CHAPTER 5

NATURAL SOLUTIONS OF THE EQUATION OF TRANSFER

5.0 Introduction

The natural solution of the equation of transfer plays a fundamental and unique role in the theory of radiative transfer. The role is fundamental in the sense that the natural solution may be used in the systematic construction of the principles of invariance, the invariant imbedding principle, and all other instances of the interaction principle encountered in radiative transfer theory. This facet of the natural solution was explored in an earlier study [251] and so need not be considered in detail in the present work. The uniqueness of the role of the natural solution of the equation of transfer lies in its remarkably wide-ranged interpretation. On the one hand, the natural solution affords a simple intuitive picture of multiply scattered light in natural media; on the other hand it forms a link with certain general iterative solution procedures of functional equations in modern operator mathematics. No other extant mode of solution of the equation of transfer possesses such a combination of intuitive and formal features. In the present chapter we shall concentrate on these features of the natural solution, with particular emphasis on the intuitive insight into the concept of multiply scattered light in optical media supplied by the natural solution.

We shall first consider the intuitive features of the natural solution. These features will be of help to the reader in the task of following all the formal developments of the present chapter and will also help build a working intuition about natural light fields in general. We begin by observing that the natural solution of the equation of transfer is based on the idea of the scattering order decomposition of a light field. This idea in turn is based on the premise that radiant flux pouring into a medium past its boundaries generates multiply scattered radiant flux within the medium and that this radiant flux is subject to a precise mathematical analysis. It is the task of the natural mode of solution of the equation of transfer to first of all unravel the apparently chaotic resultant jumble of radiant flux of all scattering orders and arrange the flux in an orderly, countably infinite sequence of indexed flows, i.e., of integer-numbered scattering orders, and then to relate each of the indexed flows by means of well-defined formulas to the other indexed flows representing the higher and lower scattering orders. These features of the natural solution can be seen in detail with the help of a simple analogy which we shall now consider.

Consider a lake on a clear sunny day. Sunlight and skylight stream down and enter the lake surface, penetrate into the body of the medium, are partially absorbed and partially scattered throughout the body of the lake, and eventually the scattered light comes to a general steady state of flow in the various directions about each point of the medium. Now we may liken the incident radiant flux on the lake surface to a family of tiny colored particles (the geometric vestige of photons of given frequency), and we may liken the substance of the lake, in reality an aggregate of molecules of water, minerals, and organic materials, to a set X of stationary bodies distributed in space, and relatively massive with respect to the incident particles. The interaction of the photons with the lake molecules may then be envisioned, for the purposes of the present discussion, in terms of the interactions of tiny colored particles with the members of an aggregate of relatively massive stationary bodies. Then within this setting, the caroming of a tiny colored particle off the side of a massive body without change in color of the particle may be interpreted as a scattering operation not unlike the elastic scattering of a photon by a molecule; and the permanent absorption of a particle of given color by a body may be thought of as the analog of an absorption by a molecular field of a photonic field's energy.

Within the present simplified setting consisting of a swarm of colored particles migrating through a maze of relatively massive stationary bodies, the natural mode of solution of the equation of transfer takes the following form. The natural mode of solution partitions the complex steady state flow of an arbitrary given set P of monochromatic particles through the space X into sets of separate families P_n of particles. Each family P_n of particles is a subfamily of P and is identified by its scattering order, n, that is by an integer n representing the common number of scatterings undergone by each particle in the family since the particle entered the medium X. Thus at some arbitrary fixed instant t in time let $P_o(t)$ be the family of particles throughout X which have not undergone any scattering in X subsequent to entering X. In general, let $P_n(t)$ be the family of swarming particles throughout X which have undergone precisely n scatterings in X, since the particles entered X, where n is a nonnegative integer. Hence at each instant t we conceptually partition the collection P(t) of colored particles within X into an ordered, pairwise disjoint collection $P_o(t)$, $P_1(t)$, ..., $P_n(t)$, ..., of particles. This ordered collection is called the *scattering order decomposition* of the light field. Whenever a member of $P_n(t)$ undergoes a scattering event at time $t + \Delta t$ where $\Delta t \geq 0$, it enters the family $P_{n+1}(t + \Delta t)$. In the steady state, the number of members of $P_n(t)$ is independent of t.

Now the members of $P_n(t)$ are generally to be found flowing in every direction within the neighborhood of any point within X. This flow in the neighborhood of the point has assignable, at least on the conceptual level, a radiance

$N^n(t)$. The natural representation of the radiance field in this setting is then defined as the sum $\sum_{n=0}^{\infty} N^n(t)$ of the radiances associated with all the $P_n(t)$. A radiance function obtained in this manner in an optical medium will be shown to be a solution--the natural solution--of the equation of transfer for that optical medium.

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5.1 The n-ary Radiometric Concepts

In this section we shall define those radiometric concepts associated with the scattering order decomposition of a light field which will be needed in the developments of the present chapter. Throughout this section we work with a general source-free optical medium X in the steady state irradiated by a steady incident radiance function N_o defined on the boundary of X. The medium X is generally inhomogeneous, of arbitrary shape and extent, and with general volume attenuation and scattering functions defined throughout. The incident radiance associated with N_o penetrates into X and generates radiant flux of arbitrarily great scattering orders, which we now proceed to analyze.

n-ary Radiance

The systematic construction of the radiance functions associated with the families $P_n(t)$ of photons described in the introductory section starts with the incident radiance N_o on the boundary of X. In particular, the radiance $N_o(x_o,\xi)$ defined for a boundary point x_o and the direction ξ at x_o can be extended to each point x of X by writing:

$$\text{"}N^o(x,\xi)\text{"} \quad \text{for} \quad N_o(x_o,\xi)T_r(x_o,\xi) \tag{1}$$

where $x = x_o + r\xi$. The meanings of these terms are shown in Fig. 5.1. In this way we can construct a radiance distribution $N^o(x,\cdot)$ at each point x inside and on the boundary of X. We call N^o the *initial (residual or unscattered or reduced) radiance function* within X. N^o represents radiance which, relative to the radiance N_o incident on the boundary of X, has undergoine no scattering operations within X.

When some of the flux which comprises the initial radiance distribution $N^o(x\cdot)$ at x undergoes a scattering operation there is generated first order (or primary) scattered radiant flux. The amount generated per unit length in the direction ξ at x is represented by writing:

$$\text{"}N_*^1(x,\xi)\text{"} \quad \text{for} \int_{\Xi} N^o(x,\xi')\sigma(x;\xi';\xi)\, d\Omega(\xi') \tag{2}$$

This may be written succinctly in operator form using the path function operator $\mathbf{R}$ of Sec. 3.17:

$$N_*^1 = N^o\mathbf{R} \quad . \tag{3}$$