ is expressed essentially in the form (5), with the exponential term in (5) also negligible. Hence, at relatively great depths in deep natural hydrosols we have essentially:

$$N(z,\xi) = \frac{N_q(z,\xi)}{1 + \frac{K}{\alpha}\cos\theta} \qquad (6)$$

where "r" has now been dropped from the notation as being inessential. Thus the radiance distribution $N(z,\xi)$ at relatively great depths z is basically an ellipsoid of revolution with vertical axis and with eccentricity $\varepsilon = K/\alpha$, which is modified, as shown in (6), by the equilibrium radiance distribution $N_q(z,\xi)$ at the same depth.

There is a special class of homogeneous optical media for which (6) reduces to precisely the ellipsoid of revolution of eccentricity ε, namely media for which $\sigma(z,\xi';\xi)$ is independent of ξ' and ξ. For such media we have from the definition (3) of Sec. 4.2:

$$\sigma(z,\xi',\xi) = s/4\pi \quad , \qquad (7)$$

so that from (8) of Sec. 3.14:

$$N_*(z,\xi) = s\,h(z)/4\pi \quad , \qquad (8)$$

where h(z) is the scalar irradiance induced by $N(z,\xi)$ (Sec. 2.7).

If we write:

$$\text{"}\rho\text{" for } s/\alpha \quad ,$$

which is the *albedo for single scattering, or scattering-attenuation ratio,* then (6) becomes:

$$N(z,\xi) = \frac{\rho h(z)}{4\pi(1 + \frac{K}{\alpha}\cos\theta)} \qquad (9)$$

It is quite clear from (8) that $N_*(z,\xi)$ is independent of ξ, and that:

$$h(z) = h(z_0)e^{-K(z-z_0)} \tag{10}$$

From this we see that there is in $N(z,\xi)$ a multiplicative uncoupling of depth (z) and directional (θ or ξ) parameters and that scalar irradiance and path function values both decrease exponentially with depth and at equal rates. This multiplicative uncoupling of z and ξ can be represented as a product of a function of z only and a function of ξ only; it is of far-reaching importance in the general theory of solutions of the equation of transfer. (See Sec. 6.6.) Furthermore, we shall return to (6) and to (9) once again in Sec. 10.5, when the problem of the asymptotic radiance distribution at great depths is examined in a more rigorous fashion.

The preceding observations point up the versatility of the classical canonical form of the equation of transfer and suggest that of all the various equations encountered in practice, (2) is perhaps the most handy and succinct rule of thumb on natural light field behavior to carry around in one's memory. To add to the evidence of the utility of (2) we now deduce from it two further features of natural light fields.

First, we may ask: What is the behavior of path radi- $N_r^*(z,\xi)$ for very short paths of sight? This question directs attention to a situation which complements that centered around (6). Now from elementary calculus it is at once clear that:

$$1-e^{-(\alpha + K\cos\theta)r} = (\alpha + K\cos\theta)r + o(r)$$

where $o(r)$ is a function such $\lim_{r\to 0} o(r)/r = 0$, so that for small r, $o(r)$ is an infinitesimal of order higher than r. Therefore (3) reduces, within first order terms in r, to:

$$\boxed{N_r^*(z,\xi) = N_*(z,\xi)r} \tag{11}$$

Hence the answer to the question posed above is that for short paths of light $N_r^*(z,\xi)$ varies linearly with r, the proportionality factor being $N_*(z,\xi)$.

Finally, we may ask: What is the structure of the apparent radiance distribution near the air-water boundary, i.e., for very shallow depths? This query rounds out the complementary situation to that in (6) which describes the light field at relatively great depths. We take a simple case to illustrate the manner in which such questions may be answered using the canonical equation for radiance. Suppose the sky above the natural hydrosol is a deep blue and the sun is the only bright source of light in the azure hemisphere. Let attention be directed at a relatively dark point of sky away from the sun's disc. Hence the radiance $N_0(0,\xi)$ (with ξ in Ξ_-) from that portion of the sky as seen from just below the surface is very small compared to the sun's radiance.

Now keeping ξ fixed, let depth z increase. If the term $N_o(0,\xi)e^{-\alpha r}$ is negligible, as we now wish it to be, then $N_r(z,\xi)$ is given essentially by $N_r^*(z,\xi)$. For small depths z (and hence small path lengths r), $N_r^*(z,\xi)$ is essentially 0. As z increases through small depths, $N_r^*(z,\xi)$ builds up linearly in magnitude according to (11). For still further increases in z, $N_r^*(z,\xi)$ eventually levels off, reaches a maximum, and then subsequently plunges toward zero exponentially with rate K as $z\to\infty$. All this information is read off during an inspection of (3). We can obtain an estimate of the depth z at which the maximum path radiance is reached. Thus, from elementary calculus we find the maximum of $N_r^*(z,\xi)$ with respect to z by holding ξ fixed, differentiating with respect for z, and setting the derivative equal to zero. First, recalling that:

$$\frac{dr}{dz} = -\sec\theta \quad ,$$

we then use (3) to differentiate $N_r^*(z,\xi)$:

$$\begin{aligned}\frac{dN_r^*(z,\xi)}{dz} &= \frac{dN_*(z,\xi)/dz}{\alpha + K\cos\theta}\left[1-e^{-(\alpha + K\cos\theta)r}\right] + \\ &+ \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\,\frac{d}{dz}\left[1-e^{-(\alpha + K\cos\theta)r}\right] \\ &= \frac{-KN_*(z,\xi)}{\alpha + K\cos\theta}\left[1-e^{-(\alpha + K\cos\theta)r}\right] \\ &+ \frac{N_*(z,\xi)}{\alpha + K\cos\theta}\cdot(\alpha + K\cos\theta)e^{-(\alpha + K\cos\theta)r}\cdot\frac{dr}{dz} \quad .\end{aligned}$$

Setting the derivative to zero, and solving for z_m, the value of z which maximizes $N_r^*(z,\xi)$, we have:

$$\boxed{z_m = \frac{-\ln(-\varepsilon/\sec\theta)}{\alpha(\varepsilon + \sec\theta)}} \qquad (12)$$

where "ε" is again written for K/α.

Still further, more realistic models can be constructed for the radiance patterns at shallow depths in natural waters using similar procedures but now based on the full form of the classical canonical equation (2); the explorations of such models and that of (12) are still in their early stages of development and are left to interested students of the subject. Figure 4.2, taken from [298], depicts a comparison plot between some computed values of $N_r(z,\xi)$ (solid curve) using (2) with actual observed radiances and path function values at the surface obtained in a real situation, and thereby illustrates graphically the predictive power of the simple model of natural light fields summarized in (2). Observe in particular the reasonably good agreement between the predicted and observed value of the depth z_m at which maximum radiance occurs.

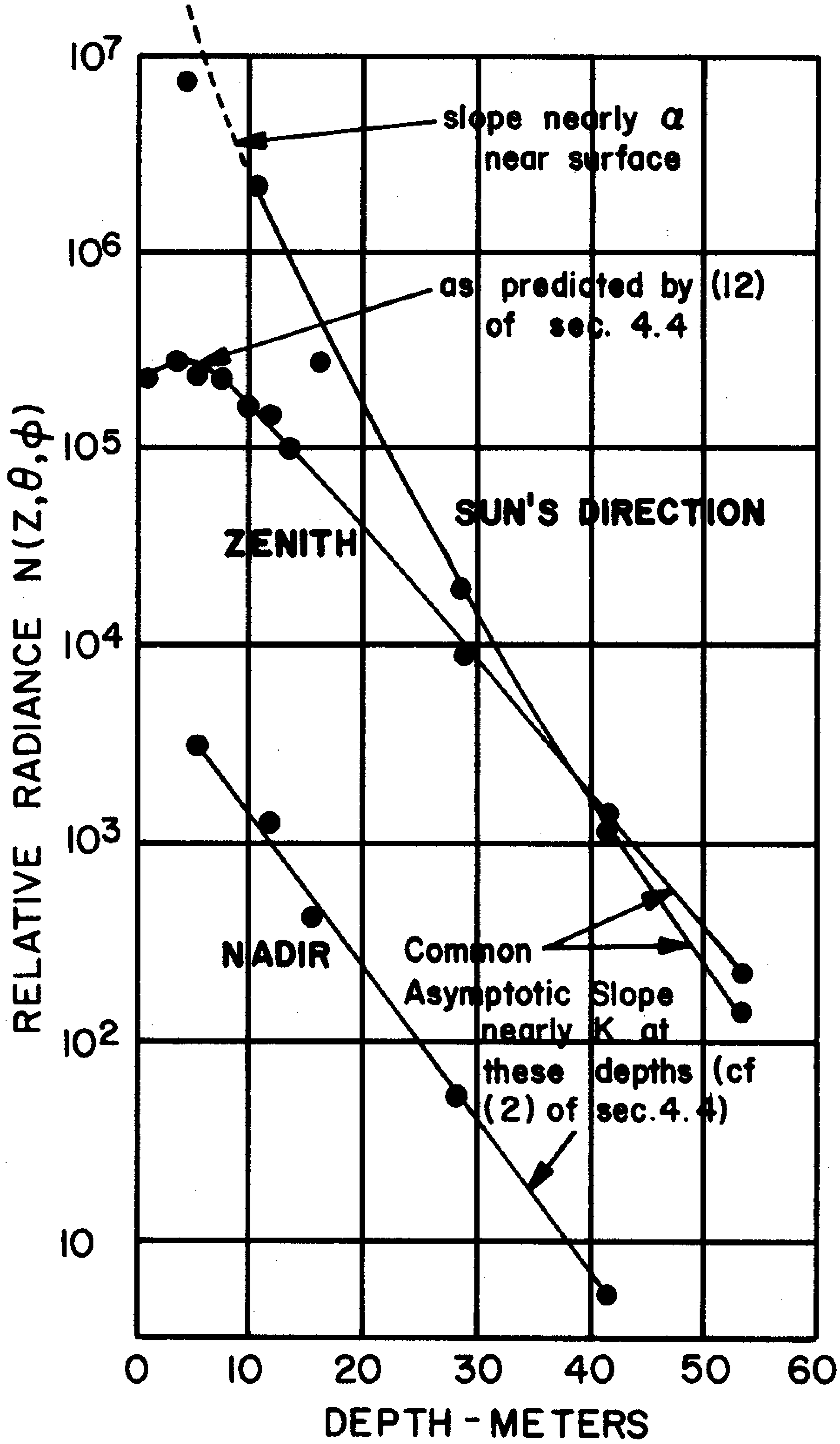

FIG. 4.2 Tyler's experimental verification (dots) of the curves predicted by the classical canonical equation for radiance.

Further models beyond (2) may now be explored by letting K vary in various known ways with depth, so that slightly more general forms than (1) are the starting points for the integration of the equation of transfer. In view of the fact that N_* generally behaves very nearly in an exponential manner with depth, these departures of K from constancy need only be very slight to cover most real situations. The basis for these generalizations is given in (19) of Sec. 4.5.