

that given by (2) of Sec. 4.4. It is clear from the discussions of Sec. 5.1 and (5) that:

$$N_{*}^{\dagger} = N_{*}^{\dagger} R \quad (8)$$

We shall return to these ideas in Sec. 5.4, especially to that of N , as defined in (3), wherein we will show that N so defined is a solution of the equation of transfer.

Finally, by applying the operator R to each side of the equation of transfer for radiance, we find:

$$\xi \cdot \nabla N_{*} = -\alpha N_{*} + N_{**} \quad (9)$$

which is the equation of transfer for the path function, and where we have written:

$$"N_{**}" \text{ for } N_{*} R \quad (10)$$

5.3 Canonical Equations for n-ary Radiance

We pause in the present development of the natural solution of the equation of transfer to present the canonical form of the transfer equation for n-ary radiance. We shall be particularly interested in the case of $n = 1$, that is, in the case of the canonical equation for primary radiance. From this case we can derive an expression which has often formed an integral part of expressions which attempt to approximately represent radiance distribution with a modicum of analytic complications. The derivations below are patterned on those in Sec. 4.5. Hence we can proceed with a minimum of motivation and explanation for the present discussion. Let us write:

$$"K^n" \text{ for } -\frac{\nabla N^n}{N^n} \quad (1)$$

Then (1) of Sec. 5.2 becomes:

$$-\xi \cdot K^n N^n = -\alpha N^n + N_{*}^n,$$

whence, for every integer n with $n \geq 1$:

$$N^n = \frac{N_{*}^n}{\alpha - \xi \cdot K^n} \quad (2)$$

and consequently:

$$N_r^n = N_o^n T_r[-\alpha] + \frac{N_{*}^n}{\alpha - \xi \cdot K^n} \left(1 - T_r[-(\alpha - \xi \cdot K^n)] \right) \quad (3)$$

which are respectively, the desired *canonical form of the equation of transfer* for n-ary radiance and its *canonical representation for a path* $\mathcal{Q}_R(x, \xi)$.

If the medium X is assumed to be a plane-parallel stratified optical medium, then following the pattern established in equations (16) - (19) of Sec. 4.5, (2) and (3) reduce to:

$$N^n = \frac{N_*^n}{\alpha + K^n \cos \theta} \quad (4)$$

and the associated canonical representation of N_R^n over a path $\mathcal{Q}_R(x, \xi)$, analogous to N_R of Sec. 4.5 is:

$$N_R^n = N_O^n T_R[-\alpha] + \frac{N_*^n}{\alpha + K^n \cos \theta} \left(1 - T_R[-(\alpha + K^n \cos \theta)] \right) \quad (5)$$

Equations (2) and (3), and their special cases (4) and (5), are the alternate (canonical) ways of representing N^n ; the usual way being summarized in (14) of Sec. 5.1 by:

$$N^n = N^O S^n \quad (6)$$

To see how (2), (3), and (6) throw light on one another, let us consider the case of a homogeneous source-free plane parallel medium X irradiated by narrow beams of radiance N_O incident at each point of its upper boundary through a small solid angle Ξ_O of magnitude Ω_O , as shown in Fig. 5.2. The radiant flux from N_O initiates a multiple scattering process within X and eventually all scattering orders of radiant flux are present within X. We direct attention now to N^1 and first compute its value at depth z in the direction ξ using (6). Thus, from (6) with $n = 1$:

$$N^1 = (N^O R) T$$

For the present case $N^O R$ is readily evaluated:

$$N_*^1(z, \xi) = N^O R = \int_{\Xi} N^O(z, \xi') \sigma(z; \xi'; \xi) d\Omega(\xi') .$$

Since for each ξ' in Ξ_O ,

$$\begin{aligned} N^O(z, \xi') &= N_O(0, \xi') T_R(0, \xi') \\ &= N_O e^{\alpha z \sec \theta_O} \end{aligned}$$

where:

$$\cos \theta_O = \xi_O \cdot \mathbf{k}$$

$$\begin{aligned}
N^1(z, \xi) &= N_{*T}^1 = \int_0^r N_*(z', \xi) T_{r-r'}(z', \xi) dr' \\
&= -\sec\theta N_0 \sigma(\xi_0; \xi) \Omega_0 \int_0^z e^{\alpha z' \sec\theta_0} e^{\alpha(z-z') \sec\theta} dz' \\
&= -\sec\theta N_0 \sigma(\xi_0; \xi) \Omega_0 e^{\alpha z \sec\theta} \int_0^z e^{\alpha z' (\sec\theta_0 - \sec\theta)} dz'
\end{aligned}$$

Therefore:

$$N^1(z, \xi) = - \frac{N_0 \sec\theta \sigma(\xi_0; \xi) \Omega_0 e^{\alpha z \sec\theta} [1 - e^{-\alpha z (\sec\theta_0 - \sec\theta)}]}{\alpha (\sec\theta_0 - \sec\theta)} \quad (8)$$

This canonical representation of $N^1(z, \xi)$, in which $\cos\theta = \xi \cdot K$, holds for all paths such that $\theta \neq \theta_0$. For the case $\theta = \theta_0$, we return to the penultimate equality and evaluate the integral anew, or use L'Hospital's rule in (8). Clearly, the new integral value is simply z for the case $\theta_0 = \theta$. Comparing (8) with the canonical form, with the latter now evaluated for the case $n = 1$:

$$N^1(z, \xi) = \frac{N_{*T}^1(z, \xi)}{\alpha + K^1 \cos\theta} \quad (9)$$

we see that the following equality must hold:

$$\alpha + K^1 \cos\theta = \frac{-\alpha (\sec\theta_0 - \sec\theta)}{\sec\theta [1 - e^{-\alpha z (\sec\theta_0 - \sec\theta)}]} \quad (10)$$

From this we can, if required, solve for K^1 (which generally is a function of z, θ , and, also in the present case, the parameter θ_0). Observe that for $\sec\theta_0 \geq \sec\theta$, i.e., for $\theta \leq \theta_0$.

$$\lim_{z \rightarrow \infty} K^1(z, \xi) = -\alpha \sec\theta_0$$

and for $\theta > \theta_0$,

$$\lim_{z \rightarrow \infty} K^1(z, \xi) = -\alpha \sec\theta$$

This shows directly that the K -function for primary radiance eventually, i.e., for sufficiently great depths, becomes

independent of ξ over large sets of directions (i.e., when $\theta < \theta_0$). This phenomenon of the eventual partial independence of K^1 with respect to direction, presages an analogous behavior of the complete K-function for observable radiance; we will study this depth behavior of K in more detail in Chapter 10.

We now summarize the main results of our illustrative example: By evaluation of (6) for the case of $n = 1$ and comparing the resultant representation of N^1 with that given by the canonical form (4), we deduce the necessary form of the K-function K^1 for N^1 . The usual classical method of looking at N^1 is by means of formulas of the structure of (8). Our studies of the canonical equation of transfer in Chapter 4, extended to the present setting, now show that (8) is but a special form of the canonical equation for primary radiance N^1 , as given in (9). Hence (8) may be given the compact and intuitively useful canonical form (9) provided K^1 is as given implicitly by (10).

Concluding Observations

In conclusion we note that the integrations leading to (8) may be redone now over a path $\mathcal{P}_r(z_0, \xi)$ with initial point at depth $z_0 > 0$. The result will be a path radiance N_r^1 expression, the special case of N_r^n for $n = 1$, leading to an instance of (6). Observe that $N^n(z, \xi)$ in (4) and $N_r^n(z, \xi)$ in (5) are equal for every z and ξ , being but two ways of expressing the same radiance: Whereas (4) expresses the radiance $N^n(z, \xi)$ as a value of the radiance distribution $N^n(z, \cdot)$ at depth z for the direction ξ , equation (5), on the other hand, expresses the same radiance now by conceptually partitioning it into two parts associated with an arbitrary path $\mathcal{P}_r(z_0, \xi)$ in the medium. In other words, we can carry over without change from the discussions of Chapter 4 to the present setting of n -ary concepts, all interpretations of path radiance N_r^* , transmitted residual radiance N_r^0 , and apparent radiance N_r , arrived at in those earlier discussions. It is of interest to emphasize in particular a powerful but simple model for radiance distributions that arises when we represent N^* rather than N by means of the general equation (2) of Sec. 4.4. For such a model " N_* " in (1) of Sec. 4.4 is replaced by " N_r^* ". The correct basis for this model is (7) of Sec. 5.2.

5.4 The Natural Solution for Radiance

We return now to the main thread of the argument, begun in 5.2, leading to the development of the natural solution of the equation of transfer. Our most basic intuitions about light fields in the sea and the air and generally for any optical medium, lead us to think of the radiance perceived by our eyes and our instruments as consisting of multiply-scattered light, i.e., light which has undergone one, two, three, and generally very large numbers of scattering operations after its entrance into the medium and before its incidence on the retina or photocell located somewhere in the medium. It is natural then (hence the name of the present