$N^n(t)$. The natural representation of the radiance field in this setting is then defined as the sum $\sum_{n=0}^{\infty} N^n(t)$ of the radiances associated with all the $P_n(t)$. A radiance function obtained in this manner in an optical medium will be shown to be a solution--the natural solution--of the equation of transfer for that optical medium.

5.1 The n-ary Radiometric Concepts

In this section we shall define those radiometric concepts associated with the scattering order decomposition of a light field which will be needed in the developments of the present chapter. Throughout this section we work with a general source-free optical medium X in the steady state irradiated by a steady incident radiance function N_0 defined on the boundary of X. The medium X is generally inhomogeneous, of arbitrary shape and extent, and with general volume attenuation and scattering functions defined throughout. The incident radiance associated with N_0 penetrates into X and generates radiant flux of arbitrarily great scattering orders, which we now proceed to analyze.

n-ary Radiance

The systematic construction of the radiance functions associated with the families $P_n(t)$ of photons described in the introductory section starts with the incident radiance N_0 on the boundary of X. In particular, the radiance $N_0(x_0,\xi)$ defined for a boundary point x_0 and the direction ξ at x_0 can be extended to each point x of X by writing:

$$\text{"}N^0(x,\xi)\text{"} \quad \text{for} \quad N_0(x_0,\xi)T_r(x_0,\xi) \tag{1}$$

where $x = x_0 + r\xi$. The meanings of these terms are shown in Fig. 5.1. In this way we can construct a radiance distribution $N^0(x,\cdot)$ at each point x inside and on the boundary of X. We call N^0 the *initial (residual or unscattered or reduced) radiance function* within X. N^0 represents radiance which, relative to the radiance N_0 incident on the boundary of X, has undergoine no scattering operations within X.

When some of the flux which comprises the initial radiance distribution $N^0(x\cdot)$ at x undergoes a scattering operation there is generated first order (or primary) scattered radiant flux. The amount generated per unit length in the direction ξ at x is represented by writing:

$$\text{"}N^1_*(x,\xi)\text{"} \quad \text{for} \quad \int_{\Xi} N^0(x,\xi')\sigma(x;\xi';\xi)\, d\Omega(\xi') \tag{2}$$

This may be written succinctly in operator form using the path function operator $\mathbf{R}$ of Sec. 3.17:

$$N^1_* = N^0\mathbf{R} \quad . \tag{3}$$

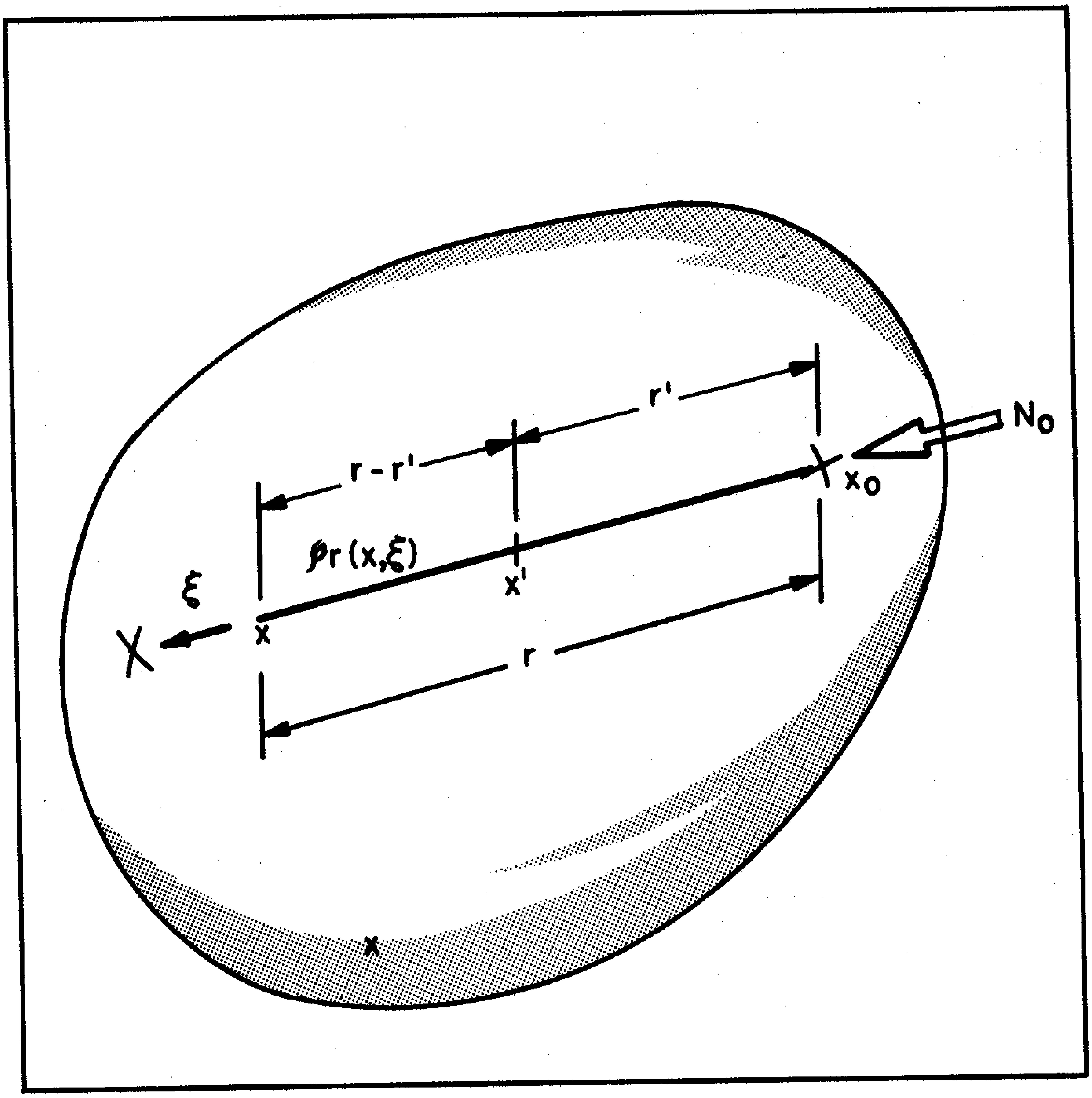

FIG. 5.1 Geometric details for computing n-ary radiance.

In other words, the operator $\mathbf{R}$ acts on N^o to generate N_*^1; alternately, we may say that $\mathbf{R}$ maps N^o into N_*^1. The amount of primary scattered radiance accumulated over a path $\mathcal{P}_r(x,\xi)$ in X is then represented by writing:

$$\text{"}N^1(x,\xi)\text{"} \quad \text{for} \quad \int_0^r N_*^1(x',\xi)\, T_{r-r'}(x',\xi)\, dr' \quad . \tag{4}$$

This may also be written succinctly using the path radiance operator $\mathbf{T}$ of Sec. 3.17:

$$N^1 = N_*^1 \mathbf{T} \quad . \tag{5}$$

The general pattern of construction of the radiance functions comprising the scattering order decomposition of the light field should now be clear. Thus, for every integer $n = 0,1,2, \ldots$, we agree to write:

$$\text{"}N_*^{n+1}\text{"} \quad \text{for} \quad N^n\mathbf{R} \tag{6}$$

and

$$\text{"}N^{n+1}\text{"} \quad \text{for} \quad N_*^{n+1}\mathbf{T} \tag{7}$$

The function N_*^n is called the *n-ary path function* and N^n is the *n-ary radiance function* in X relative to N^0. By means of (6) and (7) we can construct the (n+1)-ary radiance function on X once we know the n-ary radiance function on X, for $n \geq 0$ thus:

$$\begin{aligned} N^{n+1} &= N_*^{n+1}\mathbf{T} = (N^n\mathbf{R})\mathbf{T} \\ &= N^n(\mathbf{RT}) \end{aligned} \tag{8}$$

for every scattering order $n \geq 0$. The composition $\mathbf{RT}$ of the two operators $\mathbf{R}$ and $\mathbf{T}$ occurs often in our studies of radiative transfer theory. We shall then write, for brevity:

$$\text{"}\mathbf{S}^1\text{"} \quad \text{for} \quad \mathbf{RT} \tag{9}$$

The reader should verify that:

$$\boxed{\mathbf{S}^1 = \int_0^r \left[\int_\Xi \left[\;\right] \sigma(x';\xi';\xi)\,d\Omega(\xi') \right] T_{r-r'}(x',\xi)\,dr'} \tag{10}$$

Now, using the notation for $\mathbf{S}^1$, (8) may be written:

$$\boxed{N^{n+1} = N^n\mathbf{S}^1} \quad , \tag{11}$$

and if n is an arbitrary integer greater than 0, then it follows that we can apply the statement (8), or statement (11), once again to obtain:

$$N^{n+1} = (N^{n-1}\mathbf{S}^1)\mathbf{S}^1 \tag{12}$$

If $n-1 \geq 1$, then we can apply (11) again, with the eventual conclusion that N^{n+1} is represented as the result of operating on N^0 with $\mathbf{S}^1$ at total of n+1 times in succession. That is, if we write:

$$\text{"}\mathbf{S}^{n+1}\text{"} \quad \text{for} \quad \mathbf{S}^1\mathbf{S}^n \tag{13}$$

for every integer n, $n \geq 0$, then it is an easy application of the principle of complete induction to show that:

$$\boxed{N^n = N^0 \mathbf{S}^n} \qquad (14)$$

for every scattering order (nonnegative integer) n. The sense in which (13) and (14) are to be understood is the obvious one: Operate on N^0 and $\mathbf{S}^1$ to obtain N^1; then once N^1 is obtained, operate on N^1 with $\mathbf{S}^1$ to obtain N^2; and so on until N^n is obtained. The total combined integration operation of obtaining n-ary radiance N^n from the initial radiance N^0 is summarized by the operator $\mathbf{S}^n$ defined recursively in (13).

n-ary Scalar Irradiance

Now that the n-ary radiance functions have been defined it is a relatively easy matter to define the n-ary counterparts to all the radiometric concepts. For example, by recalling the integral representation of scalar irradiance h(x) at a point x in the optical medium X (cf. Sec. 2.7), i.e., the definition in which we have written:

$$\text{"}h(x)\text{"} \quad \text{for} \quad \int_{\Xi} N(x,\xi)\, d\Omega(\xi) \quad ,$$

we are then led to write analogously:

$$\text{"}h^n(x)\text{"} \quad \text{for} \quad \int_{\Xi} N^n(x,\xi)\, d\Omega(\xi) \qquad (15)$$

for every nonnegative integer n. We call $h^n(x)$ the *n-ary scalar irradiance* in X relative to N^0.

n-ary Radiant Energy

The connection between scalar irradiance h(x) and radiant density u(x) at each point x of X was seen in Sec. 2.7 to be:

$$h(x) = v(x)u(x)$$

where v(x) is the speed of light at x in X. Furthermore the definition of the radiant energy content U(x) of X was defined by writing:

$$\text{"}U(X)\text{"} \quad \text{for} \quad \int_{X} u(x)\, dV(x) \quad .$$

This leads us to write analogously:

$$\text{"}U^n(x)\text{"} \quad \text{for} \quad \int_X u^n(x)\,dV(x) \tag{16}$$

for every nonnegative integer n where, in turn, we have written:

$$\text{"}u^n(x)\text{"} \quad \text{for} \quad h^n(x)/v(x) \tag{17}$$

for every nonnegative integer n. Combining the definitions of h^n, u^n and U^n, we have the following representation of U^n:

$$U^n(X) = \int_X \frac{1}{v(x)} \left[\int_\Xi N^n(x,\xi)\,d\Omega(\xi) \right] dV(x) \tag{18}$$

for every nonnegative integer n, and where the n-ary radiance N^n is represented in terms of the initial radiance N^0 throughout X by means of (14).

General n-ary Radiometric Functions

The n-ary radiance and radiant energy functions constructed above will not be the only n-ary radiometric concepts used in the present work. For example the two-flow equations of Sec. 8.4 are studied by means of n-ary irradiance concepts. It is a simple matter to extend the type of definition exhibited for h^n and U^n to an arbitrary function C obtained from the radiance function by an appropriate linear operator $\mathcal{L}$ associated with C; that is:

$$C = N\mathcal{L} \quad . \tag{19}$$

For example, the operator $\mathcal{L}$ in the case where C is scalar irradiance was:

$$\mathcal{L} = \int_\Xi [\ ]\,d\Omega(\xi) \quad .$$

Then in general we write analogously:

$$\text{"}C^n\text{"} \quad \text{for} \quad N^n\mathcal{L} \quad , \tag{20}$$

for every nonnegative integer n. We call C^n the *n-ary radiometric function* of C, in X, and relative to N^0. It follows from (14) and (2) that:

$$C^n = N^0(\mathbf{s}^n\mathcal{L}) \tag{21}$$

is the representation of the n-ary radiometric function C^n associated with the general radiometric concept C. In particular, we write:

$$\text{"}C^*\text{"} \quad \text{for} \quad N^*\mathcal{L} \tag{22}$$

where N^* is the path radiance (the scattered) component of N, as it occurs in (5) of Sec. 3.13. C^* is the *diffuse radiometric function of C* in X and relative to N^o. Together, C^* and C^n are the *decomposed* radiometric functions. Radiometric functions which have not been decomposed are call *undecomposed.*

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5.2 Equation of Transfer for n-ary Radiance, Diffuse Radiance, and Path Function

The equation of transfer for n-ary radiance will now be derived. The equation is an elementary consequence of relation (11) of Sec. 5.1. To see this, suppose we fix attention on an arbitrary path $\mathcal{P}_r(x,\xi)$. Then holding the initial point x and the direction ξ of the path fixed, and differentiating N^n along the path with respect to path length r, we have:

$$\frac{dN^n}{dr} = \frac{d}{dr}\,(N^{n-1}s^1)$$

$$= \frac{d}{dr}\int_0^r \left[\int_\Xi N^{n-1}(x',\xi')\sigma(x';\xi';\xi)\,d\Omega(\xi')\right] T_{r-r'}(x',\xi)\,dr'$$

$$= \int_\Xi N^{n-1}(x,\xi')\sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$+ \int_0^r \left[\int_\Xi N^{n-1}(x',\xi')\sigma(x';\xi';\xi)\,d\Omega(\xi')\right] \frac{dT_{r-r'}(x',\xi)\,dr'}{dr}$$

At this point we observe that, by (3) of Sec. 3.11:

$$\frac{dT_{r-r'}(x',\xi)}{dr} = -\alpha(x,\xi)T_{r-r'}(x',\xi)$$

Then using (6) and (11) of Sec. 5.1 we arrive at:

$$\boxed{\xi\cdot\nabla N^n = \frac{dN^n}{dr} = -\alpha N^n + N_*^n} \tag{1}$$

which is the requisite *equation of transfer for n-ary radiance* with $n \geq 1$. Observe that the equation of transfer for N^n is not an integrodifferential equation for N^n; rather it