independent of ξ over large sets of directions (i.e., when $\theta < \theta_0$). This phenomenon of the eventual partial independence of K^1 with respect to direction, presages an analogous behavior of the complete K-function for observable radiance; we will study this depth behavior of K in more detail in Chapter 10.

We now summarize the main results of our illustrative example: By evaluation of (6) for the case of $n = 1$ and comparing the resultant representation of N^1 with that given by the canonical form (4), we deduce the necessary form of the K-function K^1 for N^1. The usual classical method of looking at N^1 is by means of formulas of the structure of (8). Our studies of the canonical equation of transfer in Chapter 4, extended to the present setting, now show that (8) is but a special form of the canonical equation for primary radiance N^1, as given in (9). Hence (8) may be given the compact and intuitively useful canonical form (9) provided K^1 is as given implicitly by (10).

Concluding Observations

In conclusion we note that the integrations leading to (8) may be redone now over a path $\mathcal{P}_r(z_0,\xi)$ with initial point at depth $z_0 > 0$. The result will be a path radiance N_r^1 expression, the special case of N_r^n for $n = 1$, leading to an instance of (6). Observe that $N^n(z,\xi)$ in (4) and $N_r^n(z,\xi)$ in (5) are equal for every z and ξ, being but two ways of expressing the same radiance: Whereas (4) expresses the radiance $N^n(z,\xi)$ as a value of the radiance distribution $N^n(z,\cdot)$ at depth z for the direction ξ, equation (5), on the other hand, expresses the same radiance now by conceptually partitioning it into two parts associated with an arbitrary path $\mathcal{P}_r(z_0,\xi)$ in the medium. In other words, we can carry over without change from the discussions of Chapter 4 to the present setting of n-ary concepts, all interpretations of path radiance N_r^*, transmitted residual radiance N_r^o, and apparent radiance N_r, arrived at in those earlier discussions. It is of interest to emphasize in particular a powerful but simple model for radiance distributions that arises when we represent N^* rather than N by means of the general equation (2) of Sec. 4.4. For such a model "N_*" in (1) of Sec. 4.4 is replaced by "N_*^*". The correct basis for this model is (7) of Sec. 5.2.

5.4 The Natural Solution for Radiance

We return now to the main thread of the argument, begun in 5.2, leading to the development of the natural solution of the equation of transfer. Our most basic intuitions about light fields in the sea and the air and generally for any optical medium, lead us to think of the radiance perceived by our eyes and our instruments as consisting of multiply-scattered light, i.e., light which has undergone one, two, three, and generally very large numbers of scattering operations after its entrance into the medium and before its incidence on the retina or photocell located somewhere in the medium. It is natural then (hence the name of the present mode of solution) to attempt to construct a solution of the equation of transfer for radiance by constructing all the n-ary radiance functions N^n within a given optical medium X and to sum them to obtain the requisite radiance field throughout the medium. Thus we are led to write:

$$\boxed{\text{"}N(z,\xi)\text{"} \quad \text{for} \quad \sum_{j=0}^{\infty} N^j(x,\xi)} \tag{1}$$

and hope that the function N so defined satisfies the equation of transfer. We call N defined by (1) the *natural solution of the equation of transfer*. We now show that the word "solution" in the name for N is indeed justified.

We begin by using (14) of Sec. 5.1 to write $N(x,\xi)$ in (1) as:

$$N(x,\xi) = \sum_{j=0}^{\infty} N^o\mathbf{s}^j(x,\xi)$$

or more compactly in functional form as:

$$N = \sum_{j=0}^{\infty} N^o\mathbf{s}^j$$

In this way we come to define the basic operator $\mathbf{s}$ for the natural solution, i.e., we can now write:

$$\text{"}\mathbf{s}\text{"} \quad \text{for} \quad \sum_{j=0}^{\infty} \mathbf{s}^j \quad , \tag{2}$$

where "S^o" denotes the *identity operator I*, with the property $fI = f$ for every radiance function. With this definition the natural solution representation takes the form:

$$\boxed{N = N^o\mathbf{s}} \tag{3}$$

By means of this representation, the formal verification that N in (3) is a solution of the equation of transfer is readily forthcoming via the following eight main steps:

$$N = N^o\mathbf{s} = N^o(I + \sum_{j=1}^{\infty} \mathbf{s}^j)$$

$$= N^o(I + (\sum_{j=0}^{\infty} \mathbf{s}^j)\mathbf{s}^1)$$

$$= N^o + (N^o\mathbf{s})\mathbf{s}^1$$

$$= N^o + N\mathbf{S}^1$$

$$= N^o + (N\mathbf{R})\mathbf{T}$$

$$= N^o + N_*\mathbf{T}$$

$$= N^o + N^*$$

We have therefore shown that:

$$N = N^o + N^* \quad , \tag{4}$$

which is the integral form of the equation of transfer (re: (1) of Sec. 3.15). An alternative approach to the above demonstration is to show that N as defined by (1) is a solution of the integrodifferential equation of transfer. The basis for such a demonstration is given by (7) of Sec. 5.2. It remains only to add (2) of Sec. 5.2 to each side of (7) and reduce the results.

To summarize our findings: We have shown that the natural mode of constructing the radiance function N from the n-ary radiance functions N^n, $n \geq 0$, leads to a solution--the *natural solution*--of the equation of transfer. It also may be seen that N so constructed is a unique solution in the sense that whenever N' is also a solution of (4), then N' = N. The mathematical basis for the existence and uniqueness of the natural solution will be described in Sec. 5.12.

We conclude by observing that the natural solution of the equation of transfer is not only fundamental from an intuitive physical point of view, but that it in essence exemplifies a mode of function construction which has been of increasing importance in the logical foundations of mathematics in recent years. This mode of construction--the enumerably recursive mode of construction--is very closely related to the natural mode of construction defined above and is coming under intensive study principally because of the current strides in developing ultrafast mechanical aids to numerical and logical computations. These developments will eventually make feasible the computation of relatively high scattering orders n for N^n, so that finite sums of the form

$$N^o + N^1 + N^2 + \ldots + N^n$$

will constitute appropriately adequate approximations to the ideal natural solution N. Thus we will eventually be able to go far beyond the first order solutions

$$N^o + N^1 = N^o + \frac{N_*^1}{\alpha + K^1 \cos\theta} \tag{6}$$

(cf. (8), (9) of Sec. 5.3) to which many classical studies in atmospheric and hydrologic optics were hitherto limited because of the relatively heavy demand on manipulative skill (and time!) needed to evaluate N^2, N^3 and higher order n-ary radiance functions.

<< Table of Contents << Title Page >> Next Page >>

5.5 Truncated Natural Solutions for Radiance

We now investigate the effect of truncating the natural solution of the equation of transfer after a finite number of terms. While the natural solution is an ideal conceptual tool in the study of radiative transfer theory, as has been demonstrated at length in Chapter III of Ref. [251], the solution can almost never be evaluated completely either numerically or theoretically, because of the infinite number of terms comprising the solution. We are then in practice obliged to stop the accumulation of the terms after a finite number of them have been evaluated. The question then arises as to the closeness of the resultant truncated solution to the natural solution. We shall now consider this question in detail.

Throughout the remainder of this section we shall choose as our setting a source-free homogeneous plane parallel optical medium X of arbitrary depth with a steady internal light field induced by arbitrary incident radiance distributions N_o at each point of the upper boundary of the medium. The volume scattering function σ and the volume attenuation function α are otherwise arbitrary.

Now, starting with the natural solution N of the equation of transfer as defined in (1) of Sec. 5.4, we write:

$$N = \sum_{j=0}^{k} N^j + \sum_{j=k+1}^{\infty} N^j \qquad (1)$$

The central question of the present discussion may now be phrased as follows. Writing:

$$\text{"}N^{(k)}\text{"} \quad \text{for} \quad \sum_{j=0}^{k} N^j \quad ,$$

we ask: by how much does the finite sum $N^{(k)}$ differ from the infinite sum N; or in other words, what is the general order of magnitude of

$$\sum_{j=k+1}^{\infty} N^j \quad ?$$

To answer this question we shall obtain an upper bound on the values of the difference $N-N^{(k)}$. This upper bound shall serve as a measure of the difference between the functions N and N(k).

We begin by letting "$\bar{N}^o$" denote the upper bound of the initial radiance function N^o within X (re: (1) of Sec. 5.1). This upper bound is easily evaluated in general, and in particular in all natural hydrosols this upper bound is actually