

$$T_r[-\alpha] = \exp \{-\alpha r\}$$

$$T_r[-(\alpha + K \cos \theta)] = \exp \{-(\alpha + K \cos \theta)r\}$$

This points up one of the primary reasons for using the logarithmic derivative in (1) for the definition of K . In most natural hydrosols all radiometric quantities (radiance, path function, irradiance, etc.) have potentially constant logarithmic derivatives with respect to depth. Indeed, in Secs. 7.9, 7.10, and 7.11, it will be shown that this fact holds for quite wide geometrical and physical settings. This observation suggests further models of natural light fields that may be derived from (19). For by postulating a certain depth dependence of K suggested by experiment or theory (these are usually relatively mild dependences) and placing that depth dependence in (19), new models of N_r and $N_\#$ can be obtained which will fall somewhere between (2) of Sec. 4.4 and (19) as regards tractability in computation and fidelity of description of light fields.

4.6 Canonical Representation of Polarized Radiance

In this section we shall extend the notion of the canonical representation of apparent radiance to the polarized context. One consequence will be a representation of polarized radiance distributions in stratified natural hydrosols comparable in simplicity and utility to the scalar equation (2) of Sec. 4.4. The resultant polarized canonical form also suggests some interesting experimental programs that may be performed for polarized light fields in natural hydrosols. These will be briefly outlined at the conclusion of the section.

In order to establish the polarized version of (15) of Sec. 4.5, it seems natural to try to repeat the constructions between (1) and (15) of Sec. 4.5, now for each of the four components ${}_iN$ of the polarized observable radiance vector N (Sec. 2.10). Thus let us write:

$$"K_i" \text{ for } -\nabla_i N / {}_iN \quad (1)$$

for each component ${}_iN$ of N , $i = 1, 2, 3, 4$, and let us write (7) of Sec. 3.15 as:

$$\xi \cdot \nabla N = -\alpha N + N_\# \quad (2)$$

where we have written:

$$"N_\#" \text{ for } \int_{\Xi} N p \, d\Omega \quad (3)$$

where p is the standard observable volume scattering matrix. All that we need know about the standard observable volume scattering matrix p in the present derivation is that it is a 4 by 4 matrix with entry p_{ij} in the i th row and j th column.

In particular the problem of how the p_{ij} are obtained in principle or in practice is immaterial for the present derivation, since we are concerned only with the mathematical process of constructing the vector counterpart to (15) of Sec. 4.5. The matrix p is defined and discussed in detail in Sec. 112 of Ref. [251].

The canonical equation of transfer in the scalar context now becomes four coupled scalar equations in the polarized context as follows. We first write:

$$"p_i" \text{ for } (p_{1i}, p_{2i}, p_{3i}, p_{4i}) \quad (4)$$

Next we read off the i th component of (2), $i = 1, 2, 3, 4$:

$$\xi \cdot \nabla_i N = -\alpha_i N + {}_i N_* \quad (5)$$

where we have written:

$${}_i N_* \text{ for } \int_E \mathbf{N} \cdot \mathbf{p}_i d\Omega \quad (6)$$

It follows from (3) and (6) that:

$$\mathbf{N}_* = ({}_1 N_*, {}_2 N_*, {}_3 N_*, {}_4 N_*) \quad (7)$$

Using (1) in (5) and solving the result for ${}_i N$:

$$\boxed{{}_i N = \frac{{}_i N_*}{\alpha - \xi \cdot \mathbf{K}_i}} \quad (8)$$

This is the canonical equation of transfer for polarized radiance, which holds for each $i = 1, 2, 3, 4$. Continuing as in Sec. 4.5, we deduce for $i = 1, 2, 3, 4$:

$${}_i N_r / {}_i N_o = \exp \left\{ - \int_0^r \xi \cdot \mathbf{K}_i dr' \right\} \quad (9)$$

which is the vector component counterpart to (2) of Sec. 4.5. Applying the notation " $T_r[f]$ " of Sec. 4.5 to the present context, (9) may be written:

$${}_i N_r = {}_i N_o T_r[-\xi \cdot \mathbf{K}_i] \quad (10)$$

and we observe that:

$${}_i N_r^0 = {}_i N_o T_r[-\alpha] \quad (11)$$

It now follows readily that for every $i = 1, 2, 3, 4$:

$$\boxed{{}_i N_r = {}_i N_o T_r[-\alpha] + \frac{{}_i N_*}{\alpha - \xi \cdot \mathbf{K}_i} (1 - T_r[-(\alpha - \xi \cdot \mathbf{K}_i)])} \quad (12)$$

which is the desired *canonical representation of polarized radiance*.

The set of four equations (12) are coupled by means of the terms ${}_i N_*$. For example, the representation for ${}_1 N_r$ uses ${}_1 N_*$ where

$${}_1 N_* = \int_{\Xi} [{}_1 N_{p11} + {}_2 N_{p21} + {}_3 N_{p31} + {}_4 N_{p41}] d\Omega$$

A Simple Model for Polarized Light Fields

We now give some attention to the construction of a simple model for polarized light fields in stratified natural hydrosols, the constructions being guided by the successful scalar prototype in Sec. 4.4. In the scalar case, the effective step was to assume that there was a nonnegative number K , less than α , such that:

$$N_*(z, \xi) = N_*(z_0, \xi) e^{-K(z-z_0)} \quad (13)$$

This suggests that we take each ${}_i N_*$, $i = 1, 2, 3, 4$, which by (6) has the form:

$${}_i N_* = \int_{\Xi} ({}_1 N_{p1i} + {}_2 N_{p2i} + {}_3 N_{p3i} + {}_4 N_{p4i}) d\Omega \quad (14)$$

and agree to write:

$${}_j N_* \text{ "for" } \int_{\Xi} {}_j N_{pji} d\Omega \quad (15)$$

so that ${}_i N_*$ will have the representation:

$${}_i N_* = {}_1 i N_* + {}_2 i N_* + {}_3 i N_* + {}_4 i N_* \quad (16)$$

Then, still being guided by the prototype (13) we agree to make the following assumption: the four nonnegative real numbers K_i , as defined in (1), are each less than α , and such that:

$$\boxed{{}_j i N_*(z, \xi) = {}_j i N_*(z_0, \xi) e^{-K_j(z-z_0)}} \quad (17)$$

for every $i, j = 1, 2, 3, 4$, where K_j now is the z -component of \mathbf{K}_j --the only nonzero component of \mathbf{K}_j by virtue of our current assumption about the stratification of the light field in natural waters. Under these assumptions, (12) reduces to:

$${}_i N_r(z, \xi) = {}_i N_o(z_o, \xi) e^{-\alpha r} + \frac{{}_i N_*(z, \xi)}{\alpha + K_i \cos \theta} \left[1 - e^{-(\alpha + K_i \cos \theta) r} \right] \quad (18)$$

for $i = 1, 2, 3, 4$, and where:

$$\begin{aligned} {}_i N_*(z, \xi) = & {}_1 i N_*(z_o, \xi) e^{-K_1(z-z_o)} \\ & + {}_2 i N_*(z_o, \xi) e^{-K_2(z-z_o)} \\ & + {}_3 i N_*(z_o, \xi) e^{-K_3(z-z_o)} \\ & + {}_4 i N_*(z_o, \xi) e^{-K_4(z-z_o)} \end{aligned}$$

or more compactly:

$${}_i N_*(z, \xi) = \sum_{j=1}^4 {}_j i N_*(z_o, \xi) e^{-K_j(z-z_o)} \quad (19)$$

Experimental Questions

The derivation of the canonical representation (18) for polarized radiance incorporated several assumptions which, even though suggested by the well-established scalar case of Sec. 4.4, require some critical examination before they are fully accepted. These assumptions in turn raise certain specific questions concerning the nature of polarized light fields in natural hydrosols in general and the nature of the K -functions in particular. We shall conclude the present section with a brief statement and discussion of these questions.

First of all, the definition of each K_i as given in (1), is a constructive definition and hence presents no difficulty in being translated into operational terms, so that actual experimental determinations of the K_i are possible in principle. These determinations should parallel very closely those already developed for the function K in (20) of Sec. 4.5, because K_i , as K , is a logarithmic derivative of a radiance function. The main difference between K_i and K is simply that each K_i is associated with the component of a vector valued function while K is associated with a scalar valued function. Thus with the extra attachments of wave plates and polarizers on the radiance meter required to measure the polarized radiance, one performs essentially those

operations with the radiance meter that one performs to find K , but now four times over for each K_i , $i = 1, 2, 3, 4$.

With the matter of the measurability of the K_i settled, at least in principle, we now ask the first question that comes to mind concerning the K_i : Is there some observable regular pattern in the individual depth-behavior and in the relative magnitudes of the four functions K_1, K_2, K_3, K_4 ? This is actually two questions in one, and it may be simpler to phrase them separately. The first question may be phrased: *Is there some observable regularity in the depth behavior of each K_i ?* The second question may then be rendered as: *Is there some observable regularity in the relative magnitudes of the K_i ?* As far as the first question is concerned, it is expected on simple physical grounds that the individual depth behavior of each K_i should follow very closely that of the scalar K defined in (20) of Sec. 4.5. In particular the depth behavior of the K_i at relatively great depths in homogeneous media should be quite regular and should follow the patterns discussed in Sec. 7.10 and Sec. 10.6 dealing with the asymptotic radiance theorem. Some attention to this question has been given by Lenoble [157]. The second question is more difficult to answer and, in view of the present state of development of the theory of polarized light fields in natural optical media, it appears likely that a definitive answer will be forthcoming first from experimental investigations. Nevertheless, it is interesting to speculate on the possible interrelations among the K_i . Thus, suppose that the K_i are all equal to a common value, then the set of equations in (18) assumes a particularly simple form. It follows that any differences between ${}_iN_r$ and ${}_jN_r$ will depend solely on the state of affairs between them at the surface of the medium. On the other hand, if there are two K_i 's which differ at all depths then the radiance component associated with the larger K_i will decay with depth more quickly than the other. As a result, those components of N with the smallest K_i 's will persist down to greater depths than the others with larger K_i 's. By contemplating these possibilities and by taking into account the known properties of the unpolarized light field, the general state of affairs for the functions K_i will most likely turn out as follows: Near the surface the K_i 's will differ, and there will be some permanent characteristic pattern of relative sizes discernible among them which is related to the state of the sea surface, and to the polarized state of the sky; however, the transmitted sky-polarization and under-surface reflection-induced pattern will eventually disappear with increasing depth in such a manner that in the limit, all the values K_i tend to a common value independent of the state of the sky's polarization, with an attendant asymptotic value of the polarization of the light field. This common value of the K_i 's will be that of the depth decay rate of scalar irradiance $h(z)$, which should be determined only by the inherent optical properties of the medium, just as in the scalar case. It remains to be seen how this conjecture is borne out by experimental studies. Our review of the experimental work of Ivanoff and Waterman in 1.2 shows some encouraging agreement in this direction.

While attention is directed toward the possible structure of the functions K_i at great depths and while conjectures

about the K_i are being made, it might be in place to add some further conjectures about the light field itself in addition to its depth-rates of decay K_i . When one imagines the natural light field at great depths one is led to picture a predominantly downward feeble flow of light, the radiance pattern being graphically depicted by an ellipsoid-like surface with vertical axis. If this light field is conceptually analyzed for polarization features, it seems--on intuitive grounds--that the radiance vector for vertical downward or upward flux should have the form $(1/2)(N, N, N, N)$, i.e., vertical downward or upward-radiance should be unpolarized. Furthermore, it seems that the horizontal radiance should be horizontally linearly polarized, i.e., have the form $(1/2) \times (0, 2N, N, N)$. This follows from the fact that the flow is predominantly vertical and beamlike (and of course very feeble) at great depths. Since natural light fields change continuously rather than abruptly in most macroscopic settings, we would expect the radiance vector components to vary continuously between these two extremes as the angle of the radiance direction varies from $\theta = 0$ (vertical upward), or π (vertical downward) to $\theta = \pi/2$ (horizontal). A simple model for this radiance $N(\theta)$ which comes readily to mind and which satisfies these conditions is:

$$N(\theta) = \frac{1}{2} (N \cos^2 \theta, N(1 + \sin^2 \theta), N, N)$$

where θ is measured from the zenith and N is the fixed reference radiance for $\sigma = 0$ at each depth. All these assertions are at this stage of our knowledge of course conjectural, being based on a modicum of physical experience with polarized radiance fields in natural waters, and are intended primarily to perform a heuristic service. It will be left to interested researchers to carry this matter to a more satisfactory state of affairs, both theoretically and experimentally. A possible theoretical approach can be based on the polarized version of (21) of Sec. 10.7, or on (29), (31) of Sec. 7.10. These approaches may show that the preceding conjecture must be modified to take into account the structure of the volume scattering matrix (cf. (24) of Sec. 13.6) of the medium.

4.7 Abstract Versions of Canonical Equations

The discussions of the present chapter have carried the notion of canonical radiance forms over a great conceptual distance, starting from the rudimentary canonical representation (1) of Sec. 4.0 discovered by Bouguer nearly two centuries ago and up to the representation in (12) of Sec. 4.6. Such a task could not have been carried out in the indicated manner without the convenient milestones in the development of the theory provided by early workers such as Schuster, Koschmieder, and others. It seems that the representations finally reached in Secs. 4.5 and 4.6 constitute the most general forms for radiance concepts attainable which are physically meaningful. Their basic forms remain essentially intact by allowing more general physical features to appear such as the time-dependent radiance terms