about the K_i are being made, it might be in place to add some further conjectures about the light field itself in addition to its depth-rates of decay K_i. When one imagines the natural light field at great depths one is led to picture a predominantly downward feeble flow of light, the radiance pattern being graphically depicted by an ellipsoid-like surface with vertical axis. If this light field is conceptually analyzed for polarization features, it seems--on intuitive grounds--that the radiance vector for vertical downward or upward flux should have the form (1/2)(N,N,N,N), i.e., vertical downward or upward-radiance should be unpolarized. Furthermore, it seems that the horizontal radiance should be horizontally linearly polarized, i.e., have the form (1/2) × (0,2N,N,N). This follows from the fact that the flow is predominantly vertical and beamlike (and of course very feeble) at great depths. Since natural light fields change continuously rather than abruptly in most macroscopic settings, we would expect the radiance vector components to vary continuously between these two extremes as the angle of the radiance direction varies from $\theta = 0$ (vertical upward), or π (vertical downward) to $\theta = \pi/2$ (horizontal). A simple model for this radiance $N(\theta)$ which comes readily to mind and which satisfies these conditions is:

$$N(\theta) = \frac{1}{2}\,(N \cos^2 \theta,\ N(1 + \sin^2 \theta),\ N,\ N)$$

where θ is measured from the zenith and N is the fixed references radiance for $\sigma = o$ at each depth. All these assertions are at this stage of our knowledge of course conjectural, being based on a modicum of physical experience with polarized radiance fields in natural waters, and are intended primarily to perform a heuristic service. It will be left to interested researchers to carry this matter to a more satisfactory state of affairs, both theoretically and experimentally. A possible theoretical approach can be based on the polarized version of (21) of Sec. 10.7, or on (29), (31) of Sec. 7.10. These approaches may show that the preceding conjecture must be modified to take into account the structure of the volume scattering matrix (cf. (24) of Sec. 13.6) of the medium.

4.7 Abstract Versions of Canonical Equations

The discussions of the present chapter have carried the notion of canonical radiance forms over a great conceptual distance, starting from the rudimentary canonical representation (1) of Sec. 4.0 discovered by Bouguer nearly two centuries ago and up to the representation in (12) of Sec. 4.6. Such a task could not have been carried out in the indicated manner without the convenient milestones in the development of the theory provided by early workers such as Schuster, Koschmieder, and others. It seems that the representations finally reached in Secs. 4.5 and 4.6 constitute the most general forms for radiance concepts attainable which are physically meaningful. Their basic forms remain essentially intact by allowing more general physical features to appear such as the time-dependent radiance terms and emission terms in the basic equation of transfer. In view of the apparent ubiquity of the canonical representation throughout the domain of pure and applied radiative transfer theory (e.g., see the canonical equations in Chapter 11) and in view of the seeming ease with which the equation of transfer is molded into its canonical form, we are led to inquire whether the notion of a canonical representation is indigenous only to radiative transfer theory or whether in our labors in this special field we have touched upon merely the shadow or projection, so to speak, of a more general analytic phenomenon in modern operator theory. It appears that the latter possibility is the case and we pause briefly here to sketch in outline the general mathematical setting in which the notion of the canonical representation appears to take a natural place.

Let L be a general (not necessarily linear) operator defined on a domain $\mathfrak{D}$ of functions such that for each function f in $\mathfrak{D}$ there is a function g in $\mathfrak{D}$ and a number λ such that:

$$\boxed{Lf = \lambda f + g} \tag{1}$$

This is the abstract counterpart to the equation of transfer with L replacing the derivative operation $\xi\cdot\nabla$, and g replacing N_*, and where f replaces N. The number λ is nonzero and may be real or complex and is evidently a replacement of $-\alpha$. Now let us write:

$$\text{"}f_q\text{"} \quad \text{for} \quad -g/\lambda \quad .$$

Then (1) can be written:

$$\boxed{Lf = \lambda(f-f_q)} \tag{2}$$

and this should be compared with (4) of Sec. 4.3. Hence f_q is the abstract vestige of equilibrium radiance, so that $Lf = 0$ if and only if $f = f_q$. Next write

$$\text{"}\kappa\text{"} \quad \text{for} \quad -Lf/f \tag{3}$$

so that κ is the abstract vestige of $\mathbf{K}$, and (1) becomes:

$$-\kappa f = \lambda f + g \quad .$$

Solving this for f:

$$\boxed{f = \frac{-g}{(\lambda + \kappa)}} \tag{4}$$

which is the requisite abstract canonical form of equation (1) associated with the operator L. An alternate form of (4) is obtained by using f_q:

$$\boxed{f = \frac{f_q}{1 + (\kappa/\alpha)}} \quad . \tag{5}$$

This basic form is applicable to all manners of radiometric concepts and optical properties. See, e.g., the various specific forms of (5) appearing throughout Chapter 11.

The abstract version of the canonical representation of f now follows readily from (4) or (5) by emulating (10) of Sec. 4.5. Now that a decomposition of f into "reduced" and "diffuse" may not be natural, we simply represent f by the identity:

$$f = fT + f(1-T) \quad , \tag{6}$$

where T is any suitable operator on $\mathfrak{D}$ and "1" denotes the identity transformation on $\mathfrak{D}$. Then using (4), this becomes:

$$\boxed{f = fT + \frac{g}{(\lambda + \kappa)} (T-1)} \quad , \tag{7}$$

which is an abstract canonical representation of f with respect to the operators T and L, via equation (1), and is to be compared to (15) of Sec. 4.5.

A more direct generalization of (15) of Sec. 4.5 (which retains the idea of "diffuse" and "reduced" components) follows upon replacing (6) by:

$$f = f_o + (f - f_o) \tag{8}$$

and defining two operators S and T such that there exists a function ϕ_o with the property that

$$f_o = \phi_o \, T \qquad \text{(cf. (12) of 4.5)} \tag{9}$$

$$f = \phi_o \, S \qquad \text{(cf. (13) of 4.5)}$$

With these definitions (8) becomes

$$f = \phi_o \, T + (\phi_o S - \phi_o T)$$

whence

$$f = \phi_o T + f(1-S^{-1} \, T) \quad . \tag{10}$$

Let us write

$$\text{"}\tau\text{"} \quad \text{for} \quad S^{-1} \, T$$

then we obtain,

$$f = \phi_o T + f(1-\tau) \tag{11}$$

which with (4) becomes:

$$\boxed{f = \phi_o T + \frac{-g}{(\lambda + \kappa)} (1-\tau)} \quad . \tag{12}$$

This is the requisite abstract version of (15) of Sec. 4.5, and the ultimate generalization of (1) of Sec. 4.0 to be attempted here. We say that (12) is the *canonical representation of f with respect to the operators L, T, S, via the equation (1)*. The operator τ turns out to be the abstract counterpart to the contrast transmittance function (Sec. 9.5).

By performing the preceding constructions of the abstract version of the canonical representation we gain a deep insight into the essential mathematical structure of the canonical representations in radiative transfer theory. Our constructions show us, in particular, that the essential physical kernel of (12) is bound up in the term $-g/(\lambda + \kappa)$, and that the overall general structure of (12), as given by (8) or (11), is a mere mathematical tautology. It seems somewhat noteworthy, therefore, that Bouguer, who discovered the first definitive trace of the canonical equation in the form (1) of Sec. 4.0, managed to light upon the essential form but yet with only partial realization of the significance of the two key physical terms a and b of the canonical form. The lessons of this chapter and hindsight now let us see that within the apparently insignificant term b, as it occurs in (1) of Sec. 4.0, resides not only the notion of equilibrium radiance, but actually the equation of transfer for radiance, the basic law of all of radiative transfer theory.

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4.8 Bibliographic Notes for Chapter 4

One of the earliest known instances of the canonical form of the equation of transfer was written down by Bouguer in his classical treatise on light, recently translated by Middleton [28]. The equation appears in essentially the form it is closest to the basic integral representation of the equation of transfer as given in (5) of Sec. 3.13. Soon after Schuster formulated his celebrated two-flow equations [279], Schwarzschild [281] in 1906 formulated an expression for what we now call "path radiance", and later, in 1914, Schwarzschild [282] incorporated it into an expression for radiance, which is essentially (6) of Sec. 3.13. The latter equation was our point of departure from which we deduced the classical form of the canonical equation, as given in (2) of Sec. 4.4.

It appears from a perusal of the literature that the canonical form of the equation of transfer, as embodied, say in (2) of Sec. 4.4, took its first definitive general form in [212] and [250] which in turn grew out of the hydrologic optics researches recorded in [82] and [5]. However, as noted in the introductory remarks, the canonical form in one