with $1-\rho$, i.e., as $1/(1-\rho)$. Similar observations can be made using (6) or (7). We shall return to the matter of truncated natural solutions in the following section and reconsider them for transient light fields. The reader wishing radiance bounds in a slightly more general steady state case than that considered in this section, may consult Sec. 22 of Ref. [251].

5.6 Optical Ringing Problem. One-Dimensional Case

The object of this section is to formulate the optical ringing problem in the context of radiative transfer theory and to indicate how the natural mode of solution may be used to solve the problem. In order to explain the ideas behind the optical ringing problem and its natural mode of solution without too many geometrical complications, we consider first the one-dimensional case of the problem. The three-dimensional case will be discussed in the following section.

The term, "optical ringing" has an analogous meaning to the term "reverberation" as used in the theory of sound. In fact the well-known term "reverberate" applies in principle equally to optical and acoustical phenomena. However, until recently, the relative difficulty of producing and recording optical reverberation because of the immeasurably short periods of time involved has given the acoustical discipline almost exclusive use of the term. We can use the popular acoustical meaning of the term "reverberation" to give the following nontechnical definition of the phenomenon at hand: *Optical ringing* in an optical medium is the optical reverberation of the medium set up by a narrow short pulse of monochromatic light. Hence the appropriate acoustical analogy to optical ringing would be the reverberation set up by a directional, short clap of one-note thunder. In more technical parlance the *optical ringing problem* in a medium X is the problem of determining at time $t \geq 0$, the time-dependent radiance function over X which is the solution of the equation of transfer, given a directional, spatial, and temporal Dirac-delta function input of radiance to the medium at time $t = 0$. This problem has applications to the description of time-dependent radiance fields set up by laser beams with their characteristic high power, narrow-beam, short-pulse shafts of monochromatic radiant flux. While interest in the optical ringing problem has reawakened because of the advent of the laser, it should be noted that the problem is a venerable one in radiative transfer theory and neutron transport theory, and was first studied purely for its intrinsic interest and as a fundamental block on which to build solutions with arbitrary initial time-varying, inputs (see, e.g., [211], [235], [236]).

Geometry of the Time-Dependent Light Field

The formulation of the time-dependent radiant flux problem in an optical medium X will be facilitated by finding an efficient means of depicting the space-time disposition of the radiant flux throughout the optical medium. We shall now construct such a means. In the present discussion the medium X is one-dimensional and is represented in Fig. 5.3(a) as a line segment. We shall consider the medium to extend indefinitely on either side of the origin point 0 of the medium, with distance measured as positive toward the right.

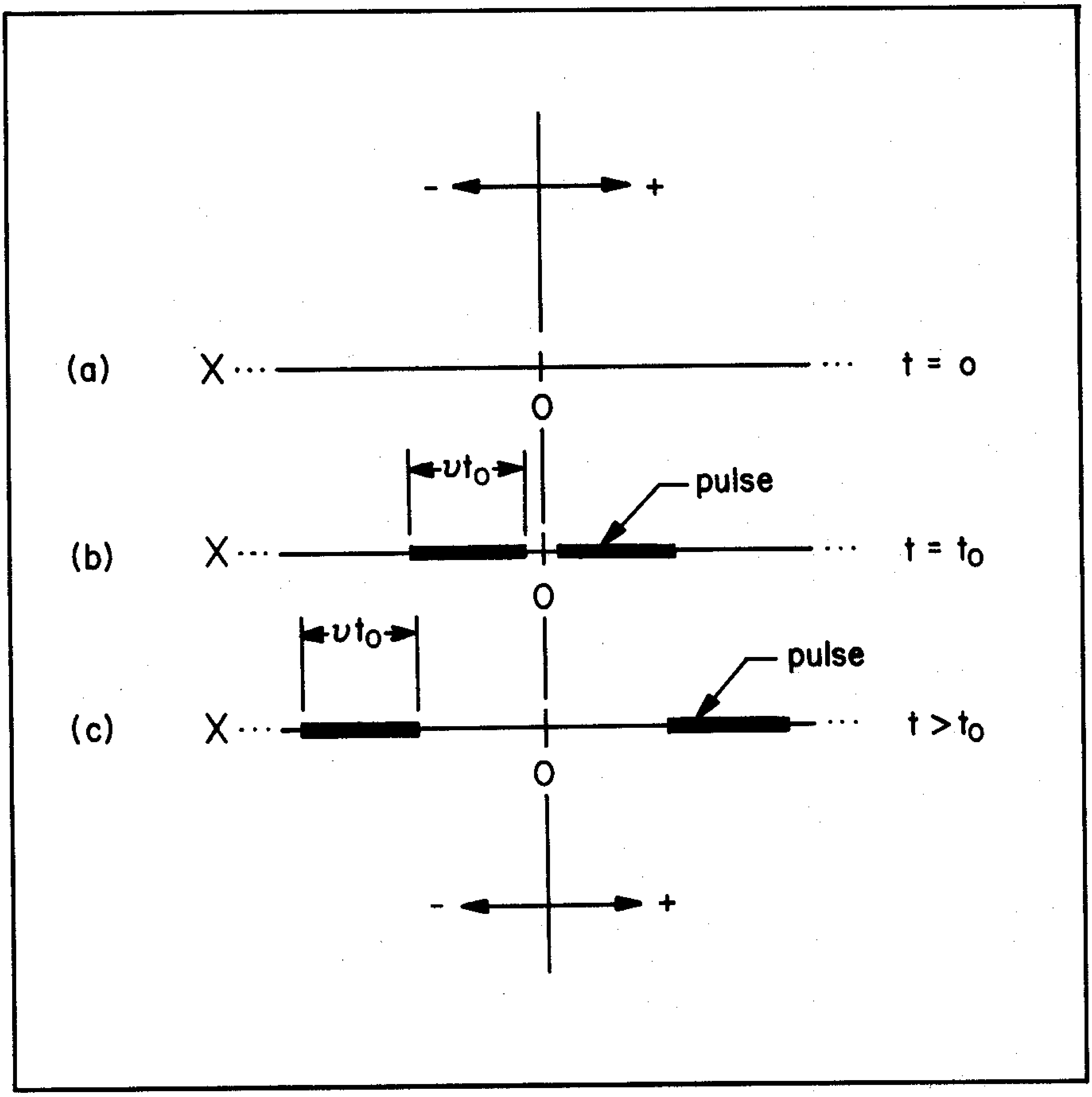

FIG. 5.3 Positions of a finite light pulse along a one-dimensional medium.

Now suppose that point 0 becomes a source of radiant flux starting at time $t = 0$ and that 0 continues to emit flux in an arbitrary fashion in both directions about 0 until time $t = t_o$, at which time the source at 0 is shut off. Let "$N_o(0,t,+)$" and "$N_o(0,t,-)$" denote these radiances of 0 at time t in the positive and negative directions, respectively. Figure 5.3(b) shows the position of the pulse emitted by 0 just after time t_o. The pulse is speeding away from point 0 into the medium on either side of 0. Figure 5.3(c) shows the position of the pulse some time later than t_o. Figures 5.3(a) through 5.3(c) are like three snapshots of the medium

X at three separate instants of time subsequent to the emission of the pulse. It would be quite instructive if instead of still shots of X at discrete time instants, we could have a moving picture of the pulse as it moves out into X from 0 and generates the field of scattered light within X. Such a means of communication is obviously unfeasible for the present work. However, an alternate and in some ways superior means of visualizing the time-dependent light field in X consists in a static space-time diagram of the pulse in X of the kind depicted in Fig. 5.4.

The description of the pulse of radiant flux from point 0 becomes relatively simple when given in terms Fig. 5.4. The space-time portrait of the pulse is given by the shaded V-shaped region in the space-time diagram. To find the instantaneous position of the pulse in X at time t', first go along the time axis erected perpendicular to X until time

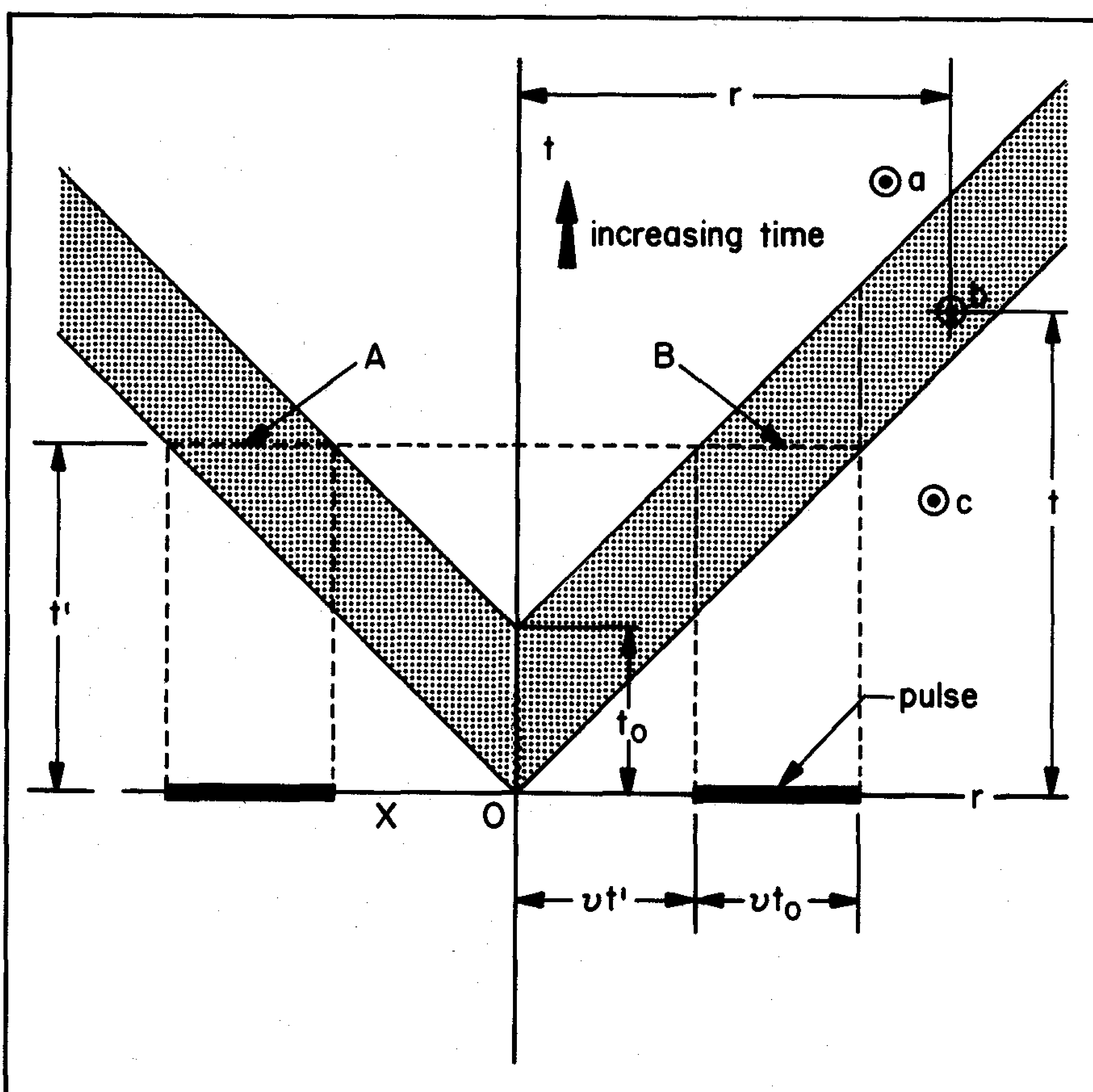

FIG. 5.4 A space-time portrait of the pulse in Fig. 5.3. The world region of the pulse is shaded.

point t' is reached. Then draw a straight line through t' parallel to X. This line will intersect the shaded region in generally two segments A and B. The perpendicular projection of these segments down onto X will then give the location of the pulse in X at time $t' \geq 0$. The slope of any straight line segments parallel to the boundaries of the shaded region of the pulse are such that, as t' units of the time axis are traversed, vt' units of the space axis are traversed, where v is the speed of light in X. We assume v to be constant over X. The shaded region of Fig. 5.4 is called the *world region* of the pulse.

It follows from the axioms of special relativity that, relative to the frame at 0, the space-time line traced out by a material particle in X cannot have an arbitrary slope, but rather one which is bounded as follows. If r(t) is the distance of the particle from 0 at time t, then:

$$\left| \frac{dr(t)}{dt} \right| \leq v$$

for every t for which r(t) is defined in the frame anchored at 0. In particular, the slopes of the *world lines* (i.e., space-time trajectories) of the photons comprising the pulse of light from 0 are exactly of magnitude v, with respect to the time axis. Thus on the one hand, the world line of a particle stationary in X is a vertical line, and on the other hand, that of a photon is parallel to one of the boundary lines of the shaded region in Fig. 5.4. All naturally moving particles in X must therefore have the tangents to their world lines always between (or coincident) with these two extremes, with respect to the r,t frame of reference at 0.

The space-time diagram also aids in visualizing the various possibilities of radiometric interactions between points of X. Thus, points a, b, and c in Fig. 5.4 depict the three possible dispositions of points in space time with respect to the pulse from 0. Point b(=(r,t)) is in the world region of the pulse, and so represents a point of X at distance r from 0 which at time t is being irradiated by radiant flux comprising some of the pulse from 0. Points a and c on the other hand are not in the world region of the pulse. Point a in particular represents a point in X *after* the pulse has gone by it (to find the contemporaneous pulse to a, draw a horizontal line through a, and the segment it determines with the world region of the pulse is the requisite position of the pulse). Point c represents a point in X *before* the pulse has gone by it. Points a and c thus have the property in common that they do not lie on the world region of the pulse from 0; however, points a and c differ from one another in a fundamental sense. Indeed, the point in X corresponding to a may eventually feel the effects of the pulse through scattering of flux from the pulse; however, the point in X corresponding to point c in the space time plane is "forever" immune to the direct or indirect effects of the pulse. Here we are implicitly adopting another empirical fact of macroscopic physics: Effects of an event may propagate futureward in space-time but not pastward. When this fact is combined with that about the limits on the slopes of the world lines of particles mentioned above, we can readily delimit those parts of the space-time plane over (or through) which they can effect or be effected by a given event (represented as a point) in the plane. These regions are shown in Fig. 5.5(a) for an arbitrary point a. In general, for two points a and b in the space-time diagram associated with X, the common region of possible interaction is the shaded intersection of the futureward sector of b with the pastward sector of a, as shown, in Fig. 5.5(b). If the intersection region is empty, then the two points cannot interact.

With these preliminary observations in mind, we may now use the the general space-time diagram to help in the study of the time-dependent radiant flux problem on X. Starting

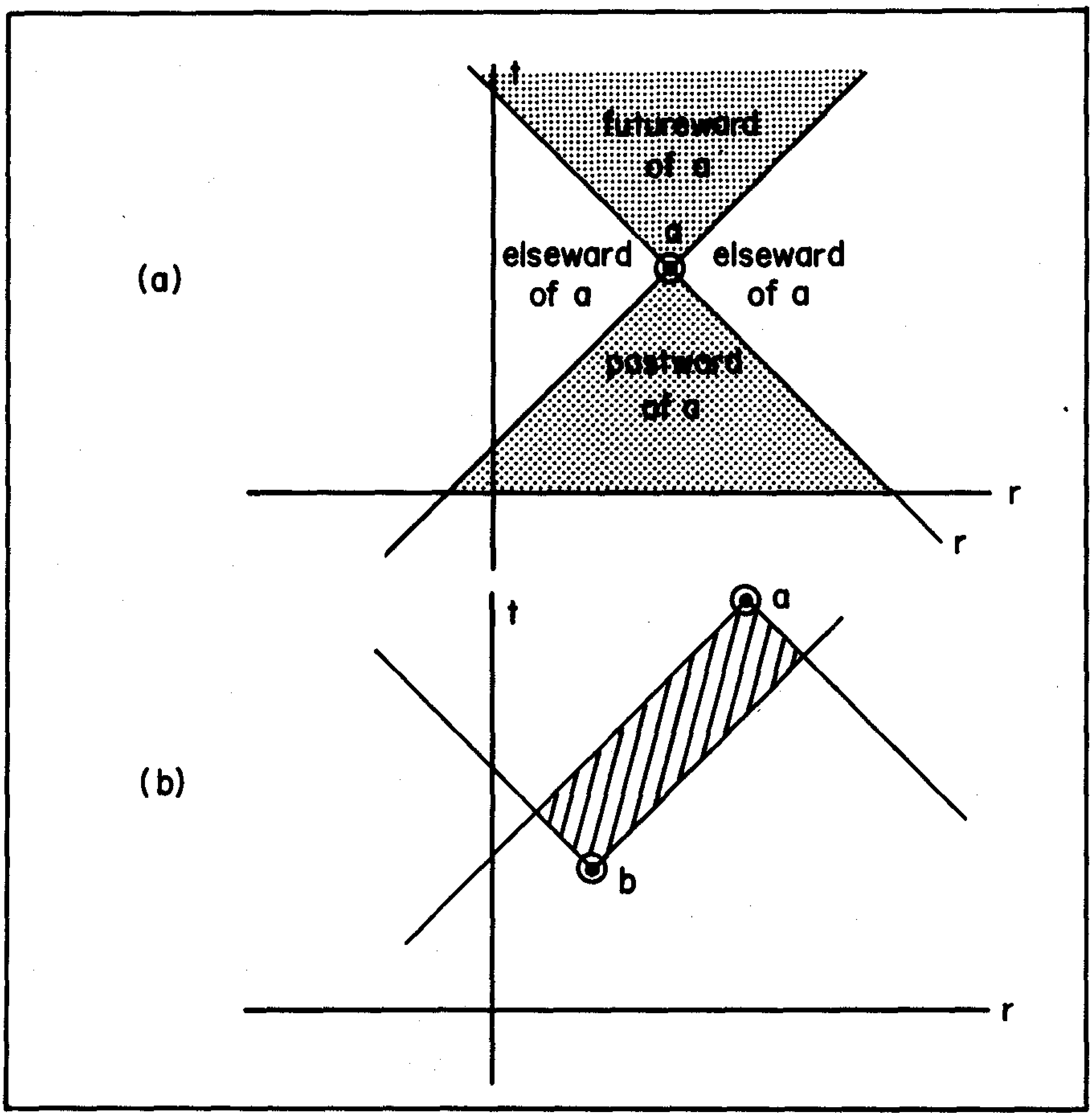

FIG. 5.5 Part (a) depicts those points of space-time about point a which lie in a's future, past, and elsewhere from a. Part (b) shows the common region (shaded) shared by the future cone of b and the past cone of a. When this shaded region exists, then b can send a light signal to a.

with a fresh space-time diagram of the pulse emitted by point 0 in X, as in Fig. 5.6, we see that the pulse effects at time t at some point a distance r from 0 in the medium arrive through the pastward sector of the point (r,t). In particular, the region of X contributing scattered flux of all orders to (r,t) is bounded by a(r,t), b(r,t), where we have written:

"a(r,t)" for (r-vt)/2 (1)

"b(r,t) for (r+vt)/2 (2)

For example, if r = 0, then the interaction region of X at each time t is an interval on X of length vt centered on 0. The route of radiant flux from 0 to point (r,t) may be quite devious. Two sample routes from 0 to (r,t) are shown by the

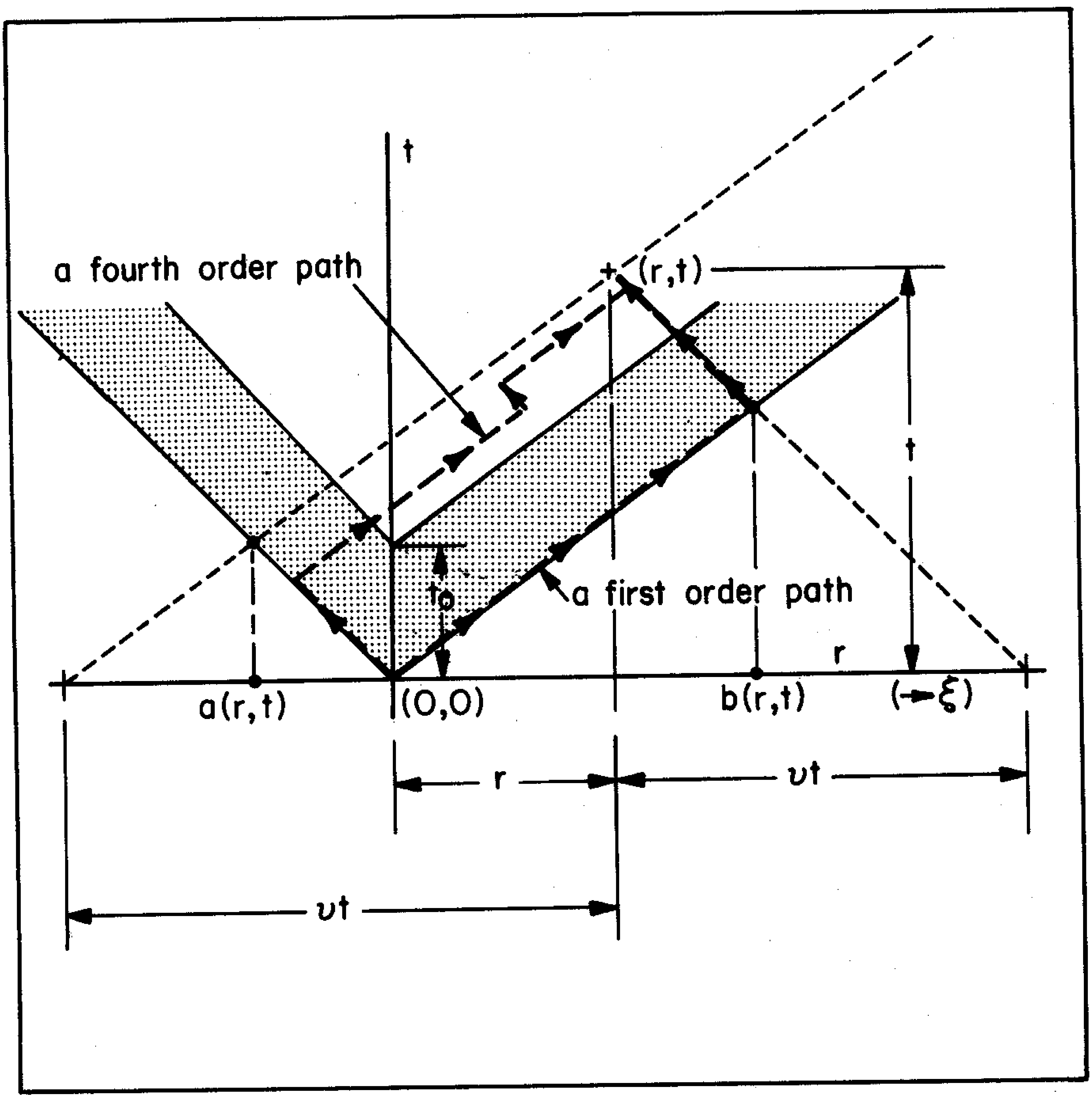

FIG. 5.6 Computing the scattered light reaching space-time point (r,t) after starting from the origin (0.0).

dashed lines in Fig. 5.6. In one of the cases the flux reaching (r,t) is intended to be fourth order radiant flux. The spatial component of the path taken by this sample of radiant flux is obtained by projecting the space-time path onto X. Observe that in this particular example the only way radiant flux can reach (r,t) from 0 is by undergoing at least one back scattering operation.

The Equation of Transfer

The integral form of the equation of transfer for the one-dimensional optical medium X defined above will now be derived. Before going into the details, however, it may be well to reemphasize that the significance of a one-dimensional optical medium lies not so much in its power to represent an actual physical setting as it does in its ability to depict with a minimum of geometric complication the essential algebraic structures of the associated three-dimensional problem. Therefore, the resultant equation of transfer derived below for the present one-dimensional setting will, in all its algebraic essentials, be representative of the full three-dimensional case, but will not be encumbered with details arising from the latter's relatively complex geometrical structure. These details will be faced in the following section.

Under suitably adapted definitions of the radiance function and inherent optical properties for X, the equation of transfer for the one-dimensional optical medium X follows formally from the integral form of (4) of Sec. 3.15. In this way we extend the logical chain from the interaction principle of Chapter 3 to the present radiative transfer discussion. In particular the present equation of transfer is obtained by postulating the characteristic form of the volume scattering function for one-dimensional media:

$$\sigma(x;\xi';\xi,t) = \rho(x,t)\ \delta(\xi+\xi') + \tau(x,t)\ \delta(\xi'-\xi)$$

where ξ is one of the two directions $(\pm\xi)$ along the medium, and δ is the well-known Dirac-delta function. The functions ρ and τ are, respectively backward and forward scattering functions for X. Furthermore, the values of the radiance function are now of the form $N(x,t,+)$ or $N(x,t,-)$, where "+" and "-" denote flux in the direction + or -, respectively. That is, we have written:

$$\text{"}N(x,\xi',t)\text{"} \quad \text{for} \quad N(x,t,+)\ \delta(\xi'-\xi) + N(x,t,-)\ \delta(\xi'+\xi)$$

Since the points x in X are located by one number only, namely the signed distance r from 0 to x, we will write "r" in place of "x" throughout the one-dimensional setting. It now follows from (8) of Sec. 3.14 with the adopted form of σ and N (and assuming here only that δ is idempotent, i.e., $\delta^2 = \delta$, at least formally) that the path function values $N_*(r,t,\pm)$ associated with directions $\pm$ are:

$$N_*(r,t,+) = N(r,t,+)\ \tau(r,t) + N(r,t,-)\ \rho(r,t) \qquad (3)$$

$$N_*(r,t-) = N(r,t,-)\ \tau(r,t) + N(r,t,+)\ \rho(r,t) \qquad (4)$$

The time-dependent integral form of the equation of transfer for the one-dimensional case therefore consists of the following two equations (one for each direction (+,-):

$$N(r,t,+) = u(r)N_o(0,t - |r/v|, +)\ T_r + \int_{a(r,t)}^{r} N_*(r',t',+)T_{r-r'}\,dr' \qquad (5)$$

$$N(r,t,-) = u(-r)N_o(0,t-|r/v|, -)\ T_{-r} + \int_{r}^{b(r,t)} N_*(r',t',-)T_{r'-r}dr' \qquad (6)$$

where $u(r) = 1$ if $r \geq 0$, and $u(r) = 0$ if $r < 0$. All terms except the transmittance terms in these two equations have been defined in the present section. The transmittances are represented as in (3) of Sec. 3.11; thus for the present case we have:

$$\text{"}T_{s-r}\text{" for } \exp\left\{-\int_r^s \alpha\,dr'\right\}$$

in which matters are arranged so that $r \leq s$.

Operator Form of the Equation of Transfer

We next cast the pair of transfer equations (5), (6) into an operator form which at once suggests the appropriate instance of the natural solution for the present case. Thus, we agree to write:

$$\text{"}N_+^o(r,t)\text{" for } u(r)N_o(0,t-|r/v|,+)\ T_r$$

$$\text{"}N_-^o(r,t)\text{" for } u(-r)N_o(0,t-|r/v|,-)\ T_{-r}$$

and further, we write:

$$\text{"}T_+\text{" for } \int_{a(r,t)}^{r} [\ ]\tau T_{r-r'}\,dr'$$

"R_-" for $\int_{a(r,t)}^{r} [\ ]\rho T_{r-r'}\, dr'$

"T_-" for $\int_{r}^{b(r,t)} [\ ]\tau T_{r'-r}\, dr'$

"R_+" for $\int_{r}^{b(r,t)} [\ ]\rho T_{r'-r}\, dr'$

With these assignations, (5), (6) become:

$$N(r,t,+) = N_+^o(r,t) + NT_+(r,t) + NR_-(r,t)$$

$$N(r,t,-) = N_-^o(r,t) + NT_-(r,t) + NR_+(r,t)$$

The notation "$NT_+(r,t)$", e.g., denotes the value of the function NT_+ at (r,t), and NT_+ is the result of acting on N with the operator T_+. These equations may be made more compact and at the same time more algebraic in appearance by writing:

"N_+" for $N(\cdot,\cdot,+)$

"N_-" for $N(\cdot,\cdot,-)$

"N_+^o" for $N_+^o(\cdot,\cdot)$

"N_-^o" for $N_-^o(\cdot,\cdot)$.

With these abbreviations for the four radiance functions we then can write (5) and (6) as:

$$N_+ = N_+^o + N_+T_+ + N_-R_- \qquad (7)$$

$$N_- = N_-^o + N_-T_- + N_+R_+ \qquad (8)$$

This form of the equation of transfer now suggests that we write:

$$"S" \quad \text{for} \quad \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix} \tag{9}$$

along with:

$$"N" \quad \text{for} \quad (N_+, N_-) \tag{10}$$

and

$$"N^o" \quad \text{for} \quad (N_+^o, N_-^o) \tag{11}$$

so that the system (7) and (8) written in vector notation becomes:

$$(N_+ N_-) = (N_+^o, N_-^o) + (N_+, N_-)S \tag{12}$$

or, succinctly:

$$\boxed{N = N^o + NS} \tag{13}$$

In this way we have reattained the basic structure of the integral equation of transfer, now for the simple one-dimensional context (recall, e.g., the derivation of (4) of Sec. 5.4). It follows that we may at once apply the natural solution procedure to (13) and thereby compute directly the scattering order components of N to as great a degree of accuracy as desired. This will now be done.

The Natural Solution

Starting with equation (13), and treating N as if it were an unknown in a simple linear algebraic equation we obtain:

$$N = N^o(I-S)^{-1}$$

where $(I-S)^{-1}$ may be shown to be expandable into an infinite series:

$$(I-S)^{-1} = I+S+S^2+S^3+\ldots \tag{14}$$

We have encountered such a type of expansion several times before in the present work. For instance it was used in Example 15 of Sec. 2.11, and it occurred many times in the examples of Chapter 3. Finally, closely related series were encountered earlier in this chapter (see (2) of Sec. 5.4). Hence the requisite solution of the time-dependent equation of transfer for the one-dimensional optical medium takes the form:

$$(N_+,N_-) = \sum_{j=0}^{\infty} (N_+^o,N_-^o)S^j \tag{15}$$

An Example

As an illustration of the natural solution for the present one-dimensional optical ringing problem suppose the medium X is homogeneous and in the steady state, so that ρ and τ are constant valued functions over space and time. Suppose further that N_+^o and N_-^o are each constant valued and over a time period from $t = 0$ to $t = t_o > 0$ (a slight simplification occurs if these are of Dirac-delta temporal structure; however, a temporally finite pulse, is at present a more useful and realistic input for X, and accordingly is adopted). Then, carrying out the expansion (15) to second order scattering, we have:

$$(N_+,N_-) = (N_+^o,N_-^o) + (N_+^o,N_-^o)S + (N_+^o,N_-^o)S^2 \tag{16}$$

Since

$$S^2 = \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix}^2 = \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix} \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix}$$

$$= \begin{bmatrix} T_+^2+R_+R_- & T_+R_++R_+T_- \\ R_-T_++T_-R_- & R_-R_++T_-^2 \end{bmatrix} ,$$

we have from (16) for the first component N_+ of the vector (N_+N_-):

$$N_+ = N_+^o + \left[N_+^oT_+ + N_-^oR_-\right] + \left[N_+^o(T_r^2 + R_+R_-) + N_-^o(R_-T_+ + T_-R_-)\right] \tag{17}$$

and for the second component N_- of the vector (N_+,N_-):

$$N_- = N_-^o + \left[N_+^oR_+ + N_-^oT_-\right] + \left[N_+^o(T_+R_+ + R_+T_-) + N_-^o(R_-R_+ + T_-^2)\right] \tag{18}$$

Equations (17) and (18) show how the natural solution (15) can be constructed order by order for an evolution of (N_+,N_-). If still another scattering order is needed, we include S^3:

$$S^2S = \begin{bmatrix} T_+^2 + R_+R_- & T_+R_+ + R_+T_- \\ R_-T_+ + T_-R_- & R_-R_+ + T_-^2 \end{bmatrix} \begin{bmatrix} T_+ & R_+ \\ R_- & T_- \end{bmatrix}$$

$$= \begin{bmatrix} T_+^3 + R_+R_-T_+ + T_+R_+R_- + R_+T_-R_- & T_+^2R_+ + R_+R_-R_+ + T_+R_+T_- + R_+T_-^2 \\ R_-T_+^2 + T_-R_-T_+ + R_-R_+R_- + T_-^2R_- & R_-T_+R_+ + T_-R_-R_+ + R_-R_+T_- + T_-^3 \end{bmatrix}$$

To show how the second order operators in (17) and (18) are applied in practice, let us assume explicitly that $N_o(0,t,-) = 0$ for all t, and that N is the constant value of the radiance pulse $N_o(0,t,+)$ of duration t_o, starting at $t = 0$, in the direction ξ, i.e., of increasing r. The present situation then constitutes an approximate model of the light field generated by a laser-like beam pulse of duration t_o seconds in the immediate vicinity of the beam. The outgoing field N_+ evaluated at $r = 0$ for every $t \geq 0$ is then, according to (17):

$$N(0,t,+) = N_o(0,t,+) + N_+^oT_+(0,t) + N_+^o(T_+^2 + R_+R_-)(0,t) \tag{19}$$

The incoming field N_- evaluated at $r = 0$ for every $t \geq 0$ is, according to (18):

$$N(0,t,-) = N_+^oR_+(0,t) + N_+^o(T_+R_+ + R_+T_-)(0,t) \tag{20}$$

In each of these equations, we have $N_o(0,t,+) = N$ for $0 \leq t \leq t_o$ and $N_o(0,t,+) = 0$ for every other t.

Let us consider (20) in more detail. The first order scattering term, unraveled, becomes:

$$N_+^oR_+(r,t) = \int_r^{b(r,t)} N_+^o(r',t')\ \rho(r',t')T_{r',-r}\ dr' \ , \tag{21}$$

in which we are to set $r = 0$, and $t' = t-|r'/v|$. A study of part (a) of Fig. 5.7, which depicts the present situation, and a study of the definitions N_+^o and N_-^o, shows that this integral is best evaluated by establishing two cases: Case (i) pertains whenever $t < t_o$; Case (ii) pertains whenever $t > t_o$. The particular forms of (21) for these two cases are as follows. Case (i), ((0,t) in the pulse):

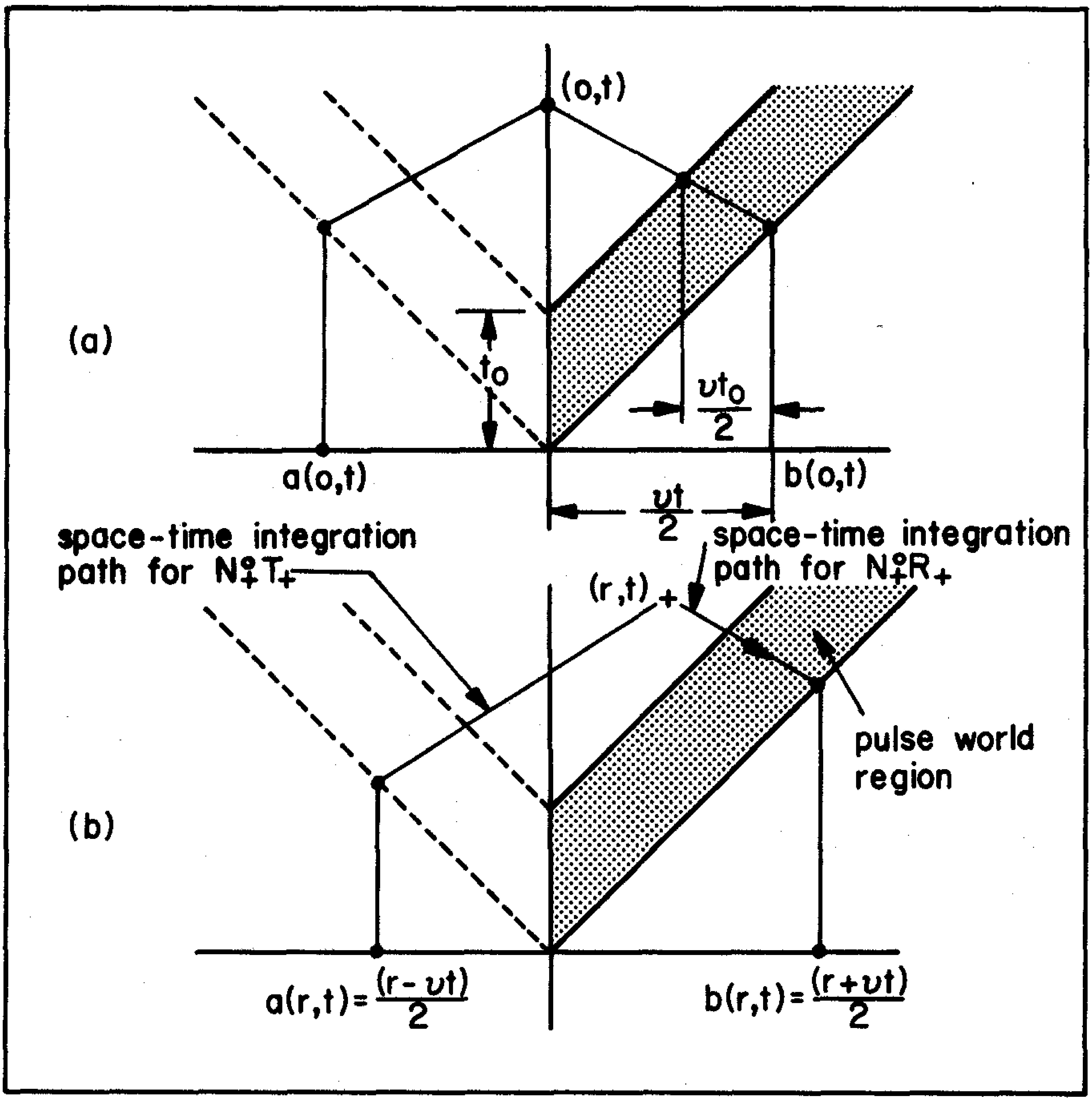

FIG. 5.7 Space-time path integration details.

$$N_+^oR_+(0,t) = N\rho \int_0^{vt/2} e^{-2\alpha r'} \, dr'$$

$$= \frac{N\rho}{2\alpha}\left(1 - e^{-\alpha vt}\right) \tag{22}$$

Case (ii), ((0,t) after the pulse):

$$N_+^o R_+(0,t) = N\rho \int_{(vt/2)-(vt_o/2)}^{vt/2} e^{-2\alpha r'}\, dr'$$

$$= \frac{N\rho\, e^{-\alpha vt}}{2\alpha}\left[e^{\alpha vt_o} - 1\right] \qquad (23)$$

Equations (22) and (23) describe the first order scattered radiance flowing in the negative direction of X, at $r = 0$.

For the radiance at a general space-time (r,t), we once again require two cases: Case (i) pertains when $(t-t_o)v \leq |r| \leq vt$; and Case (ii) pertains when $|r| < (t-t_o)v$. These cases reduce to the special instances considered above when $r = 0$. In general, Case (i) holds when the space-time point (r,t) is in the world region of the pulse; Case (ii) holds when (r,t) is futureward (above or after) the world region of the pulse. Returning now to (21), we evaluate it for a general point (r,t), according to the two cases ((b) of Fig. 5.7): Case (i), ((r,t) in the pulse):

$$N^o R_+(r,t) = N\rho \int_r^{(r+vt)/2} e^{-\alpha r'}\, e^{-\alpha(r'-r)}\, dr'$$

$$= \frac{N\rho}{2\alpha}\left[e^{-\alpha r} - e^{-\alpha vt}\right] \qquad (24)$$

Case (ii), ((r,t) after the pulse):

$$N_+^o R_+(r,t) = N\rho \int_{\frac{(r+vt)}{2} - \frac{vt_o}{2}}^{(r+vt)/2} e^{-\alpha r'}\, e^{-\alpha(r'-r)}\, dr'$$

$$= \frac{N\rho e^{-\alpha vt}}{2\alpha}\left[e^{\alpha vt_o} - 1\right] \qquad (25)$$

Equations (24) and (25) describe the primary scattered radiance in the direction $-\xi$ in X at a general space-time point (r,t) such that $r \leq vt$. For $r > vt$, the primary radiance is clearly zero, as may be seen by reviewing the geometry of the space-time plane discussed earlier. Furthermore, this value is approached by (24) as (r,t) approaches the lower boundary of the pulse's world region, i.e., the line defined by $r = vt$. Hence $N_+^o R_+$ is uniquely defined throughout the whole space-time diagram.

We turn next to illustrate the evaluation of the second order scattering terms in (20). We first consider $N_+^o T_+ R_+$. This is interpreted to be the result of the operation of R_+

on $N_+^oT_+$. The latter, in turn, gives the primary scattered radiance in the direction + for a general space-time point (r,t):

$$N_+^oT_+(r,t) = \int_{a(r,t)}^{r} N_+^o(r',t')\ \tau(r',t')\ T_{r-r'}\ dr'$$

in which we are to set t' = t-r'/v. A study of Fig. 5.7 shows that, for the present source condition, we have $N_+^o(r',t') = 0$ for r' < 0 (no source radiant flux in the direction + at any time for points r' < 0). Hence the integration may begin at r' = 0, instead of a(r,t) (=(r-vt)/2). Furthermore, $\tau(r',t')$ is constant of fixed value τ for all r' and t'. Hence, Case (i), ((r,t) in the pulse):

$$N_+^oT_+(r,t) = N\tau \int_0^r e^{-\alpha r'}\ e^{-\alpha(r-r')}\ dr'$$

Hence:

$$N_+^oT_+(r,t) = N\tau r\ e^{-\alpha r} \tag{26}$$

Case (ii), ((r,t) after the pulse):

$$N_+^oT_+(r,t) = 0 \tag{27}$$

Equations (26) and (27) give the primary scattered radiance in the direction $+\xi$ at a general space-time point (r,t) futureward of the origin (0,0).

We are now ready to evaluate the second order terms. Thus we have, Case (i), ((r,t) in the pulse):

$$N_+^oT_+R_+(r,t) = \int_r^{(r+vt)/2} N_+^oT_+(r',t')\ \rho(r',t')\ T_{r'-r}\ dr'$$

$$= N\tau\rho \int_r^{(r+vt)/2} r'\ e^{-\alpha r'} \cdot e^{-\alpha(r'-r)}\ dr'$$

$$= N\tau\rho\ e^{\alpha r} \int_r^{(r+vt)/2} r'\ e^{-2\alpha r'}\ dr'$$

$$= \frac{N\tau\rho}{4\alpha^2} \left\{ e^{-\alpha r}[1+2\alpha r] - e^{-\alpha vt}[1+\alpha(r+vt)] \right\} \tag{28}$$

Case (ii), ((r,t) after the pulse):

$$N_+^oT_+R_+(r,t) = \int_{\frac{(r+vt)}{2}-\frac{vt_o}{2}}^{\frac{(r+vt)}{2}} N_+^oT_+(r',t')\ \rho(r',t')\ T_{r'-r}\ dr'$$

$$= N\tau\rho\ e^{\alpha r} \int_{\frac{(r+vt)}{2}-\frac{vt_o}{2}}^{\frac{(r+vt)}{2}} r'\ e^{-2\alpha r'}\ dr'$$

$$= \frac{-N\tau\rho}{4\alpha^2}\ e^{-\alpha vt} \left\{ [1+\alpha(r+vt)] - e^{\alpha vt_o}[1+\alpha(r+vt)-\alpha vt_o] \right\}$$

(29)

The final term in the second order expansion of $N(0,t,-)$ as given in (20) is $N_+^oR_+T_-$, that is, the result of operating on $N_+^oR_+$ with T_-. Once again it is convenient to consider two cases: Case (i), ((r,t) in the pulse):

$$N_+^oR_+T_-(r,t) = \int_r^{b(r,t)} N_+^oR_+(r',t')\ \tau(r',t')\ T_{r'-r}\ dr'$$

$$= \frac{N\rho\tau}{2\alpha} \int_r^{b(r,t)} [e^{-\alpha r'} - e^{-\alpha vt'}]\ e^{-\alpha(r'-r)}\ dr'$$

$$= \frac{N\rho\tau e^{\alpha r}}{2\alpha} \int_r^{\frac{(r+vt)}{2}} \left[e^{-2\alpha r'} - e^{-\alpha vt'}\right] dr'$$

$$= \frac{N\rho\tau}{4\alpha^2} \left\{ e^{-\alpha r} - e^{-\alpha vt} + \alpha e^{\alpha(r-vt)}[r-vt] \right\} \quad (30)$$

Case (ii), ((r,t) after the pulse):

$$N_+^oR_+T_-(r,t) = \int_r^{b(r,t)} N_+^oR_+(r',t')\ \tau(r',t')\ T_{r'-r}\ dr'$$

$$= \int_r^{b(r,t)-\frac{vt_o}{2}} N_+^oR_+(r',t')\ \tau(r',t')\ T_{r'-r}\ dr' +$$

$$+ \int_{b(r,t)-\frac{vt_o}{2}}^{b(r,t)} N_+^oR_+(r',t')\ \tau(r',t')\ T_{r'-r}\ dr' \qquad (31)$$

The integration in Case (ii) is shown split into two parts: that part of the integration over the segment of the space-time path after the pulse, and that over the segment of the space-time path in the pulse. The result of an integration over the futureward region of the pulse is in general not zero for secondary and higher order scattering.

The first integral in (31) uses Case (ii) for $N_+^oR_+$ evaluated in (25), and the second integral uses Case (i) above by replacing the lower limit in (30) by $(r+vt/2)-(vt_o/2)$. The requisite value N(0,t,-) is now obtained by setting $r = 0$ in the appropriate cases in (24), (25), (28), (29), (30), (31) and adding the appropriate terms, in accordance with (20).

Concluding Observations

We have carried the evaluation of N(0,t,-) far enough to show the essentials of the natural solution procedure for the one-dimensional time-dependent problem. It should be particularly noted how each step builds on the preceding step and--manipulative difficulties aside--how each step is in principle directly constructable in a finite number of operations using elementary calculus. With the advent of ever more sophisticated symbolic manipulation programs for general purpose electronic computers, it should eventually be possible to have a program which would permit the *symbolic term-by-term integration of the natural solution series* (15). We have carried the solution of the present problem far enough to show that only integrals of the type

$$\int_b^c r^n e^{-ar}\ dr$$

will be encountered in the natural solution for one-dimensional time-dependent radiative transfer problems on homogeneous spaces. With such general information a program should in principle be possible which combines simple algebraic and calculus manipulations, and which will give the two components of the nth term of (15) mechanically and relatively quickly. By having the machine run out several more terms than the second order, obtained so laboriously above, a trained human looking at the emerging terms could perhaps discern a pattern in this (or subsequently more complex problems) and thereby prepare for an inductive leap to the general term of the series. The advantages of *symbolic* over numerical integration are obvious. The former is exact at each stage whereas the latter is plagued by cumulative round-off errors. Once a symbolic integration has been performed, it may then be evaluated for the particular numerical case of interest.

One final observation can be made about the natural solution of one-dimensional time-dependent problems. This concerns extension of the analogy between the class of acoustical and optical reverberations, or as they are more commonly called, "electrical circuit transients." By studying the Laplace transform techniques of solving the problems of transients in electrical circuits (see, e.g., Chapter IX of Ref. [39]), one sees the possibility of interpreting certain terms in the final solution as analogous to the nth order scattering terms developed above. This suggests the possibility of a thoroughgoing theory, built along natural-solution lines, which should underlie and unify the particular ringing problems in the fields of optics, acoustics, transmission-line theory and electromagnetics. Mathematicians can view this as extensions of the Neumann series to space-time linear settings. An approach to such a unification can be based on the formalities developed in the present chapter since many of the operator equations appearing here are clearly interpretable in terms of the concepts of each of the preceding fields.

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5.7 Optical Ringing Problem. Three-Dimensional Case

We examine next how the natural mode of solution of the equation of transfer can be applied to the problem of determining the time-dependent radiance field in a natural optical medium. The program to be followed here is that which systematically generalizes the developments of Sec. 5.1 to the time-dependent case; in particular the generalizations of the **R** and **T** operators will be the key steps in the present discussion. We begin by introducing an important geometrical concept connected with the time-dependent problem.

The Characteristic Ellipsoid

Let x and y be two points in an extensive natural optical medium X. Suppose that at time $t = 0$, a spherical pulse of light is emitted from x. This pulse expands about x as center and at time r/v passes point y, where r is the distance from x to y. Here v is the speed of light in X, assumed independent of location and time throughout this discussion. Just after the wave front of the pulse passes y, a