will be encountered in the natural solution for one-dimensional time-dependent radiative transfer problems on homogeneous spaces. With such general information a program should in principle be possible which combines simple algebraic and calculus manipulations, and which will give the two components of the nth term of (15) mechanically and relatively quickly. By having the machine run out several more terms than the second order, obtained so laboriously above, a trained human looking at the emerging terms could perhaps discern a pattern in this (or subsequently more complex problems) and thereby prepare for an inductive leap to the general term of the series. The advantages of *symbolic* over numerical integration are obvious. The former is exact at each stage whereas the latter is plagued by cumulative round-off errors. Once a symbolic integration has been performed, it may then be evaluated for the particular numerical case of interest.

One final observation can be made about the natural solution of one-dimensional time-dependent problems. This concerns extension of the analogy between the class of acoustical and optical reverberations, or as they are more commonly called, "electrical circuit transients." By studying the Laplace transform techniques of solving the problems of transients in electrical circuits (see, e.g., Chapter IX of Ref. [39]), one sees the possibility of interpreting certain terms in the final solution as analogous to the nth order scattering terms developed above. This suggests the possibility of a thoroughgoing theory, built along natural-solution lines, which should underlie and unify the particular ringing problems in the fields of optics, acoustics, transmission-line theory and electromagnetics. Mathematicians can view this as extensions of the Neumann series to space-time linear settings. An approach to such a unification can be based on the formalities developed in the present chapter since many of the operator equations appearing here are clearly interpretable in terms of the concepts of each of the preceding fields.

5.7 Optical Ringing Problem. Three-Dimensional Case

We examine next how the natural mode of solution of the equation of transfer can be applied to the problem of determining the time-dependent radiance field in a natural optical medium. The program to be followed here is that which systematically generalizes the developments of Sec. 5.1 to the time-dependent case; in particular the generalizations of the **R** and **T** operators will be the key steps in the present discussion. We begin by introducing an important geometrical concept connected with the time-dependent problem.

The Characteristic Ellipsoid

Let x and y be two points in an extensive natural optical medium X. Suppose that at time $t = 0$, a spherical pulse of light is emitted from x. This pulse expands about x as center and at time r/v passes point y, where r is the distance from x to y. Here v is the speed of light in X, assumed independent of location and time throughout this discussion. Just after the wave front of the pulse passes y, a multiply-scattered radiant flux field is generally incident on y from all directions about y. We now ask: What is the region of points in X which can send radiant flux to y at an arbitrary time $t > r/v$? It is easy to see that at exactly $t = r/v$, this region is the straight line segment between x and y. Any points x of X off this line segment could not send scattered flux to y because the detour, however, slight, would delay the scattered flux's arrival time at y. For times t of arrival at y such that $t > r/v$, such detours are possible to some extent. The region in which the scattering detours are possible and which allow arrival at y at time t is generally an ellipsoid of revolution with x and y as foci. This may be seen by studying Fig. 5.8, and recalling that definition of an ellipsoid which characterizes it as the locus of points z such that the sum of distances $d(x,z) + d(z,y)$ is a constant.

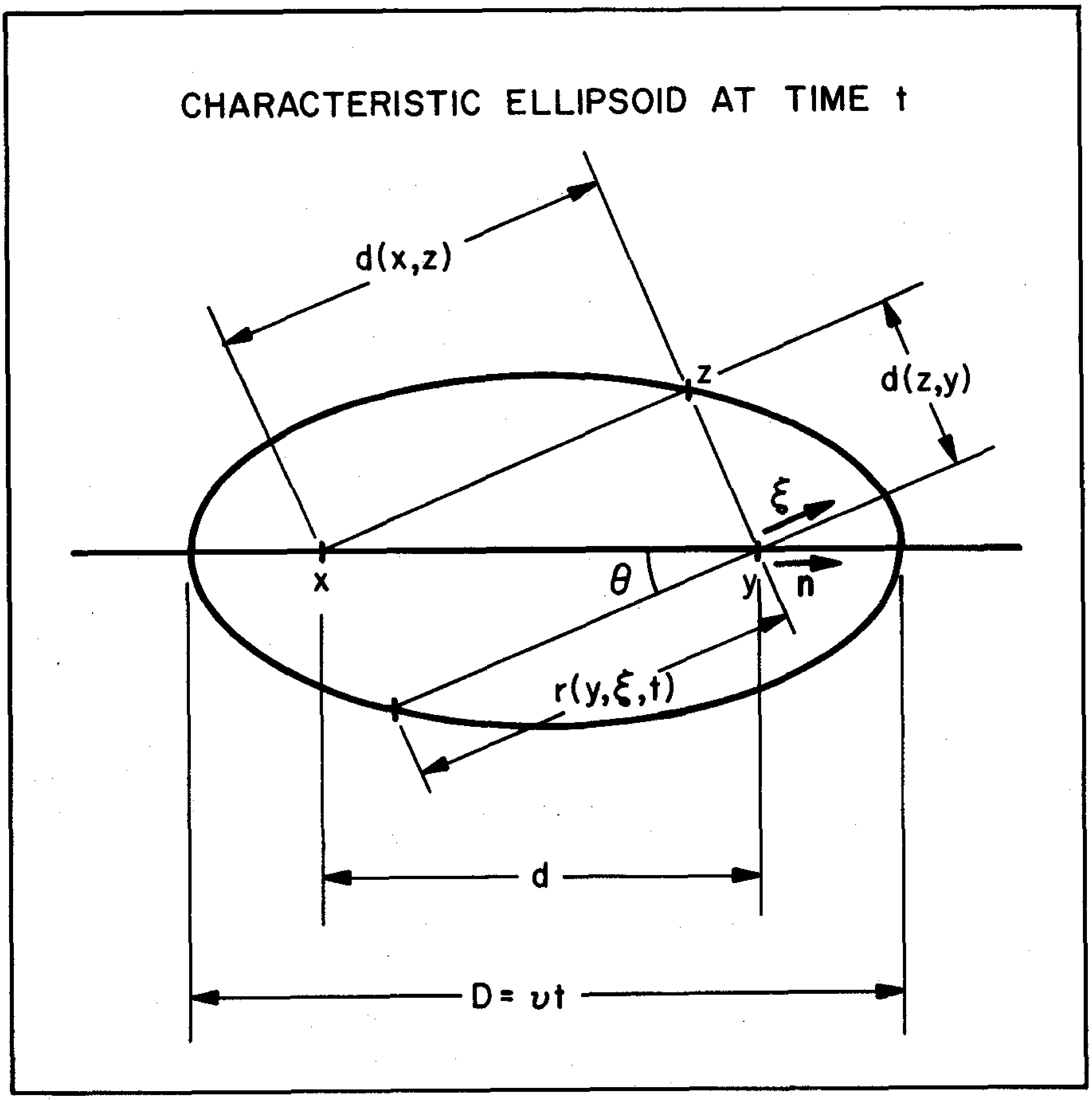

FIG. 5.8 The characteristic ellipsoid relative to the source at x and receiver at y at time t.

For the case at hand these distances are all initially considered in terms of times of travel $t(x,z)$ and $t(z,y)$ across the respective distances and we are interested in all those points z in X such that:

$$d(x,z) + d(z,y) = vt \qquad (1)$$

This defines at each instant $t \geq r/v$ an ellipsoid of revolution in X, with foci x and y. From (1) we see that the major axis of the ellipsoid is of length vt. We call the ellipsoid so defined, the *characteristic ellipsoid* $\mathcal{E}(x,y;t)$ associated with x and y at time $t \geq r/v$. A useful polar representation of $\mathcal{E}(x,y;t)$ with y as pole, is given by the equation:

$$r(y,\xi,t) = \frac{D^2 - d^2}{2(D - d\cos\theta)} \qquad (2)$$

where θ is the angle between the unit vectors ξ and $\mathbf{n}$, as in Fig. 5.8, and where we have written:

"D" for vt

"d" for d(x,y)

The eccentricity ε of the characteristic ellipsoid $\mathcal{E}(x,y;t)$ turns out to be d/D. At time t such that $t = d(x,y)/v = r/v$, we have $\varepsilon = 1$. As time increases indefinitely, ε decreases to zero, so that--if the space is infinite in all directions about y--the characteristic ellipsoid approaches a sphere which takes on very nearly the polar form:

$$r(y,\xi,t) \cong \frac{D}{2} = \frac{vt}{2}$$

The exact spherical form of $\mathcal{E}(x,y;t)$ occurs at finite times if $x = y$, i.e., whenever $d = 0$. In such a case, $\mathcal{E}(x,x;t)$ becomes the *characteristic spheroid* $S(x;t)$ with radius vt/2.

Time-Dependent **R** and **T** Operators and the Natural Solution

With the necessary geometrical preliminaries out of the way we can now adapt the **R** and **T** operators of Sec. 5.1 to the time-dependent case. We shall limit the present discussion to a homogeneous steady medium X with point source at a fixed point 0 and such that the characteristic ellipsoid $\mathcal{E}(0,x;t)$ is contained in X for all t under discussion. We shall then write:

$$\text{"}\mathbf{R}\text{"} \quad \text{for} \quad \int_{\Xi} [\ ]\,\sigma(x;\xi';\boldsymbol{\xi})\, d\Omega(\xi')$$

and:

$$\text{"}T\text{"} \quad \text{for} \int_{0}^{r(x,\xi,t)} [\ ]T_{r-r'}(x',\xi)\,dr' \tag{4}$$

Comparing this pair of operators with their namesakes in Sec. 5.1, we see that the essential difference between the two pairs rests in the limit of integration for **T**. Now we can limit the integration to the characteristic ellipsoid $\mathcal{E}(0,s;t)$, whereas before (see Fig. 5.1) the limit of integration for **T** was generally the distance from x to the boundary of X in the direction $-\xi$.

If we go on to write:

$$\text{"}S^1\text{"} \quad \text{for} \quad \mathbf{RT}$$

and then: (5)

$$\text{"}N^{n+1}\text{"} \quad \text{for} \quad N^n S^1$$

for every $n \geq 0$, it follows that we can construct the time-dependent natural solution for the time-dependent equation of transfer (4) of Sec. 3.15, just as in 5.4. In particular the solution verification may be repeated line for line and culminating as in (4) of Sec. 5.4, with the form:

$$\boxed{N(x,\xi,t) = N^o(x,\xi,t) + N^*(x,\xi,t)} \tag{5a}$$

but now each term has a time-dependent interpretation.

Truncated Natural Solution

Just as in the steady case in Sec. 5.5 we may now truncate the time-dependent natural solution and obtain an estimate of the accuracy of the truncated solution. It turns out that the truncation estimates of the time-dependent solution can be much sharper than their steady state counterparts, owing to the use of the characteristic ellipsoid in the time-dependent computations. In this discussion suppose the source starts at $t = 0$ and emits in an arbitrary manner thereafter. The light field sweeps out from 0 as center in the form of a spherical field, building up radiant flux of all scattering orders within the sphere as time goes on.

Let $\bar{N}^o$ be the maximum (or supremum, if need be) of the initial radiance function N^o over the sphere of radius vt, center 0. See Fig. 5.9. Then observe that:

$$N^o S^1(x,\xi,t) \leq \bar{N}^o \rho(1-e^{-\alpha r(\max)}) \tag{6}$$

for every ξ in Ξ at x and time t, where $\rho = s/\alpha$ and where we have written:

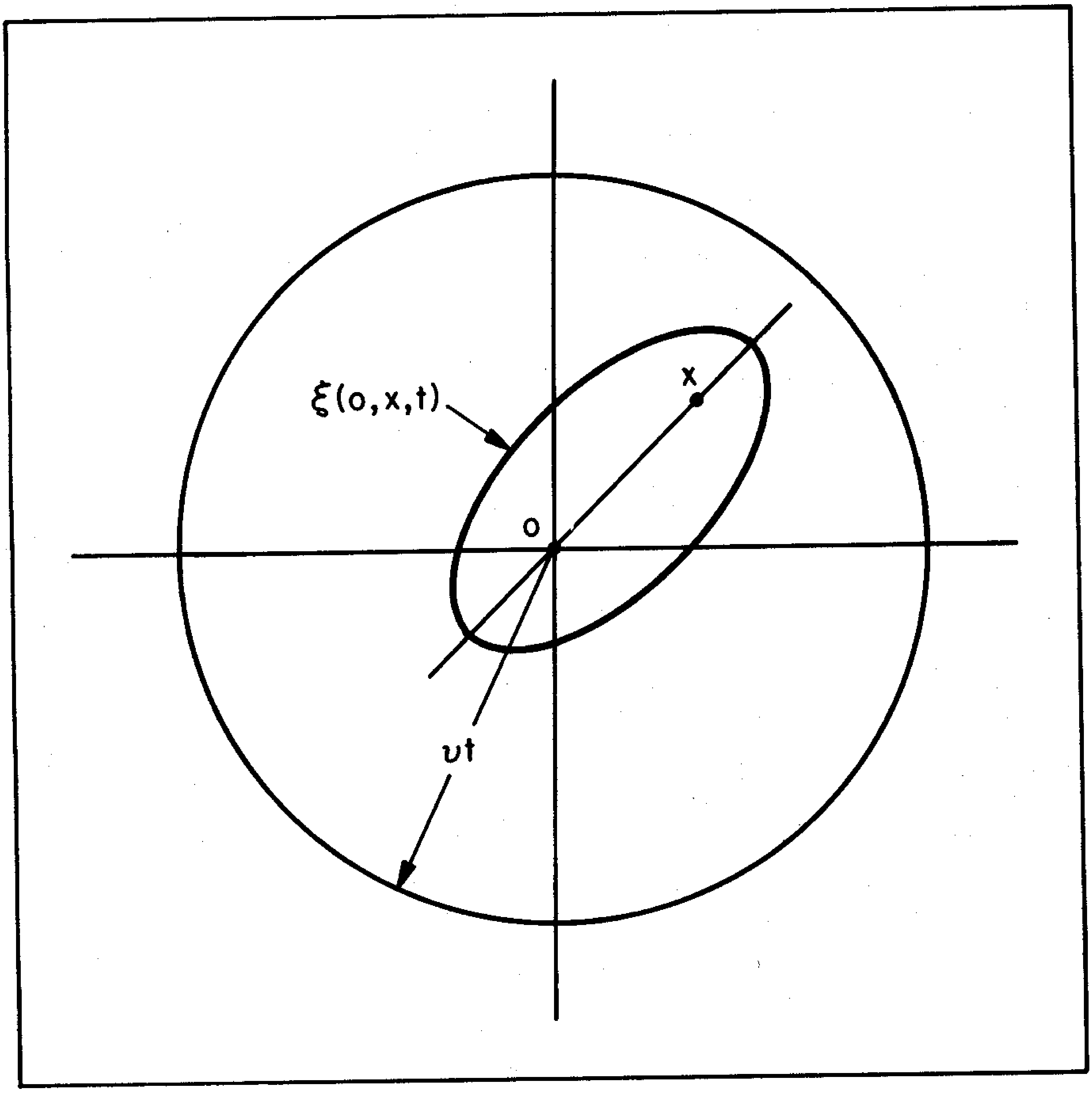

FIG. 5.9 The spherical wave front of the pulse has radius vt. The characteristic ellipsoid relative to 0 and x at time t defines those points of the medium which can send flux to x from 0 at time t.

$$\text{"r(max)"} \quad \text{for} \quad \max_{\xi \in \Xi} r(x,\xi,t)$$

Hence:

$$r(\max) = (D + d)/2 \quad , \quad D = vt$$

By letting x vary over the spherical region of radius vt, center 0, (6) leads to:

$$N^1(x,\xi,t) = N^o s^1(x,\xi,t) \leq \bar{N}^o \rho(1-e^{-\alpha vt}) \quad , \qquad (7)$$

for every x in X and ξ in Ξ. This may be compared with (3) of Sec. 5.5. Using (7) we can estimate the upper bound of primary scalar irradiance and radiant energy over X in terms of that of residual scalar irradiance or radiant energy. Using the basic idea contained in (7), we can construct a chain of inequalities for n-ary radiances. For (7) yields an upper bound of primary radiance over the sphere of radius vt, center 0, and this upper bound now can be turned around to play the role of $\bar{N}^o$ in the estimate of the next scattering order, namely, $N^2(x,\xi,t)$. Thus in general, since:

$$N^n = N^{n-1}s^1$$

it readily follows that:

$$\boxed{N^n(x,\xi,t) \le \bar{N}^o[\rho(1-e^{-\alpha vt})]^n} \qquad (8)$$

for every x in X, ξ in Ξ, and integer $n \ge 0$. This inequality reduces to (5) of Sec. 5.5 in the steady-state, i.e., when $t \to \infty$. The inequality (8) shows that for x sufficiently close to 0 and for small times t,

$$N^n(x,\xi,t) \cong (svt)^n\bar{N}^o \qquad (9)$$

where s is the total volume scattering function.

Now, just as in the steady state case of Sec. 5.5, we can estimate the error of truncation of the natural solution series. Thus using (8), we have:

$$N(x,\xi,t) - N^{(k)}(x,\xi,t) = \sum_{j=k+1}^{\infty} [\rho(1-e^{-\alpha vt})]^j$$

$$\le \bar{N}^o \sum_{j=k+1}^{\infty} [\rho(1-e^{-\alpha vt})]^j$$

Hence:

$$\boxed{N(x,\xi,t) - N^{(k)}(x,\xi,t) \le \bar{N}^o \frac{[\rho(1-e^{-\alpha vt})]^{k+1}}{1-[\rho(1-e^{-\alpha vt})]}} \qquad (10)$$

for every x in X, and ξ in Ξ at time t. For large times, (10) reduces to (6) of Sec. 5.5. The space and source conditions giving rise to this estimate are stated at the outset of this discussion.

It should now be a relatively simple matter to reduce the preceding analysis to pulselike sources at 0, such as that considered in Sec. 5.6. The general method of analysis and its results developed between (6) and (10), of course remain the same for such sources, but sharper time-dependent estimates of $\bar{N}^o$ are now possible. These truncation estimates are evidently capable of a large variety of treatments and with the general mode of analysis now clear, each special case is best left to individual treatment by the interested investigator.

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5.8 Transport Equations for Residual, Directly Observable, and n-ary Radiant Energy

In this section we shall prepare the way for the extension of the concept of the natural solution of the equation of transfer to the radiant energy field in an optical medium. We shall derive from the time-dependent equations of transfer for the n-ary radiances the corresponding time-dependent transport equations for n-ary radiant energy. We shall eventually find that the latter equations are completely solvable in terms of simple closed algebraic forms in all homogeneous optical media. This fact will allow an important insight into the structure of the associated time-dependent radiance field in the same medium, and thereby shed further light on the difficult optical ringing problem in natural optical media, introduced in Secs. 5.6 and 5.7. We begin with a discussion and solution of the transport equation for zero-order radiant energy (or alternatively, the residual radiant energy) in an optical medium with an arbitrary source. Then the transport equations for nth order radiant energy will be derived along with the transport equations for directly observable radiant energy. Throughout this section the optical medium will be homogeneous with arbitrary sources of radiant flux distributed throughout. The volume scattering function is to be arbitrary but of fixed directional dependence, and unless otherwise specified the scattering-attenuation ratio ρ is also arbitrary but fixed, with $0 < \rho < 1$.

Residual Radiant Energy

In order to help fix the main ideas in the present discussion, let the optical medium X under consideration be depicted as in Fig. 5.10, that is, as an extensive region X with a boundary Y on each point y of which is incident a radiance distribution $N_0(y,\cdot)$ which may be extended into X to obtain initial radiance distributions $N^0(x,\cdot)$ at each point x in X, after the manner of (1) of Sec. 5.1. In the terminology of Sec. 3.10 (see, e.g., (4) of Sec. 3.10) $N^0(x,\xi)$ is the transmitted (or residual) radiance at x in the direction ξ. The alternative term "residual radiance" will be particularly appropriate in the context of the present discussion, and so is singled out for special use.

Suppose now that sources of radiant flux are present within X. This is a relatively new condition since (except for the brief discussion of example 3 of Sec. 3.9), no systematic explicit use of internal sources was required. We have now arrived at a point in our developments where the advent of the special radiometric concept needed for the description of internal sources takes place naturally. We therefore hypothesize the existence of an *emission radiance function* N_η, defined for each time t in some time period and at each point x in X, and direction ξ in Ξ. The dimensions of N_η are precisely those of N_* (radiance per unit length)