we have, analogously to (16):

$$\boxed{U(t) = U(0)e^{-t/T_a} + \int_0^t e^{(t'-t)/T_a}\, P(t')\, dt'} \tag{31}$$

The quantity T_a is the *absorption time constant* for X and is related to T_α and T_s as follows:

$$\frac{1}{T_\alpha} = \frac{1}{T_a} + \frac{1}{T_s} \tag{32}$$

The natural solution procedure for radiant energy is, however, quite useful in throwing light on the inner workings of time-dependent light fields, for the solutions of the transport equations for U^n are readily obtained in simple closed forms which are quite amenable to all manners of explicit, rearrangements and manipulations. Some of the properties of time dependent radiant energy fields will be explored in the next few sections.

We conclude this section with an important observation which will facilitate the studies below. This concerns the connection between the net fluxes $\bar{P}^n(t)$, $n \geq 0$ occurring in (8) and (24), and the net flux $\bar{P}(t)$ occurring in (27). This connection is established by means of the natural solution representation of the directly observable radiant energy $U(t)$ as given in (29). Thus, by summing over all $n \geq 1$ in (24):

$$\frac{d}{dt} \sum_{n=1}^{\infty} U^n(t) = -\frac{1}{T_\alpha} \sum_{n=1}^{\infty} U^n(t) + \frac{1}{T_s} \sum_{n=1}^{\infty} U^{n-1}(t) + \sum_{n=1}^{\infty} \bar{P}^n(t)$$

and adding to these terms the corresponding terms of (8), we obtain::

$$\frac{dU(t)}{dt} = -\frac{U(t)}{T_a} + \sum_{n=0}^{\infty} \bar{P}^n(t) + P_\eta(t)$$

comparing this with (27) we conclude that:

$$\bar{P}(t) = \sum_{n=0}^{\infty} \bar{P}^n(t) \quad . \tag{33}$$

5.9 Solutions of the n-ary Radiant Energy Equations

We shall now solve the transport equation for n-ary scattered radiant energy for every $n \geq 1$, and deduce from the solutions several interesting properties of the scattering order decomposition of natural light fields. These properties are both of intrinsic interest and of use in furthering the natural solution of the radiance field in optical media. They are also helpful in studying the light storage problems in such media. These latter two applications will be considered in Secs. 5.12 and 5.13. For the present we concentrate on the immediate mathematical and physical features of the transport equations for U^n. Throughout this section, unless specifically noted otherwise, the optical medium will be as in Sec. 5.8, that is homogeneous, with arbitrary sources, arbitrary directional structure for σ, and arbitrary fixed ρ, $0 < \rho < 1$.

Natural Integral Representations of n-ary Radiant Energy

Starting with (24) of Sec. 5.8, we treat the indicated differential equation, for given $n \geq 1$, as an ordinary differential equation with unknown function U^n, and known functions U^{n-1} and P^n, and with given parameters T_α, T_s. The initial condition for U^n is:

$$U^n(0) = 0 \quad , \tag{1}$$

for every $n \geq 0$. The general solution under this condition can therefore be patterned after (16) or (31) of Sec. 5.8 with the initial values set to zero. Specifically:

$$U^n(t) = \int_0^t e^{(t'-t)/T_\alpha} \left[\frac{U^{n-1}(t')}{T_s} + P^n(t') \right] dt' \tag{2}$$

Now, to simplify matters we shall assume that:

$$P^n(t) = 0 \tag{3}$$

for every $n \geq 0$ over a given interval $(0, t_1)$ of time which is to include the time interval in which we shall be interested in the solutions of (24) of Sec. 5.8. Physically this means in effect that the collective expanding wave fronts of all sources in X are completely within the boundary Y of X over the time interval $(0, t_1)$. See Figure 5.10. With assumption (3) in force, (2) becomes:

$$U^n(t) = \frac{1}{T_s} \int_0^t e^{(t'-t)/T_\alpha}\, U^{n-1}(t')\, dt'$$

$$= \frac{e^{-t/T_\alpha}}{T_s} \int_0^t e^{t'/T_\alpha}\, U^{n-1}(t')\, dt' \tag{4}$$

which holds for $n \geq 1$ and $0 \leq t \leq t_1$. The form of (4) suggests a recursive construction of $U^n(t)$ starting with $n = 1$ and using knowledge of $U^o(t)$ as given in (16) of Sec. 5.8. By (3), $P_0(t)$ in (16) of Sec. 5.8 uses only the internal source function P_η. Hence $U^n(t)$ should be expressible in terms of $U^o(t)$ (or $P_\eta(t)$) along with T_s and T_α. Thus, starting with (4) now applied to $U^{n-1}(t)$, $n-1 \geq 1$, we have:

$$U^n(t) = \frac{1}{T_s}\int_0^t e^{(t'-t)/T_\alpha}\left[\frac{1}{T_s}\int_0^{t'} e^{(t''-t')/T_\alpha}\, U^{n-2}(t'')\, dt''\right] dt'$$

$$= \frac{e^{-t/T_\alpha}}{T_s^2}\int_0^t (t-t')\, e^{t'/T_\alpha}\, U^{n-2}(t')\, dt' \qquad (5)$$

This process can be continued as long as the scattering order in the integrand is greater than zero. The pattern forming in (4) and (5) is clear. Applying (4) once again, now to U^{n-2} the pattern is crystallized:

$$U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^3}\int_0^t \frac{(t-t')^2}{2}\, e^{t'/T_\alpha}\, U^{n-3}(t')\, dt' \quad . \qquad (6)$$

Thus, applying the representation (4) in all k times, $0 \leq k \leq n-1$, we have for $U^n(t)$:

$$U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^{k+1}}\int_0^t \frac{(t-t')^k}{k!}\, e^{t'/T_\alpha}\, U^{n-k-1}(t')\, dt' \qquad (7)$$

If in (7) we let $k = n-1$, then the desired integral representation of $U^n(t)$, $0 \leq t \leq t_1$ is obtained:

$$\boxed{U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^n}\int_0^t \frac{(t-t')^{n-1}}{(n-1)!}\, e^{t'/T_\alpha}\, U^o(t')\, dt'} \qquad (8)$$

or, in terms of P_η:

$$\boxed{U^n(t) = \frac{e^{-t/T_\alpha}}{T_s^n}\int_0^t \frac{(t-t')^n}{n!}\, e^{t'/T_\alpha}\, P_\eta(t')\, dt'} \qquad (9)$$

Equations (8) or (9) are the desired integral representations of $U^n(t)$. Observe that (8) holds for $n \geq 1$ and (9) holds for $n \geq 0$.

Natural Closed Form Representations of n-ary Radiant Energy

The formulas (8) or (9) are the requisite representations of $U^n(t)$ under the given initial conditions (1), and the conditions on the medium hypothesized in (3) and at the outset of this section. In order to evaluate the integrals we must specify the nature of U^n or P_η over the time interval $(0, t_1)$. We now illustrate the use of (9) by choosing two important instances of P_η. The first instance is where P_η is the Dirac-delta function centered at $t = 0$ and with radiant energy content U_η. The second instance is where P_η is constant valued over $(0, t_1)$ with its constant magnitude denoted by "P_η". In the first instance, we have:

$$U^n(t) = U_\eta\left(\frac{t}{T_s}\right)^n \frac{e^{-t/T_\alpha}}{n!} \qquad (10)$$

for

$$P_\eta(t) = U_\eta(t)\ \delta(t)$$

over the interval $(0,t_1)$ and for $n \geq 0$. We shall refer to this case as the *optical reverberation case* (cf. the introduction to Sec. 5.6).

The second instance yields the representation:

$$U^n(t) = \left[\frac{T_\alpha}{T_s}\right]^n U^o(\infty)\left[1 - \left(\sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!}\right) e^{-t/T_\alpha}\right] \qquad (11)$$

for

$$P_\eta(t) = P_\eta \quad .$$

over the interval $(0, t_1)$ and for $n \geq 0$. Here $U^o(\infty)$ is as defined in (13) of Sec. 5.8. These two specific instances of (9) are verified by direct integration of (9) in each case.

General Integral Representations of n-ary Radiant Energy

The integral representation (9) of U^n will now be generalized to the case for which the initial conditions on U^j, $j < n$, are arbitrary. That is, we now relax the conditions (1). However, we shall retain condition (3). The resultant representation will permit the construction of relatively general representations of the time-dependent n-ary radiant energy in a homogeneous medium for which the wave fronts of internal sources have not yet passed the boundaries. Thus, by successive applications of the type of solution displayed in (16) of Sec. 5.8, we eventually arrive at:

$$\boxed{\begin{aligned} U^n(t) &= \left[U^n(0) + \left(\frac{t}{T_s}\right) U^{n-1}(o) + \dots + \frac{1}{n!}\left(\frac{t}{T_s}\right)^n U^o(0)\right] e^{-t/T_\alpha} + \\ &\quad + \frac{e^{-t/T_\alpha}}{T_s^n} \int_0^t \frac{(t-t')^n}{n!} e^{t'/T_\alpha} P_\eta(t')\, dt' \end{aligned}} \tag{12}$$

This is the desired generalization of (9), which holds for $n \geq 0$.

Standard Growth and Decay Formulas for n-ary Radiant Energy

Of the infinite variety of possible time-dependent radiant energy fields attainable in principle via (12), two types stand out as particularly interesting. These are sufficiently instructive to isolate and set up here as standards. The first of these light fields is that given by (11) above. This equation we shall call the *standard growth formula* for U^n. Recall that in this case the initial values for the U^j, $j \leq n$, are all zero and that P_η is a positive constant over some time interval $(0, t_1)$. Suppose we write:

$$\text{"}F_n(t/T_\alpha)\text{"} \quad \text{for} \quad e^{-t/T_\alpha} \sum_{j=0}^{n} \frac{(t/T_\alpha)^j}{j!} \tag{13}$$

Then we summarize the *standard growth formula* as follows: If

(a) The optical medium is homogeneous.

(b) $U^n(0) = 0$ and $P_\eta(t) = P_\eta$ for t in $(0,t_1)$ and $n \geq 0$.

(c) $\mathbb{P}^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$\boxed{U^n(t) = U^n(\infty)[1-F_n(t/T_\alpha)]} \qquad (14)$$

for every t in $(0,t_1)$ and $n \geq 0$.

The second standard case is that which describes the decay of the n-ary light field from a given steady state level. Thus if an opaque curtain were suddenly drawn over the ocean in which previously all internal radiant sources were turned off, the following *standard decay formula* for U^n would describe very closely the decay of $U^n(t)$ for $t \geq 0$ for every $n \geq 0$ in the ocean; thus: If

(a) The optical medium is homogeneous.

(b) $U^n(0) = \rho^n U^o(0)$ and $P_\eta = 0$ for t in $(0,t_1)$ and $n \geq 0$.

(c) $\mathbb{P}^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$\boxed{U^n(t) = U^n(0)\ F_n(t/T_\alpha)} \qquad (15)$$

for every t in $(0,t_1)$ and $n \geq 0$.

A few words about condition (b), the initial condition for U^n, are in order. An examination of the general representation (11) of $U^n(t)$ shows that at steady state (i.e., the limit of $U^n(t)$ as $t \to \infty$) the various magnitudes $U^n(\infty)$ are not arbitrary. Indeed, they generally depend on P_η and the initial values $U^n(0)$, as explicitly shown in (12). Hence when a steady state light field begins to decay after sources have been turned off, the initial values $U^n(0)$, $n \geq 0$ are generally not expected to be independent of each other. For example, if the standard growth conditions are in effect, then (11) shows that:

$$U^n(\infty) = \left[\frac{T_\alpha}{T_s}\right]^n U^o(\infty) = \rho^n U^o(\infty)$$

$$= \rho^n P_\eta T_\alpha$$

for every $n \geq 0$. Thus we see that the standard decay formula is intended to describe the decay of a light field which has been attained under standard growth conditions as given by (14) for $t \to \infty$.

We can combine the standard growth and decay formulas (14) and (15) into a single standard formula as follows: If

(a) The optical medium is homogeneous.

(b) $U^n(0)$, $n \geq 0$ is given as steady state value attained under a previous standard growth condition and $P_\eta(t) = P_\eta$ for t in $(0,t_1)$.

(c) $P^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$\boxed{U^n(t) = U^n(\infty) + [U^n(0) - U^n(\infty)]\, F_n(t/T_\alpha)} \tag{16}$$

and where $U^n(\infty)$ is determined by (14) for the present source condition. As an interesting consistency check, observe that if the previous steady state condition (b) above is induced by P_η as given in (b), then $U^n(t)$ in (16) is independent of time, because $U^n(0) = U^n(\infty)$.

As a final standard type of growth and decay formula, we consider the case in which a standard growth begins at $t = 0$ and continues until time t_o, at which time the source is shut off and the existing light field decays from that point on until some arbitrary time t_1 under standard decay conditions. Equation (12) shows that the decay formula is:

$$U^n(t) = \left[U^n(t_o) + \left(\frac{t-t_o}{T_s}\right)U^{n-1}(t_o) + \dots + \left(\frac{t-t_o}{T_s}\right)^n U^o(t_o)\right] e^{-(t-t_o)/T_\alpha} \tag{17}$$

for $t_o < t \leq t_1$ and $n \geq 0$. For $t \leq t_o$, $U^n(t)$ is given by (11). Formula (17) may be used to describe the transient radiant energy fields induced in large bodies of air or water by radiant sources which are intermediate between the Dirac-delta pulse and the steady source described in (10) and (11). Since any source output P_η over a time interval $(0,t_o)$ can be approximated by a step function, we see that by superimposing fields of the type given by (17), we can represent n-ary radiant energy fields induced by finite non-constant sources under the general conditions of this section.

<< Table of Contents << Title Page >> Next Page >>

5.10 Properties of Time-Dependent n-ary Radiant Energy Fields and Related Fields

We now turn to examine in detail some of the more intuitively interesting properties of time-dependent radiant energy fields. In order to present the properties in their simplest forms, we shall adopt for study throughout this section a light field evolving under either *standard growth or decay conditions* or *optical reverberation conditions* in an