

with the general mode of analysis now clear, each special case is best left to individual treatment by the interested investigator.

5.8 Transport Equations for Residual, Directly Observable, and n-ary Radiant Energy

In this section we shall prepare the way for the extension of the concept of the natural solution of the equation of transfer to the radiant energy field in an optical medium. We shall derive from the time-dependent equations of transfer for the n-ary radiances the corresponding time-dependent transport equations for n-ary radiant energy. We shall eventually find that the latter equations are completely solvable in terms of simple closed algebraic forms in all homogeneous optical media. This fact will allow an important insight into the structure of the associated time-dependent radiance field in the same medium, and thereby shed further light on the difficult optical ringing problem in natural optical media, introduced in Secs. 5.6 and 5.7. We begin with a discussion and solution of the transport equation for zero-order radiant energy (or alternatively, the residual radiant energy) in an optical medium with an arbitrary source. Then the transport equations for nth order radiant energy will be derived along with the transport equations for directly observable radiant energy. Throughout this section the optical medium will be homogeneous with arbitrary sources of radiant flux distributed throughout. The volume scattering function is to be arbitrary but of fixed directional dependence, and unless otherwise specified the scattering-attenuation ratio ρ is also arbitrary but fixed, with $0 < \rho < 1$.

Residual Radiant Energy

In order to help fix the main ideas in the present discussion, let the optical medium X under consideration be depicted as in Fig. 5.10, that is, as an extensive region X with a boundary Y on each point y of which is incident a radiance distribution $N_0(y, \cdot)$ which may be extended into X to obtain initial radiance distributions $N^0(x, \cdot)$ at each point x in X , after the manner of (1) of Sec. 5.1. In the terminology of Sec. 3.10 (see, e.g., (4) of Sec. 3.10) $N^0(x, \xi)$ is the transmitted (or residual) radiance at x in the direction ξ . The alternative term "residual radiance" will be particularly appropriate in the context of the present discussion, and so is singled out for special use.

Suppose now that sources of radiant flux are present within X . This is a relatively new condition since (except for the brief discussion of example 3 of Sec. 3.9), no systematic explicit use of internal sources was required. We have now arrived at a point in our developments where the advent of the special radiometric concept needed for the description of internal sources takes place naturally. We therefore hypothesize the existence of an *emission radiance function* N_η , defined for each time t in some time period and at each point x in X , and direction ξ in \mathbb{E} . The dimensions of N_η are precisely those of N_* (radiance per unit length)

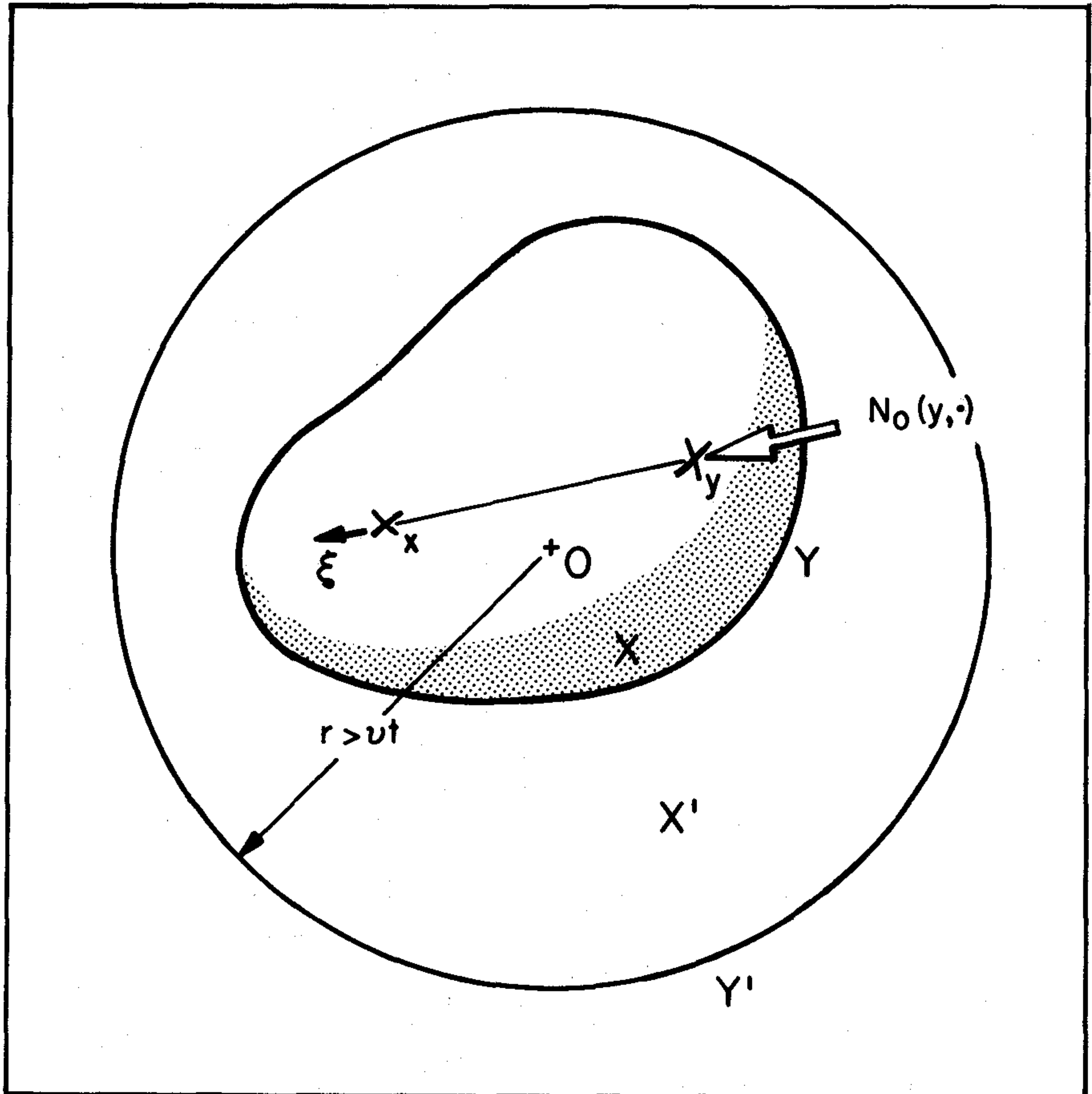


FIG. 5.10 Computing residual radiant energy in medium X.

and the use of N_η may be best understood by keeping this equality of dimensions in mind. Physically, $N_\eta(x, \xi, t)$ is intended to describe the radiance emitted at x and time t per unit length in the direction ξ . We envision $N_\eta(x, \xi, t)$ to be generated by some radiant emission mechanism in X . This mechanism generally takes two distinct forms, which may be in operation singly or simultaneously. These forms are described in Sec. 19 of Ref. [251] and therefore need not be repeated at length here. It suffices for our present purposes to observe that the radiance $N_\eta(x, \xi, t)$ arises generally either through scattering by change in frequency from an arbitrary frequency to the one under consideration, or through the emission processes of conversion of nonradiant energy to radiant energy.

When internal sources, characterized by means of an emission radiance function N_η , are present throughout a medium X , the initial radiance function N^0 is defined throughout X as follows. We write:

$$\begin{aligned}
 "N^0(z, \xi, t)" \text{ for } N_0(x, \xi, t-r/v) T_r(x, \xi) \\
 + \int_0^r N_\eta(x', \xi, t') T_{r-r'}(x', \xi) dr' \quad (1)
 \end{aligned}$$

This definition takes place in the same general geometrical setting of (2) of Sec. 3.10 and reduces to (2) of Sec. 3.10 when X is source-free and the light field is in the steady state. Here as usual $z = x + \xi r$, and $t' = t - r'/v$. A slightly more general definition can be written if X itself has changing inherent optical properties. Also, if scattering with change of frequency is to be explicitly taken into account, we may replace N_η by the true emission function N_e . The details of this more general definition of N^0 may be found in Sec. 22 of Ref. [251]. Such generality will not be required in any of our discussions, and so in the interests of simplicity of exposition, the present definition will be retained. Immediately forthcoming from (1) is the equation of transfer for initial radiance in the presence of internal sources:

$$\boxed{\frac{1}{v} \frac{\partial N^0}{\partial t} + \xi \cdot \nabla N^0 = -\alpha N^0 + N_\eta} \quad (2)$$

This is obtained by taking the lagrangian derivative of the definitional identity which (1) implies. That is, while following in imagination a photon packet along a natural path through X , we differentiate the right side of (1), by adapting the general procedure used to obtain equation (3) of Sec. 3.15 from equation (1) of that section. Now, we use D/Dt instead of d/dr , where D/Dt is defined in (5) of Sec. 3.15. Equation (2) is a direct generalization of (2) of Sec. 5.2.

We are now ready to define the notion of residual radiant energy and to establish its various analytical representations. By setting $n = 0$ in the definitions (16) and (17) of Sec. 5.1 we obtain the definitional identity:

$$U^0(X, t) = \frac{1}{v} \int_X \left[\int_{\Xi} N^0(x, \xi, t) d\Omega(\xi) \right] dV(X) \quad (3)$$

$U^0(X, t)$ is the *residual (or reduced or unattenuated) radiant energy* in X at time t . When X is understood and fixed throughout a discussion (as in the present one) its name may be dropped from the notation and we will write " $U^0(t)$ " for the residual radiant energy. The term "residual" is particularly well adapted to the photon interpretation of light. For in that interpretation, $U^0(t)$ is simply the radiant

energy content of X at time t associated with photons which have not been scattered or absorbed relative to the incident and emission sources of flux on X . Thus the photons making up $U^0(t)$ are those left over and in their original unscattered state after t units of time have elapsed since the external sources over X (represented by N_0) and the internal sources over X (represented by N_η) have been turned on.

Transport Equation for Residual Radiant Energy

The transport equation for residual radiant energy can be obtained directly from (2) by applying the integral operations occurring in (3) to each side of (2). Thus, integrating (2) term by term, the time derivative term becomes:

$$\frac{1}{v} \frac{\partial}{\partial t} \int_X \left[\int_{\Xi} N^0(x, \xi, t) d\Omega(\xi) \right] dV(x) = \frac{\partial U^0(t)}{\partial t} \quad (4)$$

Next, we observe that the spatial derivative term may be written as:

$$\nabla \cdot (\xi N^0) \quad ,$$

since ξ is a variable independent of location on X . Then we observe that the integral:

$$\int_{\Xi} \xi N^0(x, \xi, t) d\Omega(\xi)$$

defines the residual radiance counterpart to the vector irradiance function H , as developed in Sec. 2.8. If we write " $H^0(x, t)$ " for the preceding integral, we can then go on to perform the remaining integration, as required by (3), to obtain:

$$\int_X \nabla \cdot H^0(x, t) dV(x)$$

which by the divergence theorem may be written as a surface integral of H over the boundary Y of X ; thus:

$$\int_X \nabla \cdot H^0(x, t) dV(x) = - \int_Y H^0(x, t) \cdot \mathbf{n}(x) dA(x) \quad , \quad (5)$$

where $\mathbf{n}(x)$ is the unit inward normal to X at each x on Y , and A is the area measure of Y . Suppose we write:

$$"P^0(t)" \text{ or } "P^0(Y,t)" \text{ for } \int_Y \mathbf{H}^0(x,t) \cdot \mathbf{n}(x) dV(x) \quad (6)$$

Thus $P^0(Y,t)$ is the net inward flux to X across the boundary Y of X . Finally we write:

$$"P_\eta(t)" \text{ or } "P_\eta(X,t)" \text{ for } \int_X \left[\int_E N_\eta(x,\xi,t) d\Omega(\xi) \right] dV(x) \quad (7)$$

Thus $P_\eta(X,t)$ is the input radiant flux over X at time t . Assembling the results summarized in (4)-(7), equation (2) becomes:

$$\boxed{\frac{dU^0(t)}{dt} = - \frac{U^0(t)}{T_\alpha} + P^0(t) + P_\eta(t)} \quad (8)$$

where we have written:

$$"T_\alpha" \text{ for } \frac{1}{v\alpha} \quad (9)$$

Equation (8) is the requisite *transport equation for residual radiant energy* in medium X at time t .

The Attenuation Time Constant

The quantity T defined in (9) and which has the dimension of time, is the *attenuation time constant* for X . The significance of T_α will become apparent as the discussions of this section proceed. However, a preliminary insight into its significance can be obtained as follows. Imagine all of E_3 to be an infinite homogeneous three-dimensional optical medium about the origin 0 . Let the initial radiant energy content of E_3 be zero. Let the sources in E_3 be confined to a point source at 0 which is turned on at time $t = 0$ and which pours radiant flux out into X at a constant rate P_η (i.e., $P_\eta(t)$ is independent of t , $t > 0$). At any finite time $t > 0$ the spherical wave front traveling outward from 0 is of radius vt . For every $t > 0$, let Y' be any given sphere of radius $r(>vt)$, and let X' be the medium bounded by Y' , as in Fig. 5.10.

Under these conditions we have in particular $P^0(t) = 0$ for every t , $0 \leq t \leq r/v$, and (8) reduces to:

$$\frac{dU^0(t)}{dt} = -\frac{U^0(t)}{T_\alpha} + P_\eta \quad (10)$$

with initial condition:

$$U^0(0) = 0 \quad (11)$$

The solution of (10), subject to (11), is:

$$U^0(t) = U^0(\infty)(1 - e^{-t/T_\alpha}) \quad (12)$$

over the time interval $(0, r/v)$, and where we have written:

$$"U^0(\infty)" \text{ for } P_\eta T_\alpha$$

The significance of T_α now springs into view if we recall a well-known result of elementary circuit analysis concerning the charging of a simple capacitance-resistance DC circuit such as that depicted in Fig. 5.11. When switch S is closed at time $t = 0$, battery B of voltage V pumps electrons along the circuit A which has resistance R, until the capacitor of capacitance C (initially discharged) is fully charged. The amount $q(t)$ of charge on the capacitor

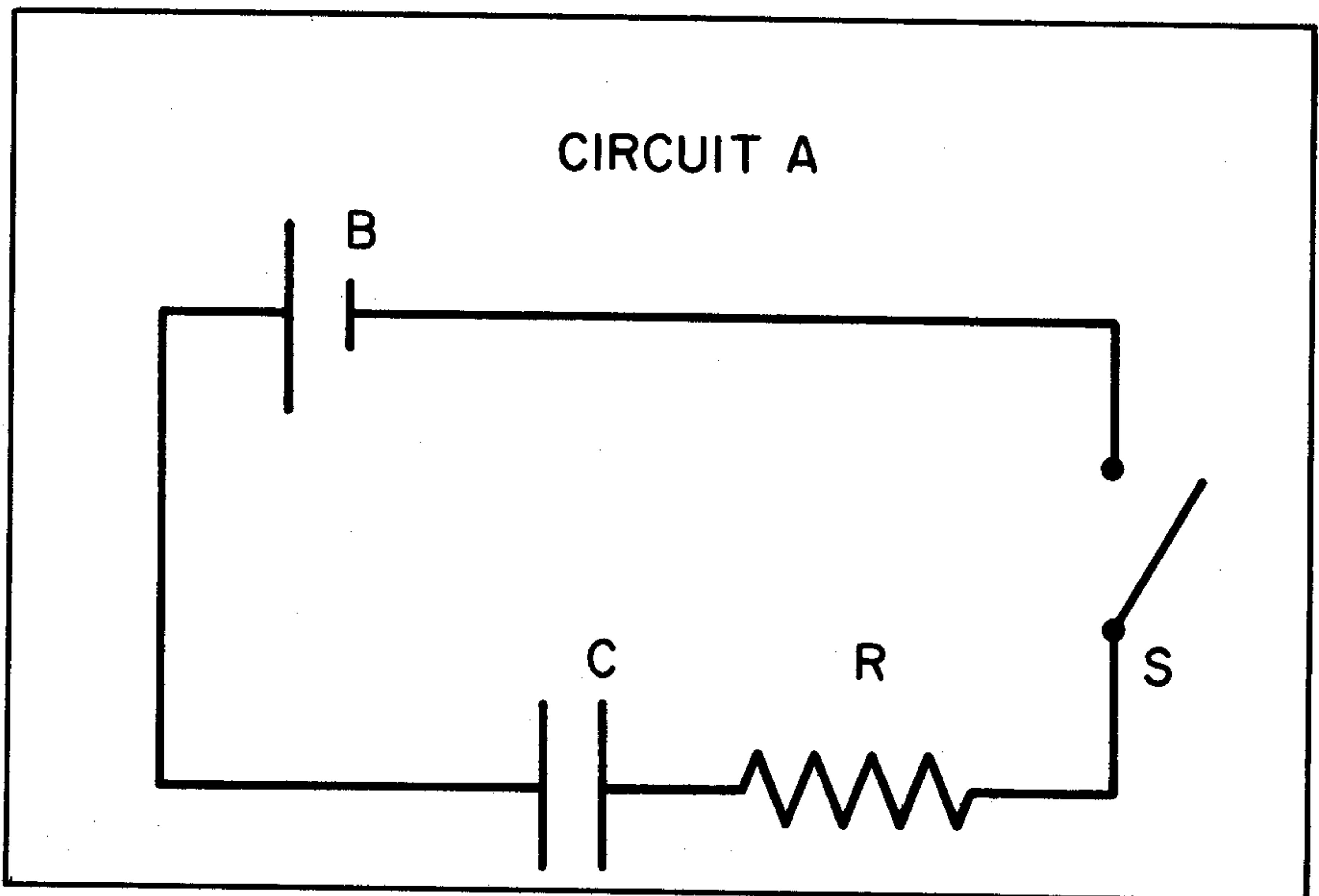


FIG. 5.11 The analogy between an electric circuit and an optical medium.

at time $t \geq 0$ is given by the equation:

$$q(t) = q(\infty) (1 - e^{-t/RC}) \quad (14)$$

where we have written:

$$"q(\infty)" \text{ for } CV \quad (15)$$

With the strong structural resemblance between (12) and (14) in mind, we can make the following pairings between the radiative transfer concepts and the electrical circuit concepts:

<u>In the Optical Medium</u>	<u>In the Electrical Circuit</u>
The medium X	The circuit A
The Source Point 0	The battery B
P_{η}	V/R
$U^0(t)$	$q(t)$
$1/v$	C
$1/\alpha$	R
T_{α} (attenuation time constant)	RC (circuit time constant)

Hence the buildup of residual radiant energy in an extensive homogeneous medium X is analogous to the charging of a capacitor in a simple DC capacitor-resistance circuit. The internal source of radiant flux P_{η} is analogous to the basic current associated with the battery voltage V and circuit resistance R. The capacitance of the circuit is, for given geometry, dependent on the materials of the plates. Thus the smaller the speed of propagation in the material, the larger the capacitance, and the larger the steady state charge $q(\infty)$. Analogously, the smaller the speed of propagation v in the optical medium, all other things being equal, the larger the steady state stored energy $U^0(\infty)$. On this basis (which is not, however, logically compelling) we pair $1/v$ with C. Furthermore, the less dense the conducting material of the circuit, the smaller is the conductance $1/R$; similarly, the less dense the material of the optical medium the smaller is α . On this basis we pair $1/\alpha$ with R. The standard circuit time constant RC then pairs off with T_{α} . This pairing of time constants is relatively strongly suggested by direct comparison of (12) and (14), whereas the suggested pairings of $1/v$ with C and $1/\alpha$ with R are not as strong and, indeed, the pairings may be switched without affecting the important pairing of T_{α} with RC, the pairing of principal interest at the moment. However, the indicated optical counterparts to R and C are quite interesting to contemplate, particularly when it appears that the analogy between the medium X and the circuit A can be extended quite far by establishing a link with the analogies summarized in the closing paragraph of Sec. 5.6. Apparently, if these analogies can be extended far enough,

then with sufficient care and ingenuity, some of the time-dependent radiative transfer problems can possibly be solved by electrical (or even acoustical) analog methods in which the time-dependent electrical (or reverberating acoustical) field replaces the radiant field.

Just as in the electrical case, the attenuation time constant T_α is the time required for the residual radiant energy to attain 63 percent of its steady state value. Below is given a table for the values of $U^0(t)/U^0(\infty)$ for various values of t in terms of multiples of T_α

TABLE 1

Values of $U^0(t)/U^0(\infty)$ for various values of t in terms of multiples of T_α

$t = nT_\alpha$	$U^0(t)/U^0(\infty)$
T_α	0.63
$2T_\alpha$	0.86
$3T_\alpha$	0.95
$4T_\alpha$	0.98
$5T_\alpha$	0.99

General Representation of
Residual Radiant Energy

The solution (12) of the differential equation for residual radiant energy is a special case of the more general solution:

$$U^0(t) = U^0(0)e^{-t/T_\alpha} + \int_0^t e^{(t'-t)/T_\alpha} P_0(t') dt' \quad (16)$$

where we have written:

$$"P_0(t)" \text{ for } P^0(t) + P_\eta(t) \quad (17)$$

The solution (15) represents the residual radiant energy in a general homogeneous optical medium X with known combined internal and external source flux function P_0 , as given by (17).

Transport Equation for n-ary
Radiant Energy

We derive next the transport equation for the second main radiometric concept of this section, the n-ary radiant energy $U^n(t)$. The definition of $U^n(t)$ was given in steady state form in Sec. 5.1. Thus we have for every nonnegative integer n ,

$$U^n(X,t) = \frac{1}{V} \int_X \left[\int_{\Xi} N^n(x,\xi,t) d\Omega(\xi) \right] dV(x) \quad (18)$$

We shall write " $U^n(t)$ " for $U^n(X,t)$ whenever X is understood.

Starting with the time-dependent radiance field in X we apply to (5) of Sec. 5.7 the lagrangian derivative operator D/Dt in exactly the way d/dr was applied to (11) of Sec. 5.1 to yield (1) of Sec. 5.2. We have, as a consequence, for every integer n , $n \geq 1$:

$$\boxed{\frac{1}{V} \frac{\partial N^n}{\partial t} + \xi \cdot \nabla N^n = -\alpha N^n + N_{*}^n} \quad (19)$$

which is the time-dependent equation of transfer for n-ary radiance N^n , and which is to be compared to (2) above and (1) of Sec. 5.2. Applying the integral operations in (18) to each member of each side of (19), we find that:

$$\frac{1}{V} \int_X \left[\int_{\Xi} \frac{\partial N^n}{\partial t} d\Omega(\xi) \right] dV(x) = \frac{\partial U^n(t)}{\partial t} \quad (20)$$

We write:

$$"H^n(x,t)" \quad \text{for} \quad \int_{\Xi} \xi N^n(x,\xi,t) d\Omega(\xi) \quad (21)$$

and

$$"P^n(t)" \quad \text{or} \quad "P^n(Y,t)" \quad \text{for} \quad \int_Y H^n(x,t) \cdot n(x) dA(x) \quad (22)$$

where $n(x)$ is defined as in (6). Finally we observe that:

$$\int_X \left[\int_{\Xi} N_{\star}^n(x, \xi, t) d\Omega(\xi) \right] dV(x) = s U^{n-1}(t) \quad (23)$$

With the results (20) through (23) in mind, (19) yields up the following *transport equation for n-ary radiant energy*:

$$\frac{dU^n(t)}{dt} = - \frac{U^n(t)}{T_{\alpha}} + \frac{U^{n-1}(t)}{T_s} + P^n(t) \quad (24)$$

for every integer $n \geq 1$. The main details of derivation of (24) thus proceed as in the case of the residual radiant energy (8). Here we have written:

$$"T_s" \text{ for } 1/vs \quad (25)$$

In equation (24), $P^n(t)$ is the *net inward radiant flux* across the boundary Y of X at time t . The radiant flux $P^n(t)$ has scattering order n relative to that of $P^0(t)$. A term by term interpretation of (24) is instructive: the time rate of change of n -ary radiant energy in X at time t is the sum of a growth term $U^{n-1}(t)/T_s$ (which is the rate of conversion of $(n-1)$ -ary scattered energy into n -ary scattered energy), a decay term $-U^n(t)/T_{\alpha}$ (which is the rate of conversion of n -ary energy into $(n+1)$ ary energy and nonradiant energy), and finally a general *net rate* of growth term giving the net balance of influx and efflux of n -ary radiant energy across the boundary of X . The quantity T_s is the *scattering time constant* for the medium X . It is a concept which helps write (24) in a uniform manner in terms of the fundamental timelike quantities T_{α} and T_s .

Transport Equation for Directly Observable Radiant Energy

The radiant energy U associated with directly observable radiance N , using a standard radiance meter is called the *directly observable* radiant energy. This energy is to be held both in conceptual and empirical contrast to the n -ary radiant energy U^n , $n > 1$, which is not directly observable in practice. (The residual radiant energy is indirectly observable using techniques alluded to in Sec. 3.10 and Sec. 16 of Ref. [251].) We now derive the transport equation for $U(t)$. We begin with the definitional identity:

$$U(x, t) = \frac{1}{v} \int_X \left[\int_{\Xi} N(x, \xi, t) d\Omega(\xi) \right] dV(x) \quad (26)$$

based on (2) and (12) of Sec. 2.7. As usual we shall drop reference to X , when X is understood.

Starting with the time-dependent radiance equation (4) of Sec. 3.15, we now apply the integral operations in (26) to each side of the transfer equation and obtain, in a manner analogous to that culminating in (8) and (24) above, the result:

$$\boxed{\frac{dU(t)}{dt} = -\frac{U(t)}{T_a} + \bar{P}(t) + P_\eta(t)} \quad (27)$$

This is the *transport equation for directly observable radiant energy*. In the equation we have written:

$$"T_a" \quad \text{for} \quad \frac{1}{va}$$

and where a in turn is the value of the constant volume absorption function in X . Furthermore, we have written:

$$"P(t)" \quad \text{or} \quad "P(Y,t)" \quad \text{for} \quad \int_Y \mathbf{H}(x,t) \cdot \mathbf{n}(x) dA(x) \quad (28)$$

The unit vector $\mathbf{n}(x)$ is defined as in Fig. 5.10, and so $\bar{P}(t)$ is the net inward radiant flux into X over the boundary Y of X .

The Natural Solution for Directly Observable Radiant Energy

It is a relatively easy matter to verify (using (5a) of Sec. 5.7) that:

$$U(X,t) = \sum_{j=0}^{\infty} U^j(X,t) \quad (29)$$

holds for every $t > 0$, where $U(x,t)$ is defined as in (26) and the $U^j(X,t)$ are defined as in (18). Thus, once each $U^j(X,t)$, $j > 0$, is known, $U(X,t)$ is known and computable. Equation (29) represents the natural solution of the directly observable radiant energy.

In the case of radiant energy the natural solution procedure is not as vitally essential in the solution of $U(t)$ as in the natural solution procedure for the case of radiance in Secs. 5.6 and 5.7. Indeed, the solution of (27) is written down quite readily, assuming $\bar{P}(t)$ and $P_\eta(t)$ given. Thus, writing,

$$"P(t)" \quad \text{for} \quad \bar{P}(t) + P_\eta(t) \quad (30)$$

we have, analogously to (16):

$$U(t) = U(0)e^{-t/T_a} + \int_0^t e^{(t'-t)/T_a} P(t') dt' \quad (31)$$

The quantity T_a is the *absorption time constant* for X and is related to T_α and T_s as follows:

$$\frac{1}{T_\alpha} = \frac{1}{T_a} + \frac{1}{T_s} \quad (32)$$

The natural solution procedure for radiant energy is, however, quite useful in throwing light on the inner workings of time-dependent light fields, for the solutions of the transport equations for U^n are readily obtained in simple closed forms which are quite amenable to all manners of explicit, rearrangements and manipulations. Some of the properties of time dependent radiant energy fields will be explored in the next few sections.

We conclude this section with an important observation which will facilitate the studies below. This concerns the connection between the net fluxes $\bar{P}^n(t)$, $n > 0$ occurring in (8) and (24), and the net flux $\bar{P}(t)$ occurring in (27). This connection is established by means of the natural solution representation of the directly observable radiant energy $U(t)$ as given in (29). Thus, by summing over all $n \geq 1$ in (24):

$$\frac{d}{dt} \sum_{n=1}^{\infty} U^n(t) = -\frac{1}{T_\alpha} \sum_{n=1}^{\infty} U^n(t) + \frac{1}{T_s} \sum_{n=1}^{\infty} U^{n-1}(t) + \sum_{n=1}^{\infty} \bar{P}^n(t)$$

and adding to these terms the corresponding terms of (8), we obtain::

$$\frac{dU(t)}{dt} = -\frac{U(t)}{T_a} + \sum_{n=0}^{\infty} \bar{P}^n(t) + P_\eta(t)$$

comparing this with (27) we conclude that:

$$\bar{P}(t) = \sum_{n=0}^{\infty} \bar{P}^n(t) \quad (33)$$

5.9 Solutions of the n-ary Radiant Energy Equations

We shall now solve the transport equation for n-ary scattered radiant energy for every $n \geq 1$, and deduce from the solutions several interesting properties of the scattering order decomposition of natural light fields. These properties are both of intrinsic interest and of use in furthering the natural solution of the radiance field in optical