We can combine the standard growth and decay formulas (14) and (15) into a single standard formula as follows: If

(a) The optical medium is homogeneous.

(b) $U^n(0)$, $n \geq 0$ is given as steady state value attained under a previous standard growth condition and $P_\eta(t) = P_\eta$ for t in $(0,t_1)$.

(c) $F^n(t) = 0$ for t in $(0,t_1)$ and $n \geq 0$.

Then:

$$U^n(t) = U^n(\infty) + [U^n(0) - U^n(\infty)]\, F_n(t/T_\alpha) \tag{16}$$

and where $U^n(\infty)$ is determined by (14) for the present source condition. As an interesting consistency check, observe that if the previous steady state condition (b) above is induced by P_η as given in (b), then $U^n(t)$ in (16) is independent of time, because $U^n(0) = U^n(\infty)$.

As a final standard type of growth and decay formula, we consider the case in which a standard growth begins at $t = 0$ and continues until time t_o, at which time the source is shut off and the existing light field decays from that point on until some arbitrary time t_1 under standard decay conditions. Equation (12) shows that the decay formula is:

$$U^n(t) = \left[U^n(t_o) + \left(\frac{t-t_o}{T_s}\right)U^{n-1}(t_o) + \dots + \left(\frac{t-t_o}{T_s}\right)^n U^o(t_o)\right] e^{-(t-t_o)/T\alpha} \tag{17}$$

for $t_o < t \leq t_1$ and $n \geq 0$. For $t \leq t_o$, $U^n(t)$ is given by (11). Formula (17) may be used to describe the transient radiant energy fields induced in large bodies of air or water by radiant sources which are intermediate between the Dirac-delta pulse and the steady source described in (10) and (11). Since any source output P_η over a time interval $(0,t_o)$ can be approximated by a step function, we see that by superimposing fields of the type given by (17), we can represent n-ary radiant energy fields induced by finite non-constant sources under the general conditions of this section.

5.10 Properties of Time-Dependent n-ary Radiant Energy Fields and Related Fields

We now turn to examine in detail some of the more intuitively interesting properties of time-dependent radiant energy fields. In order to present the properties in their simplest forms, we shall adopt for study throughout this section a light field evolving under either *standard growth or decay conditions* or *optical reverberation conditions* in an optical medium X over a time interval $(0,t_1)$(Sec. 5.9). It will be clear from the results stated below how analogous or complementary statements and properties can be formulated under still more general conditions. We begin with a study of some of the fine-structure properties of n-ary radiant energy fields and then go on to a formulation of the various representations of related radiant energy quantities.

Some Fine-Structure Properties of n-ary Radiant Energy

Property 1. *Let t be a fixed time in $(0,t_1)$. Then the sequence $U^o(t)$, $U^1(t)$, ..., $U^n(t)$, ... of n-ary radiant energies at time t is a monotonic decreasing sequence with limit 0.* The proof of this property is based on (14) of Sec. 5.9. By (13) of Sec. 5.9 we see that:

$$\lim_n F_n(t/T_\alpha) = 1 \tag{1}$$

Hence by noting that $0 < \rho < 1$, we see that:

$$\lim_n U^n(\infty) = 0$$

so that

$$\lim_n U^n(t) = 0$$

for t in $(0,t_1)$. As for the monotonicity of the sequence, it suffices to note that:

$$\frac{U^{n+1}(t)}{U^n(t)} = \rho \frac{1-F_{n+1}(t/T_\alpha)}{1-F_n(t/T_\alpha)} \tag{2}$$

and that $F_n(t/T_\alpha)$ increases monotonically, with n, to unity. This may be seen by verifying that:

$$0 < 1 - F_{n+1}(t/T_\alpha) < 1 - F_n(t/T_\alpha) < 1$$

for every $n \geq 0$ and every positive t. The limit part of property 1 follows also from (2) by using the ratio test for convergent infinite series.

Property 2. *Under standard growth conditions,*

$$\frac{dU^n(t)}{dt} = P_n \left[\frac{t}{T_s}\right]^n \frac{e^{-t/T_\alpha}}{n!} > 0$$

for every t in $(0,t_1)$. The proof is immediate. For example, one may use (14) of Sec. 5.9 directly with the calculus, or one may use algebra with the fact that $dU^n(t)/dt$ is the difference given in (24) of Sec. 5.8, with $P^n(t) = 0$. Property 2 shows in particular that each n-ary radiant energy component increases monotonically with time. Property 2 is to be compared with:

Property 3. *Under standard decay conditions*

$$\frac{dU^n(t)}{dt} = -\frac{U^n(0)}{T_\alpha}\left[\frac{t}{T_s}\right]^n \frac{e^{-t/T_\alpha}}{n!} < 0$$

for every t in $(0,t_1)$. The proof is immediately obtainable from (15) of Sec. 5.9. Hence the rates of growth and decay of n-ary radiant energy under standard conditions are, to within a constant multiplicative factor, identical in structure within a given space.

Property 4. *Under standard growth conditions,*

$$\frac{U^{n+k}(t)}{U^n(t)} < \rho^k$$

for every t in $(0,t_1)$ *and positive integers n, k.* This follows from property 2 and (24) of Sec. 5.8 with $P^n(t) = 0$. The inequality is reversed under standard decay conditions.

Property 5. *In the steady state of the standard growth process,*

$$U^n(\infty) = \rho^n U^o(\infty)$$

for every $n \geq 0$. *Hence:*

$$\frac{U^{n+k}(\infty)}{U^n(\infty)} = \rho^k$$

for every pair n, k of nonnegative integers.

Property 6. *In the optical reverberation case (equation (10) of Sec. 5.9) we have the ratio:*

$$U^n(t)/U^{n-1}(t) = \frac{vts}{n} = t/nT_s$$

for $n \geq 1$ *and t in* $(0,t_1)$. *Thus, the ratio of successive n-ary radiant energy contents increases linearly with increasing time and decreases hyperbolically with increasing n.*

Property 7. *In the optical reverberation case with point source (equation (10) of Sec. 5.9)* $U^n(t)$, *for a given scattering order, attains a maximum when the radius of the wave front is n times the attenuation length* $1/\alpha$. *Further, for any given total volume scattering value s and time t in* $(0,t_1)$, *that component* $U^n(t)$ *is maximal whose order n makes the absolute value of*

$$\left(\frac{vts}{n}\right) - 1 = (t/nT_s) - 1$$

a minimum. The geometric content of properties 6 and 7 are summarized in part (a) of Figure 5.12.

Property 8. *In the optical reverberation case, the directly observable radiant energy U(t) is given by:*

$$U(t) = U_\eta \, e^{-t/T_a}$$

The proof rests on (10) of Sec. 5.9 and (29) of Sec. 5.8 and the simple calculation:

$$U(t) = \sum_{j=0}^{\infty} U^j(t) = U_\eta \, e^{-t/T_\alpha} \sum_{j=0}^{\infty} \frac{\left(\frac{t}{T_s}\right)^j}{j!}$$

$$= U_\eta \, e^{-t/T_\alpha} \cdot e^{t/T_s} = U_\eta \, e^{-t/T_a} \quad ,$$

in which (32) of Sec. 5.8 was used. It follows immediately from property 8 that, in optical media with no absorption, i.e., for which $a = 0$, U(t) is independent of t in the reverberation case. Part (b) of Figure 5.12 gives plots of $U^n(t)$ for the first four scattering orders in the optical reverberation case in which $a = 0$ and $U_\eta = 1$. In the figure we have

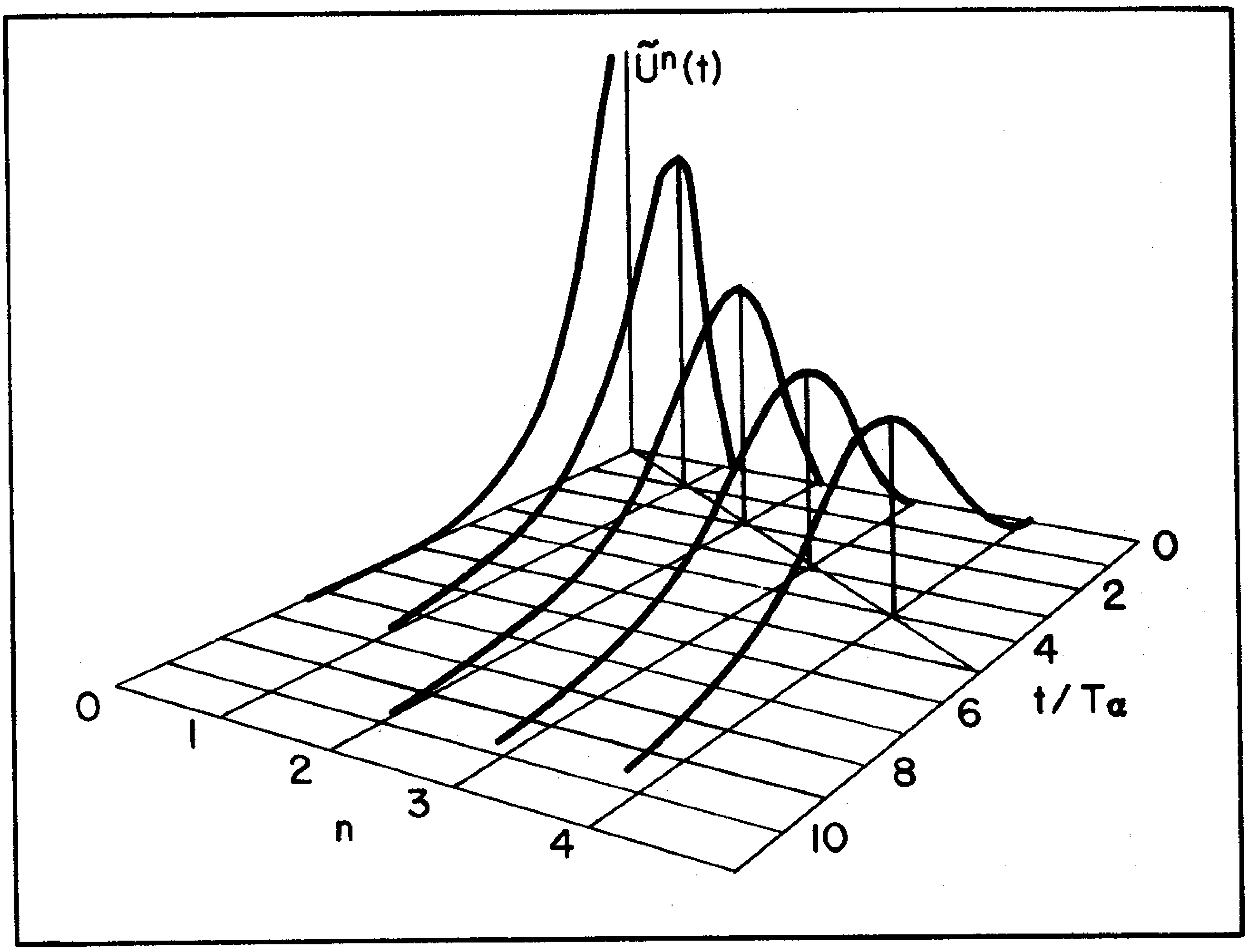

FIG. 5.12(a) The geometric version of property 7 of scattered radiant energy.

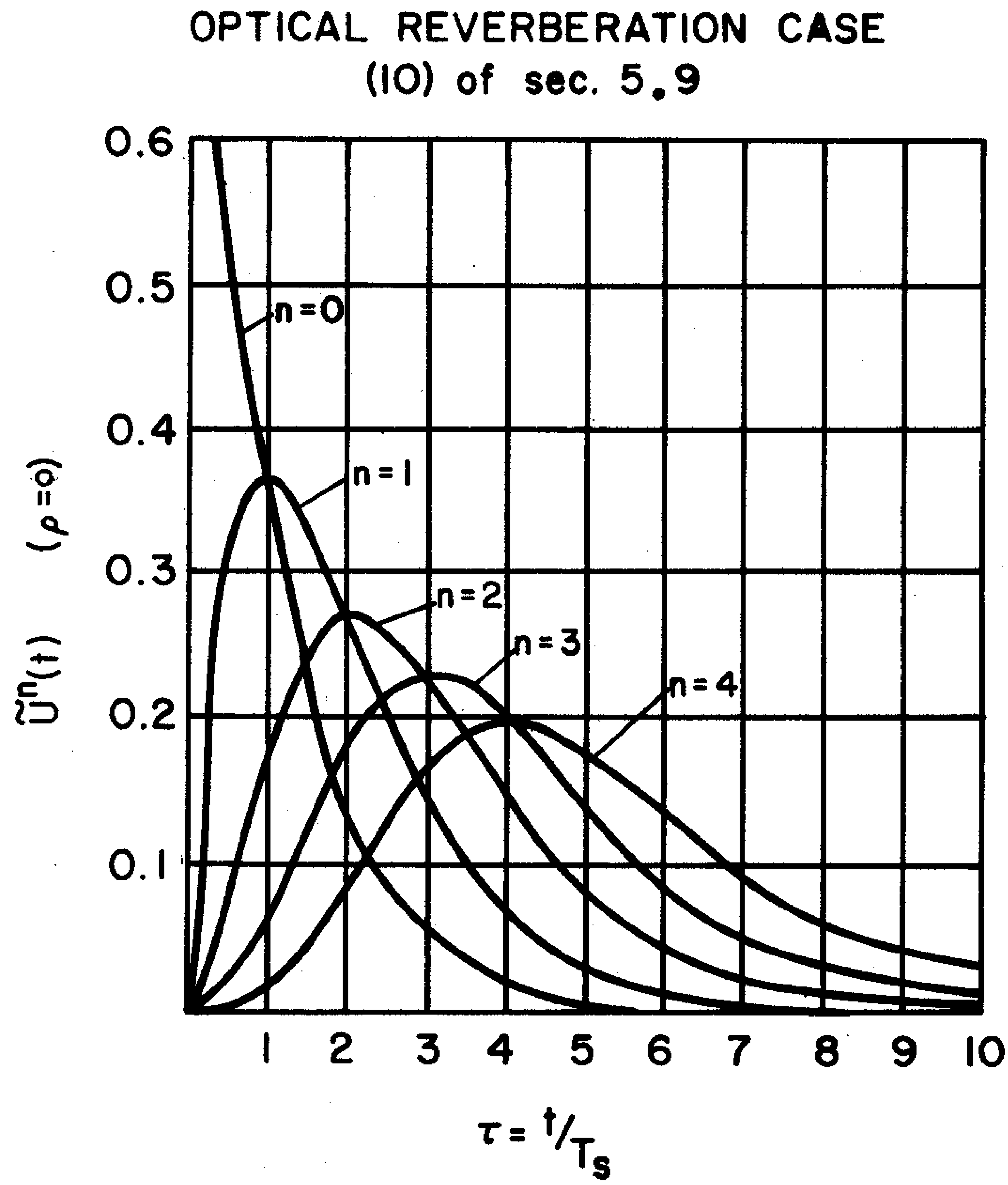

FIG. 5.12(b) The geometric version of property 7 of scattered radiant energy.--Concluded.

written "τ" for t/T_s. Thus the medium is a nonabsorbing medium ($\rho = 0$) with conserved directly observable energy. Note how the scattering order components of $U(t)$ well up one after another, reaching their maxima, as described by property 7. Finally, according to property 8, the sum of the ordinates of all the curves at each τ should add up to unity.

Scattered, Absorbed, and Attenuated Radiant Energies

We now round out the roster of the types of radiant energy fields most commonly encountered in theoretical discussions of time-dependent light fields. Until further notice, source conditions are arbitrary and with $\overline{P}(t) = 0$.

So far we have introduced the residual radiant energy ((3) of Sec. 5.8), the n-ary radiant energy ((19) of Sec. 5.8), and the directly observable radiant energy ((26) of

Sec. 5.8) with its natural representation ((29) of Sec. 5.8). By writing:

$$\text{"}U^*(t)\text{"} \quad \text{for} \quad \sum_{j=1}^{\infty} U^j(t) \tag{3}$$

we define the *scattered* (or diffuse) *radiant energy* (in X) at time t. We then have from (29) of Sec. 5.8 the following radiant energy counterpart to the time-dependent integral equation of transfer (cf. (4) of Sec. 5.4):

$$U(t) = U^o(t) + U^*(t) \tag{4}$$

Using the emission radiant flux function P_η and recalling that we have set $\overline{P}(t) = 0$ for t in $(0, t_1)$, let us write:

$$\text{"}U(t;\alpha)\text{"} \quad \text{for} \quad \int_0^t P_\eta(t')\,dt' - U^o(t) \tag{5}$$

for t in $(0, t_1)$. The meaning of this new radiant energy becomes clear when it is recalled that $U^o(t)$ is the residual (i.e., the *unattenuated*) radiant energy. Therefore, since the integral gives the total radiant energy input to the medium, the difference in (5) must be all the energy present at time t that has undergone attenuation (absorption or at least one scattering operation). We call $U(t;\alpha)$ the *attenuated radiant energy* (in the medium X) at time t. Only part of $U(t;\alpha)$ is detectable. In fact, the detectable part of $U(t;\alpha)$ is precisely $U^*(t)$. Thus let us write:

$$\text{"}U(t;a)\text{"} \quad \text{for} \quad U(t;\alpha) - U(t;s) \tag{6}$$

where, for uniformity of notation and heuristic purposes, we have agreed momentarily to write

$$\text{"}U(t;s)\text{"} \quad \text{for} \quad U^*(t) \quad . \tag{7}$$

Then from (6) we have:

$$U(t;\alpha) = U(t;a) + U(t;s) \quad , \tag{8}$$

a formula remarkably similar in structure to the basic relation:

$$\alpha = a + s$$

derived from (4) of Sec. 4.2. We call $U(t;a)$ the *absorbed radiant energy* (in X) at time t. The absorbed radiant energy is radiant energy that has disappeared from the present radiometric scene via absorption processes.

Representations of $U(t;\alpha)$, $U(t;s)$, and $U(t;a)$

The transport equations for the three auxiliarly radiant energies and their solutions are relatively easy to obtain. We shall illustrate the power of the natural solution procedure by basing the derivations of these equations and representations directly on the knowledge of the n-ary radiant energies developed so far.

We begin with the derivation of the differential equation for attenuated radiant energy $U(t;\alpha)$. From the definition (5) we have

$$\frac{dU(t;\alpha)}{dt} = P_\eta(t) - \frac{dU^0(t)}{dt} \quad .$$

From (8) of Sec. 5.8 we obtain:

$$\boxed{\frac{dU(t;\alpha)}{dt} = \frac{U^0(t)}{T_\alpha}} \tag{9}$$

recalling that the condition $\overline{P}^n(t) = 0$ is in force for every $n \geq 0$ (hence $\overline{P}^0(t) = 0$, in particular, holds). This elegant formula for the growth rate of $U(t;\alpha)$ shows perhaps most clearly the reservoir source of $U(t;\alpha)$ (namely, $U^0(t)$) and the main line which taps the reservoir (namely, T_α, i.e., attenuation). At standard steady state (9) shows that:

$$\frac{dU(\infty;\alpha)}{dt} = P_\eta \tag{10}$$

Thus in the steady state attained under standard growth conditions the rate of increase of $U(t;\alpha)$ is precisely the input rate P_η, so that attenuated radiant energy in the medium increases as fast as it is put into the medium by the source.

Next we consider the scattered radiant energy $U(t;s)$, or "$U^*(t)$" as we would call it ordinarily. The representation (3) of $U(t;s)$ gives rise to the associated differential equation for $U(t;s)$ by computing (with the help of (24) of Sec. 5.8) the following derivative:

$$\begin{aligned} \frac{dU(t;s)}{dt} &= \sum_{j=1}^{\infty} \frac{dU^j(t)}{dt} \\ &= \sum_{j=1}^{\infty}\left(-\frac{U^j(t)}{T_\alpha} + \frac{U^{j-1}(t)}{T_s}\right) \\ &= \left(-\frac{1}{T_\alpha} + \frac{1}{T_s}\right)U(t;s) + \frac{U^0(t)}{T_s} \quad . \end{aligned}$$

Hence:

$$\frac{dU(t;s)}{dt} = -\frac{U(t;s)}{T_a} + \frac{U^o(t)}{T_s} \qquad (11)$$

Here we begin to see some of the utility of the various time constants T_a, T_s, T_α. They serve to remind one of the correct dimensions of each term in an equation or representation, and they serve also to show the physical mechanism associated with that term. Thus we see at a glance from (11) that the rate of growth of U(t;s)--the scattered radiant energy--is augmented by scattering of residual radiant energy $U^o(t)$ and decreased by absorption of scattered radiant energy U(t;s).

There is no need to solve (11) since we need only sum the representations of the $U^j(t)$ in (3) to obtain the desired representation of U(t;s). Thus, under standard growth conditions ((14) of Sec. 5.9):

$$U(t;s) = \sum_{k=1}^{\infty} U^k(t) = \sum_{k=1}^{\infty} U^k(\infty)\ [1-F_k(t/T\alpha)]$$

$$= U^o(\infty) \sum_{k=1}^{\infty} \left[\frac{T_\alpha}{T_s}\right]^k \left[1 - \sum_{j=0}^{k} \frac{(t/T_\alpha)^j}{j!}\ e^{-t/T_\alpha}\right]$$

Hence:

$$U(t;s) = T_a U^o(\infty)\left[\frac{1}{T_\alpha}\ (1-e^{-t/T_a}) - \frac{1}{T_a}\ (1-e^{-t/T_\alpha})\right] \qquad (12)$$

An alternate representation of U(t;s)is obtained by distributing $T_a U^o(\infty)$ throughout the preceding representation. The result is:

$$U(t;s) = \left[\frac{T_a}{T_\alpha}\right] U^o(\infty)(1-e^{-t/T_a}) - U^o(t) \qquad (13)$$

From this we obtain immediately the representation for the directly observable radiant energy. For, by (4) and (13), we have:

$$U(t) = \left[\frac{T_a}{T_\alpha}\right] U^o(\infty)(1-e^{-t/T_a}) \qquad (14)$$

which is clearly a solution of (27) of Sec. 5.8 under standard growth conditions.

Finally the absorbed radiant energy is represented most simply as:

$$\boxed{U(t;a) = P_{\eta} t - U(t)} \tag{15}$$

under standard growth conditions. This representation follows from (4), (5), and (8). A representation under more general growth conditions is obtained by retaining the integral in (5). The differential equation for U(t;a) under standard growth conditions is readily obtained:

$$\frac{dU(t;a)}{dt} = \frac{dU(t;\alpha)}{dt} - \frac{dU(t;s)}{dt}$$

$$= \frac{U^{o}(t)}{T_{\alpha}} - \left(- \frac{U(t;s)}{T_{a}} + \frac{U^{o}(t)}{T_{s}} \right.$$

$$= \frac{U(t;s) + U^{o}(t)}{T_{a}}$$

Hence:

$$\boxed{\frac{dU(t;a)}{dt} = \frac{U(t)}{T_{a}}} \tag{16}$$

We have made a point of deriving the differential equation for U(t;a) so as to make possible the comparison between it and (9). The comparison lends valuable insight into the general roles of scattering and absorption in radiative transfer phenomena. Thus, in the case of (16), the reservoir source for U(t;a) is the directly observable radiant energy and the energy is tapped via the process of absorption.

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5.11 Dimensionless Forms of n-ary Radiant Energy Fields and Related Fields

We shall now develop the dimensionless forms of the various equations and solutions for n-ary radiant energy, residual radiant energy, directly observable radiant energy, and the related energy fields introduced in Sec. 5.10. We shall also explore the various possibilities for the definition of time constants which are to characterize time-dependent light fields in optical media. Before going on to the details of the discussion, some preliminary observations on physical theories using dimensionless concepts are in order.

When the analytical representation of a natural phenomenon can be placed into such a form that the terms of the new representation are dimensionless, this usually indicates that the given phenomenon is a member of an inclusive class of phenomena whose members exhibit a common mathematical representation, but which ostensibly may have different external appearances. The mathematics used to represent the concepts of electrical network theory is a good example of this kind; for the mathematical procedures employed in that theory are