$$f = \phi_o T + f(1-\tau) \tag{11}$$

which with (4) becomes:

$$\boxed{f = \phi_o T + \frac{-g}{(\lambda + \kappa)} (1-\tau)} \quad . \tag{12}$$

This is the requisite abstract version of (15) of Sec. 4.5, and the ultimate generalization of (1) of Sec. 4.0 to be attempted here. We say that (12) is the *canonical representation of f with respect to the operators L, T, S, via the equation (1).* The operator τ turns out to be the abstract counterpart to the contrast transmittance function (Sec. 9.5).

By performing the preceding constructions of the abstract version of the canonical representation we gain a deep insight into the essential mathematical structure of the canonical representations in radiative transfer theory. Our constructions show us, in particular, that the essential physical kernel of (12) is bound up in the term $-g/(\lambda + \kappa)$, and that the overall general structure of (12), as given by (8) or (11), is a mere mathematical tautology. It seems somewhat noteworthy, therefore, that Bouguer, who discovered the first definitive trace of the canonical equation in the form (1) of Sec. 4.0, managed to light upon the essential form but yet with only partial realization of the significance of the two key physical terms a and b of the canonical form. The lessons of this chapter and hindsight now let us see that within the apparently insignificant term b, as it occurs in (1) of Sec. 4.0, resides not only the notion of equilibrium radiance, but actually the equation of transfer for radiance, the basic law of all of radiative transfer theory.

4.8 Bibliographic Notes for Chapter 4

One of the earliest known instances of the canonical form of the equation of transfer was written down by Bouguer in his classical treatise on light, recently translated by Middleton [28]. The equation appears in essentially the form it is closest to the basic integral representation of the equation of transfer as given in (5) of Sec. 3.13. Soon after Schuster formulated his celebrated two-flow equations [279], Schwarzschild [281] in 1906 formulated an expression for what we now call "path radiance", and later, in 1914, Schwarzschild [282] incorporated it into an expression for radiance, which is essentially (6) of Sec. 3.13. The latter equation was our point of departure from which we deduced the classical form of the canonical equation, as given in (2) of Sec. 4.4.

It appears from a perusal of the literature that the canonical form of the equation of transfer, as embodied, say in (2) of Sec. 4.4, took its first definitive general form in [212] and [250] which in turn grew out of the hydrologic optics researches recorded in [82] and [5]. However, as noted in the introductory remarks, the canonical form in one guise or other from (1) of Sec. 4.0 to (11) of Sec. 4.4 (and even (8) of Sec. 5.3) appears and reappears in the work of independent researchers over the years in diverse applications of radiative transfer. One outstanding early use of the canonical equation is in the work of Koschmieder [141]. The task of tracing the subsequent manifold reappearances of the canonical form is best left to historians of the subject. One source of references for such work is Middleton's treatise [177]. For our purposes it suffices to anchor the canonical equation's first ground-form in Schwarzschild's work [282], as noted above. It has been the intention of this chapter to clarify the canonical equation's logical and conceptual roles in the general theory of radiative transfer as outlined in Secs. 4.4, 4.5, and 4.7, and to extend it to the polarized context as in Sec. 4.6. For further discussions of underwater polarized light fields, see [117], [118], [108].