is the representation of the n-ary radiometric function C^n associated with the general radiometric concept C. In particular, we write:

$$\text{"}C^*\text{"} \quad \text{for} \quad N^*\mathscr{L} \tag{22}$$

where N^* is the path radiance (the scattered) component of N, as it occurs in (5) of Sec. 3.13. C^* is the *diffuse radiometric function of C* in X and relative to N^o. Together, C^* and C^n are the *decomposed* radiometric functions. Radiometric functions which have not been decomposed are call *undecomposed*.

5.2 Equation of Transfer for n-ary Radiance, Diffuse Radiance, and Path Function

The equation of transfer for n-ary radiance will now be derived. The equation is an elementary consequence of relation (11) of Sec. 5.1. To see this, suppose we fix attention on an arbitrary path $\mathcal{P}_r(x,\xi)$. Then holding the initial point x and the direction ξ of the path fixed, and differentiating N^n along the path with respect to path length r, we have:

$$\frac{dN^n}{dr} = \frac{d}{dr}\,(N^{n-1}s^1)$$

$$= \frac{d}{dr}\int_0^r \left[\int_{\Xi} N^{n-1}(x',\xi')\sigma(x';\xi';\xi)\,d\Omega(\xi')\right] T_{r-r'}(x',\xi)\,dr'$$

$$= \int_{\Xi} N^{n-1}(x,\xi')\sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$+ \int_0^r \left[\int_{\Xi} N^{n-1}(x',\xi')\sigma(x';\xi';\xi)\,d\Omega(\xi')\right] \frac{dT_{r-r'}(x',\xi)\,dr'}{dr}$$

At this point we observe that, by (3) of Sec. 3.11:

$$\frac{dT_{r-r'}(x',\xi)}{dr} = -\alpha(x,\xi)T_{r-r'}(x',\xi)$$

Then using (6) and (11) of Sec. 5.1 we arrive at:

$$\boxed{\xi\cdot\nabla N^n = \frac{dN^n}{dr} = -\alpha N^n + N^n_*} \tag{1}$$

which is the requisite *equation of transfer for n-ary radiance* with $n \geq 1$. Observe that the equation of transfer for N^n is not an integrodifferential equation for N^n; rather it is a first order linear differential equation for N^n with known n-ary path function N_*^n, once N^{n-1} is known. This suggests a conceptually powerful natural mode of solution of the general equation of transfer for N, which we shall study throughout this chapter. In the following section we shall place (1) into its canonical form, thus rounding out the studies of the canonical equation given in Chapter 4. In Sec. 5.4, the complete natural solution will be obtained.

Before concluding this discussion on n-ary radiance equations, we mention two more transfer equations for radiometric concepts which are closely related to the family of equations in (1). Note that (1) holds only for $n \geq 1$, the case $n = 0$ being excluded. This singular case $n = 0$ is readily stated using (4) of Sec. 3.10 and (2) of Sec. 3.11. The result is:

$$\boxed{\xi \cdot \nabla N^0 = \frac{dN^0}{dr} = -\alpha N^0} \tag{2}$$

for source-free media. A generalization of (2) for media with sources is given in (2) of Sec. 5.8. The remaining transfer equation to be noted here is that for the *diffuse radiance* N^* (or path radiance when a specific path of length r is given somewhere in the medium). Thus, using the concept of n-ary radiance, let us write:

$$\text{"}N\text{" for } \sum_{j=0}^{\infty} N^j \tag{3}$$

$$\text{"}N^*\text{" for } \sum_{j=1}^{\infty} N^j \tag{4}$$

and

$$\text{"}N_*^*\text{" for } \sum_{j=2}^{\infty} N_*^j \quad . \tag{5}$$

Then summing each side of (1) over all n from 1 to ∞, we have:

$$\sum_{j=1}^{\infty} \xi \cdot \nabla N^j = \sum_{j=1}^{\infty} \frac{dN^j}{dr} = -\sum_{j=1}^{\infty} \alpha N^j + \sum_{j=1}^{\infty} N_*^j \tag{6}$$

which, on applying (4) and (5) becomes:

$$\boxed{\xi \cdot \nabla N^* = \frac{dN^*}{dr} = -\alpha N^* + N_*^* + N_*^1} \tag{7}$$

This is the equation of transfer for diffuse radiance N^*. By assuming that N_*^* obeys (1) of Sec. 4.4, i.e., N_*^* decays exponentially with depth at the rate K, then (7) supplies a somewhat more powerful description of the light field than that given by (2) of Sec. 4.4. It is clear from the discussions of Sec. 5.1 and (5) that:

$$N_*^* = N^*\mathbf{R} \tag{8}$$

We shall return to these ideas in Sec. 5.4, especially to that of N, as defined in (3), wherein we will show that N so defined is a solution of the equation of transfer.

Finally, by applying the operator $\mathbf{R}$ to each side of the equation of transfer for radiance, we find:

$$\boxed{\xi \cdot \nabla N_* = -\alpha N_* + N_{**}} \tag{9}$$

which is the equation of transfer for the path function, and where we have written:

$$\text{"}N_{**}\text{"} \quad \text{for} \quad N_*\mathbf{R} \tag{10}$$

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5.3 Canonical Equations for n-ary Radiance

We pause in the present development of the natural solution of the equation of transfer to present the canonical form of the transfer equation for n-ary radiance. We shall be particularly interested in the case of n = 1, that is, in the case of the canonical equation for primary radiance. From this case we can derive an expression which has often formed an integral part of expressions which attempt to approximately represent radiance distribution with a modicum of analytic complications. The derivations below are patterned on those in Sec. 4.5. Hence we can proceed with a minimum of motivation and explanation for the present discussion. Let us write:

$$\text{"}\mathbf{K}^n\text{"} \quad \text{for} \quad -\frac{\nabla N^n}{N^n} \tag{1}$$

Then (1) of Sec. 5.2 becomes:

$$-\xi \cdot \mathbf{K}^n N^n = -\alpha N^n + N_*^n \quad ,$$

whence, for every integer n with $n \geq 1$:

$$\boxed{N^n = \frac{N_*^n}{\alpha - \xi \cdot \mathbf{K}^n}} \tag{2}$$

and consequently:

$$\boxed{N_r^n = N_o^n T_r[-\alpha] + \frac{N_*^n}{\alpha - \xi \cdot \mathbf{K}^n}\left(1 - T_r[-(\alpha - \xi \cdot \mathbf{K}^n)]\right)} \tag{3}$$