(cf. (8), (9) of Sec. 5.3) to which many classical studies in atmospheric and hydrologic optics were hitherto limited because of the relatively heavy demand on manipulative skill (and time!) needed to evaluate N^2, N^3 and higher order n-ary radiance functions.

5.5 Truncated Natural Solutions for Radiance

We now investigate the effect of truncating the natural solution of the equation of transfer after a finite number of terms. While the natural solution is an ideal conceptual tool in the study of radiative transfer theory, as has been demonstrated at length in Chapter III of Ref. [251], the solution can almost never be evaluated completely either numerically or theoretically, because of the infinite number of terms comprising the solution. We are then in practice obliged to stop the accumulation of the terms after a finite number of them have been evaluated. The question then arises as to the closeness of the resultant truncated solution to the natural solution. We shall now consider this question in detail.

Throughout the remainder of this section we shall choose as our setting a source-free homogeneous plane parallel optical medium X of arbitrary depth with a steady internal light field induced by arbitrary incident radiance distributions N_o at each point of the upper boundary of the medium. The volume scattering function σ and the volume attenuation function α are otherwise arbitrary.

Now, starting with the natural solution N of the equation of transfer as defined in (1) of Sec. 5.4, we write:

$$N = \sum_{j=0}^{k} N^j + \sum_{j=k+1}^{\infty} N^j \qquad (1)$$

The central question of the present discussion may now be phrased as follows. Writing:

$$\text{"}N^{(k)}\text{"} \quad \text{for} \quad \sum_{j=0}^{k} N^j \quad ,$$

we ask: by how much does the finite sum $N^{(k)}$ differ from the infinite sum N; or in other words, what is the general order of magnitude of

$$\sum_{j=k+1}^{\infty} N^j \quad ?$$

To answer this question we shall obtain an upper bound on the values of the difference $N-N^{(k)}$. This upper bound shall serve as a measure of the difference between the functions N and $N(k)$.

We begin by letting "$\bar{N}^o$" denote the upper bound of the initial radiance function N^o within X (re: (1) of Sec. 5.1). This upper bound is easily evaluated in general, and in particular in all natural hydrosols this upper bound is actually attained by N^o at the air-water boundary of the medium. Indeed, for sunny days, N^o is almost invariably the apparent radiance of the run as seen just below the surface of the medium.

The upper bound of the primary radiance function N^1 is obtained by first 'bounding' N^1_*. Thus, starting with (6) of Sec. 5.1 in which n = 0, we have for every x in X and direction ξ in Ξ:

$$N^1_*(x,\xi) = \int_\Xi N^o(x,\xi')\ \sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$\leq \bar{N}^o \int_\Xi \sigma(x;\xi';\xi)\,d\Omega(\xi')$$

$$= \bar{N}^o\ s \qquad (2)$$

Here "s" denotes the value of the volume total scattering function defined in (3) of Sec. 4.2. The reader will discern that it is sufficient at this stage to assume that:

$$\sigma(x;\xi';\xi) = \sigma(x;\xi;\xi')$$

for every ξ' and ξ at each point x of X, in order that we have:

$$s(x) = \int \sigma(x;\xi';\xi)\,d\Omega(\xi') \quad .$$

This is not an unusual requirement on σ (it is called a *reciprocal* condition) and is readily met by all σ from natural hydrosols. (For related conditions on σ, see Sec. 7.12.)

Next, use is made of (7) of Sec. 5.1 and the equality (2) just deduced to obtain:

$$N^1(x,\xi) = \int_0^r N^1_*(x',\xi)\ T_{r-r'}(x',\xi)\,dr'$$

$$\leq \bar{N}^o s \int_0^r T_{r-r'}(x',\xi)\,dr'$$

$$= \bar{N}^o s \int_0^r e^{-\alpha(r-r')}\,dr'$$

$$= \frac{\bar{N}^o s}{\alpha}\ (1-e^{-\alpha r})$$

$$\leq \bar{N}^o \rho \tag{3}$$

for every point x in X and direction ξ in Ξ; and where we have written:

$$"\rho" \text{ for } s/\alpha \quad . \tag{4}$$

The ratio ρ is called the *albedo for single scattering* or more accurately the *scattering-attenuation ratio*. By our agreement in Sec. 4.2, namely that about the nonnegativity of the volume absorption function a, it follows that ρ satisfies the inequality $0 \leq \rho \leq 1$. For the present discussion we assume in particular that $0 < \rho < 1$. When we repeat the results (2) and (3), but now applied to $N^2(x,\xi)$ we obtain:

$$N^2(x,\xi) \leq \bar{N}^o \rho^2$$

for every x in X and ξ in Ξ. From this we can see a pattern emerging and we readily prove that:

$$\boxed{N^n(x,\xi) \leq \bar{N}^o \rho^n} \tag{5}$$

for every scattering order n, every point x in X and direction ξ in Ξ.

The inequality (5) is the main result needed for the determination of the upper bound for the difference $N-N^{(k)}$. Indeed, by direct computation, we have:

$$\begin{aligned} N(x,\xi) - N^{(k)}(x,\xi) &= \sum_{j=k+1}^{\infty} N^j(x,\xi) \\ &\leq \sum_{j=k+1}^{\infty} \bar{N}^o \rho^j \\ &= \bar{N}^o \sum_{j=k+1}^{\infty} \rho^j \\ &= \bar{N}^o \rho^{k+1} \sum_{j=0}^{\infty} \rho^j \\ &= \frac{\bar{N}^o \rho^{k+1}}{1-\rho} \quad , \end{aligned}$$

which holds for every x in X and ξ in Ξ.

Summarizing, we may say that:

$$\boxed{N(x,\xi) - N^{(k)}(x,\xi) \leq \frac{\bar{N}^o \rho^{k+1}}{1-\rho}} \tag{6}$$

holds for every nonnegative integer k, every point x in X and direction ξ in Ξ.

As an example of the use of (6), suppose a given lake has a scattering-attenuation ratio of $\rho = 0.4$ for wavelength 550 mμ, and that $\bar{N}^0$ for that wavelength is 10^6 watts/(m^2 × steradian). We require for a particular computation that $N(x,\xi) - N^{(k)}(x,\xi)$ be not more than 10^4 watts (m^2 × steradian) for every x and ξ. What is the least scattering order k at which the natural solution must be truncated so that this condition is met? By (6) we require k such that:

$$10^4 < \frac{10^6 (0.4)^{k+1}}{1-(0.4)}$$

or that:

$$0.6 \times 10^{-2} < (0.4)^{k+1}$$

Forming an equality for the moment, we require:

$$\log_{10}(6\times10^{-3}) = (k + 1) \log_{10}(0.4)$$

This implies that to the nearest integer, k+1=6, so that k=5. Hence the truncation solution is required to be carried out to five scattering orders, at least.

A useful alternative formula to (6) is obtained by first noting that for media in which $\rho > 0$, we certainly have the maximum value $\bar{N}$ of $N(x,\xi)$ greater than the maximum value N^0 of $N^0(x,\xi)$. Then (6) implies:

$$\frac{N(x,\xi) - N^k(x,\xi)}{\bar{N}} < \frac{\rho^{k+1}}{1-\rho} \qquad (7)$$

for every x in X and ξ in Ξ. The comparative merit of (7) over (6) consists in equation (7)'s ability to express the error of truncation in terms of a relative error, that is the error relative to the prevailing magnitude N of the light field. Hence for the medium at hand, carrying out the natural solution to five terms results in a *relative* error of less than 1 percent.

Before closing we shall examine the inequalities (5) and (6) for some insight they may yield about the relative importance of the various components of the decomposition of the natural light field. For example, (5) shows that n-ary radiances are on the whole less by a factor of ρ than (n-1)-ary radiances. Thus if $\rho = 1/2$, say, then $N^1(x,\xi)$ is on the whole, about half the magnitude of $N^0(x,\xi)$, and the magnitude of $N^2(x,\xi)$, in turn, is about half that of $N^1(x,\xi)$, and so on. Thus the overall magnitude of n-ary radiances decrease exponentially with scattering order n. Inequality (6) also shows that for small ρ (near 0), a given n-ary radiance varies directly as the nth power of ρ, whereas for large ρ (near 1), the n-ary radiances vary essentially hyperbolically with $1-\rho$, i.e., as $1/(1-\rho)$. Similar observations can be made using (6) or (7). We shall return to the matter of truncated natural solutions in the following section and reconsider them for transient light fields. The reader wishing radiance bounds in a slightly more general steady state case than that considered in this section, may consult Sec. 22 of Ref. [251].

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5.6 Optical Ringing Problem. One-Dimensional Case

The object of this section is to formulate the optical ringing problem in the context of radiative transfer theory and to indicate how the natural mode of solution may be used to solve the problem. In order to explain the ideas behind the optical ringing problem and its natural mode of solution without too many geometrical complications, we consider first the one-dimensional case of the problem. The three-dimensional case will be discussed in the following section.

The term, "optical ringing" has an analogous meaning to the term "reverberation" as used in the theory of sound. In fact the well-known term "reverberate" applies in principle equally to optical and acoustical phenomena. However, until recently, the relative difficulty of producing and recording optical reverberation because of the immeasurably short periods of time involved has given the acoustical discipline almost exclusive use of the term. We can use the popular acoustical meaning of the term "reverberation" to give the following nontechnical definition of the phenomenon at hand: *Optical ringing* in an optical medium is the optical reverberation of the medium set up by a narrow short pulse of monochromatic light. Hence the appropriate acoustical analogy to optical ringing would be the reverberation set up by a directional, short clap of one-note thunder. In more technical parlance the *optical ringing problem* in a medium X is the problem of determining at time $t \geq 0$, the time-dependent radiance function over X which is the solution of the equation of transfer, given a directional, spatial, and temporal Dirac-delta function input of radiance to the medium at time $t = 0$. This problem has applications to the description of time-dependent radiance fields set up by laser beams with their characteristic high power, narrow-beam, short-pulse shafts of monochromatic radiant flux. While interest in the optical ringing problem has reawakened because of the advent of the laser, it should be noted that the problem is a venerable one in radiative transfer theory and neutron transport theory, and was first studied purely for its intrinsic interest and as a fundamental block on which to build solutions with arbitrary initial time-varying, inputs (see, e.g., [211], [235], [236]).

Geometry of the Time-Dependent Light Field

The formulation of the time-dependent radiant flux problem in an optical medium X will be facilitated by finding an efficient means of depicting the space-time disposition of the radiant flux throughout the optical medium. We shall now construct such a means. In the present discussion the medium X is one-dimensional and is represented in Fig. 5.3(a)