4.5 The General Canonical Equation for Radiance

The purpose of this section is to draw attention to a general pattern discernible in the various expressions, derived in the foregoing sections, for the apparent radiance N_r which is the logical common denominator of the large collection of analytic expressions for N_r which occur in the everyday studies of atmospheric and hydrologic optics. No specific or general problems of applied radiative transfer theory are intended to be solved for the moment, and no new numerical methods are expected to be immediately forthcoming. We seek instead to determine a general equation which will unify and hold within its form, as special cases, the various ways of correctly representing the apparent radiances of both near and distance parts of one's radiometric environment. In short, we extract from the examples discussed above and others in the literature, the *general canonical representation* of the apparent radiance function which will hold for all cases occurring in geophysical optics.

The key concept leading to the formulation of the appropriate canonical representation of apparent radiance turns out to be that of a generalized form of *radiance transmittance* associated with a path of sight $\mathcal{P}_r(x,\xi)$ in an optical medium. This concept is suggested after a study of the integral representation of the beam transmittance $T_r(x,\xi)$ associated with $\mathcal{P}_r(x,\xi)$, as given in (3) of Sec. 3.11, while keeping in mind the basic property of $T_r(x,\xi)$ as summarized in (4) of Sec. 3.10, that is, the fact that $T_r(x,\xi)$ is the ratio of the beam-transmitted radiance $N_r^o(z,\xi)$ to the initial inherent radiance $N_o(x,\xi)$ over a path $\mathcal{P}_r(x,\xi)$. Suppose now we take the ratio of $N_r(z,\xi)$ to $N_o(x,\xi)$, i.e., of the apparent radiance to the inherent radiance over the path $\mathcal{P}_r(x,\xi)$. Let us call this ratio the *radiance transmittance* of the path $\mathcal{P}_r(x,\xi)$. It is quite evident that the beam transmittance and the radiance transmittance of a given path are generally two distinct numbers. We now ask: Can the radiance transmittance just defined be given an integral representation analogous to that for beam transmittance? For, if so, then it is quite a simple matter to construct the appropriate generalization of (2) of Sec. 4.4 without the encumbrance of special restrictive assumptions of the kind in (1) of Sec. 4.4 which, while justifiable in many useful contexts, distract from the mathematical elegance and physical completeness of the resultant canonical representation of $N_r(z,\xi)$.

The requisite integral representation of the radiance transmittance is readily obtained by building an analogy on the fact that α, the key function in the integral representation of $T_r(z,\xi)$, is the logarithmic derivative of $N_r^o(z,\xi)$ along the path. This observation is based on (3) of Sec. 3.10 and (2) of Sec. 3.11. Some preliminary experimentation leads to the following definition of the appropriate analogue of α required in the present discussion. Thus let us write:

$$\boxed{\text{"}K\text{" for } -\nabla N/N} \quad . \qquad (1)$$

Here ∇ is the spatial gradient operator and N is a general radiance function defined and differentiable in a region

X such that N does not vanish in X. If $\mathcal{P}_r(x,\xi)$ is a path in X, and N_0 and N_r are the radiances along the path at points x and $x + r\xi$, respectively, then it is a simple exercise in calculus to show that, under the preceding conditions on N,

$$\boxed{N_r/N_o = \exp\left\{-\int_0^r \xi \cdot \mathbf{K}\, dr'\right\}} \tag{2}$$

where the integration is along the path $\mathcal{P}_r(x,\xi)$. We shall call $\mathbf{K}$ the *general K-function* for radiance; it is a most useful concept not only in the present discussion, but in many practical investigations of light in natural media. By means of $\mathbf{K}$ we can cast the equation of transfer (3) of Sec. 3.15 into *canonical form* as follows: Since d/dr is the direction derivative operation along the path,

$$\frac{dN(z,\xi)}{dr} = \xi \cdot \nabla N(z,\xi) \quad , \tag{3}$$

and we have:

$$\xi \cdot \nabla N(z,\xi) = -\ \xi \cdot \mathbf{K}(z,\xi)N(z,\xi) \tag{4}$$

by (1). From this we see that an immediate effect of the introduction of $\mathbf{K}$ is to replace the differential operation occurring in the equation of transfer by an ostensibly algebraic operation. The effect of this replacement on the equation of transfer is striking, as may be seen by writing (3) of Sec. 3.15 in abbreviated form:

$$\xi \cdot \nabla N = -\ \alpha N + N_*$$

and using (1), the equation becomes:

$$-\ \xi \cdot \mathbf{K}N = -\ \alpha N + N_*$$

which, upon solving for N, becomes:

$$N = \frac{N_*}{\alpha - \xi \cdot \mathbf{K}}$$

or in more detailed notations:

$$\boxed{N(z,\xi) = \frac{N_*(z,\xi)}{\alpha(z,\xi) - \xi \cdot \mathbf{K}(z,\xi)}} \tag{5}$$

Equation (5) is the *general canonical form of the equation of transfer*. It forms a key step in the derivations of the present section, and will also be used later in our studies of optical properties of natural hydrosols (Sec. 9.5). But for the present the reader should compare (5) above with (6) of 4.4 and note the close resemblance between that earlier approximate formula and the present exact formula (5).

Canonical Representation of Apparent Radiance

We can turn now to the details of the derivation of the requisite canonical representation of apparent radiance. The derivation will be facilitated if we adopt the following notation. We write:

$$\text{"}T_r[f]\text{" for } \exp\left\{\int_0^r f\,dr'\right\} \tag{6}$$

for every admissible function f on $\mathcal{P}_r(x,\xi)$, i.e., for every f defined and integrable, over a path $\mathcal{P}_r(x,\xi)$ of an optical medium X. In this notation, beam transmittance becomes:

$$T_r(x,\xi) = T_r[-\alpha] \tag{7}$$

and radiance transmittance becomes $T_r[-\xi\cdot\mathbf{K}]$. Observe that if f and g are two admissible functions on $\mathcal{P}_r(x,\xi)$, then:

$$\left.\begin{aligned} &T_r[f + g] = T_r[f]T_r[g] \\ &\text{and that:} \\ &(T_r[f])^{-1} \equiv T_r^{-1}[f] = T_r[-f] \quad . \end{aligned}\right\} \tag{8}$$

Henceforth we shall assume that α and $\xi\cdot\mathbf{K}$ are admissible on each path.

We begin the general derivation with (5) of Sec. 3.13:

$$N_r = N_r^o + N_r^* \tag{9}$$

which is the general representation of apparent radiance in decomposed form, i.e., in terms of beam transmitted inherent radiance N_r^o and path radiance N_r^* on a path $\mathcal{P}_r(x,\xi)$.

We use (9) to suggest the construction of the following identity:

$$N_r = N_r^o + \left[N_r - N_r^o\right] \tag{10}$$

which of course has no physical content, and is logically equivalent to the statement:

$$0 = 0 \quad . \tag{11}$$

However, we next observe that:

$$N_r^o = N_o T_r[-\alpha] \tag{12}$$

and that:

$$N_r = N_o T_r[-\xi\cdot\mathbf{K}] \tag{13}$$

and with these observations, (10) is transformed with the help of (8) into:

$$N_r = N_o T_r[-\alpha] + N_r\left(1 - T_r[-\alpha]T_r^{-1}[-\xi\cdot\mathbf{K}]\right)$$

$$= N_o T_r[-\alpha] + N_r\left(1 - T_r[-(\alpha - \xi\cdot\mathbf{K})]\right) \tag{14}$$

Even though (14) is entirely devoid of physical meaning, and even though it is logically equivalent to (11), it nevertheless seems to be on the verge of saying something *physically* significant by virtue of the fact that it has the general form of (2) of Sec. 4.4. At this point the canonical form (5) of the equation of transfer makes its entrance. By using (5) to replace N_r on the right side of (14), life is breathed, so to speak, into the cold symbolic clay of (14) and we obtain:

$$\boxed{N_r = N_o T_r[-\alpha] + \frac{N_*}{\alpha \;\; \xi\cdot\mathbf{K}}\left(1 - T_r[-(\alpha - \xi\cdot\mathbf{K})]\right)} \quad \text{Errata} \tag{15}$$

This is the desired *general form of the canonical representation of apparent radiance* N_r over a path $\mathcal{P}_r(x,\xi)$. The radiance N_r in (15) is no longer arbitrary and free as in (14); now N_r in (15) is indissolubly locked to the optical properties of the medium via the equation of transfer. Equation (15) is the most general form of (2) of Sec. 4.4 attainable for unpolarized steady radiance functions in a general source-free optical medium. The quantity $T_r[-\xi\cdot\mathbf{K}]$ in (13) is called the *radiance transmittance* associated with $\mathcal{P}_r(x,\xi)$. It will be studied further, along with related transmittance concepts, in Sec. 9.5.

The Canonical Form for Stratified Media

As an application of (15) we now derive the appropriate instance of the equation in an arbitrary stratified natural hydrosol. The result will be a canonical representation for N_r about midway in generality between (2) of Sec. 4.4 and (15) above. We shall use without further explanation the terrestrially based coordinate system for hydrologic optics described in Sec. 2.4. (See Fig. 4.1.)

The reduction of (15) begins with the observation that from (1) we have in general:

$$\mathbf{K} = \mathbf{i}I + \mathbf{j}J + \mathbf{k}K \tag{16}$$

where we have written:

$$\text{"I"} \quad \text{for} \quad -\frac{1}{N}\frac{\partial N}{\partial x}$$

$$\text{"J"} \quad \text{for} \quad -\frac{1}{N}\frac{\partial N}{\partial y}$$

$$\text{"K"} \quad \text{for} \quad -\frac{1}{N}\frac{\partial N}{\partial z}$$

and where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors for a right-hand Cartesian coordinate system. For the particular coordinate system of hydrologic optics (Fig. 4.1) we must replace (16) by:

$$\mathbf{K} = \mathbf{i}I + \mathbf{j}J - \mathbf{k}K \tag{17}$$

For a stratified plane parallel medium all radiometric and optical functions are independent of x and y. Hence the x and y derivatives I and J above are zero, and so:

$$\xi \cdot \mathbf{K} = -\xi\cdot\mathbf{k}K = -K\cos\theta \tag{18}$$

and (15) becomes

$$\boxed{N_r = N_o T_r[-\alpha] + \frac{N_*}{\alpha + K\cos\theta}\left(1 - T_r[-(\alpha + K\cos\theta)]\right)} \tag{19}$$

This equation is exact and completely general for plane parallel media; α and K have general depth and direction dependence. Other than the stratification condition summarized in (18) and the current choice of coordinates summarized in (17), the canonical equation (19) holds for completely arbitrary lighting conditions and optical properties in a plane-parallel optical medium. In particular it should be noted that the function K in (19) may, according to (1) and (17), be defined within the plane-parallel context directly by writing:

$$\text{"}K(z,\xi)\text{"} \quad \text{for} \quad \frac{-1}{N(z,\xi)}\frac{dN(z,\xi)}{dz} \quad . \tag{20}$$

This is an operational definition of $K(z,\xi)$ using directly observable radiances $N(z,\xi)$; and so K, as it occurs in (19), is quite general in the plane-parallel setting. We shall study the depth behavior of $K(z,\xi)$ in some detail in Secs. 10.5 and 10.6. The reader should particularly note that (20) may serve as an *operational* definition of K in stratified plane parallel media. The associated canonical form of the equation of transfer is:

$$\boxed{N(z,\xi) = \frac{N_*(z,\xi)}{\alpha(x) + K(z,\xi)\cos\theta}} \tag{21}$$

Equation (19) reduces to (2) of Sec. 4.4 upon requiring α and K to be independent of depth z in the hydrosol. For then:

$$T_r[-\alpha] = \exp\{-\alpha r\}$$

$$T_r[-(\alpha + K \cos\theta)] = \exp\{-(\alpha + K \cos\theta)r\}$$

This points up one of the primary reasons for using the logarithmic derivative in (1) for the definition of **K**. In most natural hydrosols all radiometric quantities (radiance, path function, irradiance, etc.) have potentially constant logarithmic derivatives with respect to depth. Indeed, in Secs. 7.9, 7.10, and 7.11, it will be shown that this fact holds for quite wide geometrical and physical settings. This observation suggests further models of natural light fields that may be derived from (19). For by postulating a certain depth dependence of K suggested by experiment or theory (these are usually relatively mild dependences) and placing that depth dependence in (19), new models of N_T and N_T^* can be obtained which will fall somewhere between (2) of Sec. 4.4 and (19) as regards tractability in computation and fidelity of description of light fields.

<< Table of Contents << Title Page >> Next Page >>

4.6 Canonical Representation of Polarized Radiance

In this section we shall extend the notion of the canonical representation of apparent radiance to the polarized context. One consequence will be a representation of polarized radiance distributions in stratified natural hydrosols comparable in simplicity and utility to the scalar equation (2) of Sec. 4.4. The resultant polarized canonical form also suggests some interesting experimental programs that may be performed for polarized light fields in natural hydrosols. These will be briefly outlined at the conclusion of the section.

In order to establish the polarized version of (15) of Sec. 4.5, it seems natural to try to repeat the constructions between (1) and (15) of Sec. 4.5, now for each of the four components ${}_iN$ of the polarized observable radiance vector **N** (Sec. 2.10). Thus let us write:

$$\text{"}\mathbf{K}_i\text{" for } -\nabla_i N / {}_i N \qquad (1)$$

for each component ${}_iN$ of **N**, i = 1,2,3,4, and let us write (7) of Sec. 3.15 as:

$$\xi \cdot \nabla\mathbf{N} = -\alpha\mathbf{N} + \mathbf{N}_* \qquad (2)$$

where we have written:

$$\text{"}\mathbf{N}_*\text{" for } \int_\Xi \mathbf{N}\,\mathbf{p}\,d\Omega \qquad (3)$$

where **p** is the standard observable volume scattering matrix. All that we need know about the standard observable volume scattering matrix **p** in the present derivation is that it is a 4 by 4 matrix with entry p_{ij} in the ith row and jth column.