$$\boxed{U(t;a) = P_\eta t - U(t)} \tag{15}$$

under standard growth conditions. This representation follows from (4), (5), and (8). A representation under more general growth conditions is obtained by retaining the integral in (5). The differential equation for U(t;a) under standard growth conditions is readily obtained:

$$\begin{aligned} \frac{dU(t;a)}{dt} &= \frac{dU(t;\alpha)}{dt} - \frac{dU(t;s)}{dt} \\ &= \frac{U^o(t)}{T_\alpha} - \left(-\frac{U(t;s)}{T_a} + \frac{U^o(t)}{T_s}\right) \\ &= \frac{U(t;s) + U^o(t)}{T_a} \end{aligned}$$

Hence:

$$\boxed{\frac{dU(t;a)}{dt} = \frac{U(t)}{T_a}} \tag{16}$$

We have made a point of deriving the differential equation for U(t;a) so as to make possible the comparison between it and (9). The comparison lends valuable insight into the general roles of scattering and absorption in radiative transfer phenomena. Thus, in the case of (16), the reservoir source for U(t;a) is the directly observable radiant energy and the energy is tapped via the process of absorption.

5.11 Dimensionless Forms of n-ary Radiant Energy Fields and Related Fields

We shall now develop the dimensionless forms of the various equations and solutions for n-ary radiant energy, residual radiant energy, directly observable radiant energy, and the related energy fields introduced in Sec. 5.10. We shall also explore the various possibilities for the definition of time constants which are to characterize time-dependent light fields in optical media. Before going on to the details of the discussion, some preliminary observations on physical theories using dimensionless concepts are in order.

When the analytical representation of a natural phenomenon can be placed into such a form that the terms of the new representation are dimensionless, this usually indicates that the given phenomenon is a member of an inclusive class of phenomena whose members exhibit a common mathematical representation, but which ostensibly may have different external appearances. The mathematics used to represent the concepts of electrical network theory is a good example of this kind; for the mathematical procedures employed in that theory are often equally applicable to problems in mechanical dynamics. As a result of this common understructure, researchers in each of these fields have enriched the mathematical methods of the other by noting the applicability of the same set of techniques in each field of study. (See Sec. 5.15.)

Some of the discussions in this chapter have already indicated that the set of transient radiant energy phenomena may be treated as a member of the class of natural phenomena which includes electrical network behavior ((14) of Sec. 5.8; see also concluding comments of Sec. 5.6). We can also point out that the natural mode of solution leads to radiant energy equations which have the same mathematical structure as the equations governing the growth and decay of families radioactive substances. In this case, the counterparts to n-ary radiant energy U^n are the population counts P_n of the nth species S_n of radioactive atoms which are the decay products of species S_{n-1} and where S_n itself decays into species S_{n+1}. Still other and ostensibly different natural phenomena share the same mathematical substructure as the time-dependent radiant energy equations. For example, interacting biological species S_n often are arranged in a predatory hierarchy so that members of species S_n prey upon those in species S_{n-1} and are in turn preyed upon by those in species S_{n+1}. The time-dependent equations governing the population counts of the nth interacting species--be they animal, vegetable, or mineral--often have a common fundamental mathematical core which is obtainable by stripping away the accidental topography of the equations associated with the particular case. The advantages of attaining such dimensionless formulations lie in the resultant conceptual simplifications and economy of description of natural processes.

The casting into dimensionless form of the basic differential equations of transient radiant energy and their associated solutions has practical as well as conceptual advantages. For example, dimensionless formulas allow the inclusion of a wide range of special cases in a single tabulation or graph, the specific case being recoverable after multiplication by a suitable factor. The dimensionless forms thus compress a huge amount of particular numerical information into a relatively small space.

We turn now to the details of the discussion. For simplicity we shall adopt throughout this section the standard growth conditions in a homogeneous optical medium (re: (14) of Sec. 5.9). The developments of this section may serve as a pattern for generalizations to the nonstandard cases.

Conversion Rules for Dimensionless Quantities

An examination of the various analytic representations of $U^o(t)$, $U^*(t)$, $U(t)$, and related radiant energy concepts in Sec. 5.10, with an eye toward achieving dimensionless versions of these representations, brings to light the essential observation that, without exception, each of the representations within the standard growth context obtains its dimension of *energy* from the presence of the product $P_n T_\alpha$ in the form of $U^o(\infty)$. For example, (12) of Sec. 5.8 states that

$$U^o(t) = U^o(\infty)(1-e^{-t/T_\alpha})$$

and (11) of Sec. 5.9 states that:

$$U^n(t) = \left[\frac{T_\alpha}{T_s}\right]^n U^o(\infty)\ [1-F_n(t/T_\alpha)]$$

A perusal of $U(t;\alpha)$, $U(t;s)$, (i.e., $U^*(t)$) and $U(t;a)$ in the preceding section will corroborate the observation still further. This leads us to the following definition.

Definition of the Dimensionless form of U. Let "$U^\#$" denote any of the following radiant energy expressions: $U^n(t)$, $U(t;\alpha)$, $U(t;s)$, $U(t;a)$, $U(t)$. Then we shall write:

$$\text{"}\tilde{U}^\#\text{"} \quad \text{for} \quad U^\#/U^o(\infty)$$

and we call $\tilde{U}^\#$ the *dimensionless* form of U.

The next observation concerns the presence of terms of the form t/T_α, t/T_s, t/T_a, T_α/T_a, T_s/T_a, and T_α/T_s in the various equations constructed so far. These expressions are already dimensionless. The observation to make at present is that these six terms, which involve four separate concepts, can be represented compactly by means of only two distinct concepts, namely the ratio t/T_α and the scattering-attenuation ratio $\rho(=s/\alpha)$. To see this, let us write:

$$\text{"}\tau\text{"} \quad \text{for} \quad t/T_\alpha \tag{1}$$

We call τ the *relative time.* Its connection with steady state concepts is very close and may be stated succinctly by first writing

$$\text{"}L_\alpha\text{"} \quad \text{for} \quad 1/\alpha \quad .$$

We call L_α the *attenuation length* associated with the optical medium. Since T_α is $1/v\alpha$, we see that:

$$L_\alpha = vT_\alpha \tag{2}$$

so that:

$$\tau = t/T_\alpha = vt/L_\alpha \tag{3}$$

From (3), τ may be interpreted not only in a temporal sense (i.e., the number of attenuation times in a certain time t), but in a spatial sense, too, namely the number of attenuation lengths in a certain path (traversed by light in real time t). The representation of the six dimensionless terms displayed above may be made in terms of ρ and τ as follows:

TABLE 2

Representation of six dimensionless terms.

t/T_α	τ
t/T_s	$\rho\tau$
t/T_a	$(1-\rho)\tau$
T_α/T_s	ρ
T_α/T_a	$(1-\rho)$
T_s/T_a	$(1-\rho)/\rho$

We are now ready to state the conversion rules by which one is guided to the dimensionless differential equations and associated solutions for the various radiant energy fields. Towards this end, we note that the derivative:

$$\frac{dU\#(t)}{dt}$$

may be written as:

$$\frac{dU\#(\tau)}{d\tau} \cdot \frac{d\tau}{dt} \quad ,$$

where:

$$\frac{d\tau}{dt} = 1/T_\alpha$$

so that:

$$\boxed{\frac{dU\#(\tau)}{d\tau} = T_\alpha \frac{dU\#(t)}{dt}} \qquad (4)$$

Conversion rule 1. *To convert $dU\#(t)/dt$ to dimensionless form under standard growth conditions, multiply by $T_\alpha/U^o(\infty)$ and change all time ratios of the kind t/T_x and T_x/T_y into their equivalent forms in terms of ρ and τ, using Table 2.*

Conversion rule 2. *To convert $U\#(t)$ to dimensionless form under standard growth conditions, multiply by $1/U^o(\infty)$ and change all time ratios of the kind t/T_x and T_x/T_y into their equivalent forms in terms of ρ and τ, using Table 2.*

Dimensionless Forms for $U^o(t)$

Starting with (8) of Sec. 5.8 under the standard growth condition, we have

$$\frac{dU^o(t)}{dt} = -\frac{U^o(t)}{T_\alpha} + P_\eta$$

To apply conversion rules 1 and 2, we write this as:

$$T_\alpha \frac{d\,U^o(t)/U^o(\infty)}{dt} = -\ U^o(t)/U^o(\infty) \quad + \quad P_\eta T_\alpha/U^o(\infty)$$

and then go on to obtain:

$$\boxed{\frac{d\tilde{U}^o(\tau)}{d\tau} = -\ \tilde{U}^o(\tau) + 1} \qquad (5)$$

The solution of (5) is:

$$\boxed{\tilde{U}^o(\tau) = 1-e^{-\tau}} \qquad (6)$$

The only dimensionless parameter in the representation of $\tilde{U}^o(\tau)$ is the relative time τ. The absence of ρ from (5) and (6) indicates that the growth of residual radiant energy is basically independent of the medium in which it takes place. At any rate $\tilde{U}^o(\tau)$ will be seen to differ from $\tilde{U}^n(\tau)$, e.g., the growth and decay of which depends critically on the parameter ρ.

Dimensionless Forms for $U^n(t)$

Starting with (24) of Sec. 5.8 under the standard growth condition, we have:

$$\frac{dU^n(t)}{dt} = -\frac{U^n(t)}{T_\alpha} + \frac{U^{n-1}(t)}{T_s}$$

which we may write as:

$$T_\alpha \frac{[d\,U^n(t)/U^o(\infty)]}{dt} = -\ U^n(t)/U^o(\infty) \quad + \quad \frac{T_\alpha}{T_s}\ U^{n-1}(t)/U^o(\infty) \quad ,$$

which by conversion rules 1 and 2 become:

$$\boxed{\frac{d\tilde{U}^n(\tau)}{d\tau} = -\ \tilde{U}^n(\tau) + \rho\tilde{U}^{n-1}(\tau)} \qquad (7)$$

which has the solution:

$$\tilde{U}^n(\tau) = \rho^n \, [1-F_n(\tau)] \tag{8}$$

where F_n is defined in (13) of Sec. 5.9. From (8) we have immediately that:

$$\tilde{U}^n(\infty) = \rho^n \tag{9}$$

for every $n \geq 1$, and a study of (7) shows that this relation holds also for $n = 0$.

It is interesting to note how (7), even though defined only for $n \geq 1$, actually reduces to the correct relation when $n = 0$. A comparison of (5) and (7), suggests that we can identify the term $\rho\tilde{U}^{n-1}(\tau)$ with 1 when $n = 0$, i.e., we are encouraged to extend the meaning of $\tilde{U}^j(\tau)$ to the case where $j = -1$. Thus let us write:

$$\text{"}\tilde{U}^{-1}(\tau)\text{"} \quad \text{for} \quad 1/\rho \tag{10}$$

In full dimensional form this means that we have the definitional identity:

$$U^{-1}(t) = P_\eta \, T_s \tag{11}$$

With this extension, we may use (7) as the basic n-ary differential equation which then includes (5) as a special case.

Dimensionless Forms for U*(t)

Applying the conversion rules to (11) of Sec. 5.10, we have, under the standard growth condition:

$$\frac{d\tilde{U}^*(\tau)}{d\tau} = -\,(1-\rho)\,\tilde{U}^*(\tau) + \tilde{U}^o(\tau) \tag{12}$$

with solution:

$$U^*(\tau) = \frac{1}{1-\rho}\left[1-e^{-(1-\rho)\tau}\right] - \left[1-e^{-\tau}\right] \tag{13}$$

It is interesting to see how (13) predicts the growth of scattered radiant energy in extreme media, i.e., media for which $\rho = 0$ and for which $\rho = 1$, e.g., in purely absorbing and scattering media, respectively. To see this, observe that:

$$\lim_{\rho \to 1} \frac{1-e^{-(1-\rho)\tau}}{1-\rho} = \tau$$

Then we have from (13):

$$\tilde{U}^*(\tau) = (\tau - 1) + e^{-\tau} \tag{14}$$

Thus in purely scattering media, at $\tau = 0$, $U^*(0) = 0$, and for very small relative times τ:

$$\tilde{U}^*(\tau) \cong \frac{\tau^2}{2} \quad ,$$

so that $\tilde{U}^*(\tau)$ commences growth parabolically from $\tau = 0$. For somewhat larger τ, $\tilde{U}^*(\tau)$ grows essentially linearly with τ, as might be expected. In the case of the other extreme type of space, the purely absorbing space, i.e., one for which $\rho = 0$, equation (13) predicts $\tilde{U}^*(\tau) = 0$ for every τ, as expected. In general for *normal spaces*, i.e., for spaces in which there is present both scattering and absorption, so that $0 < \rho < 1$, (13) predicts the steady state value of U^* to be

$$\tilde{U}^*(\infty) = \frac{\rho}{1-\rho} \tag{15}$$

This agrees with the natural solution computation based on (9):

$$\tilde{U}^*(\infty) = \sum_{n=1}^{\infty} U^n(\infty) = \sum_{n=1}^{\infty} \rho^n = \frac{\rho}{1-\rho} \tag{16}$$

The growth pattern of $\tilde{U}^*(\tau)$ is relatively interesting because the rate of growth of $\tilde{U}^*(\tau)$ exhibits a maximum at a certain finite time which depends on on ρ. Thus, from (13) we have:

$$\frac{d\tilde{U}^*(\tau)}{d\tau} = (e^{\rho\tau} - 1)e^{-\tau} \quad . \tag{17}$$

For normal spaces, i.e., when $0 < \rho < 1$, this rate of growth is zero for $\tau = 0$ and $\tau = \infty$ and positive for all intermediate τ. The τ for maximum growth rate is obtained in the usual manner using calculus, and is of the form τ_{max}, where we have written:

$$\text{"}\tau_{max}\text{"} \quad \text{for} \quad \frac{-\ln(1-\rho)}{\rho} \tag{18}$$

We shall have occasion to return to this relative time in the discussion below on time constants.

Dimensionless Forms for U(t)

Applying the conversion rules to (27) of Sec. 5.8, we have, under the standard growth condition:

$$\boxed{\frac{d\tilde{U}(\tau)}{d\tau} = -\ (1-\rho)\ \tilde{U}(t)\ +\ 1} \tag{19}$$

whose solution is:

$$\boxed{\tilde{U}(\tau) = \frac{1-e^{-(1-\rho)\tau}}{1-\rho}} \tag{20}$$

Note that for purely scattering media ($\rho = 1$):

$$\frac{d\tilde{U}(\tau)}{d\tau} = 1$$

which implies:

$$\tilde{U}(\tau) = \tau$$

for all $\tau \geq 0$. For purely absorbing media, $\tilde{U}(\tau) = \tilde{U}^{o}(\tau)$. In normal spaces the steady state value of $\tilde{U}(\tau)$ is:

$$\tilde{U}(\infty) = \frac{1}{1-\rho} \tag{21}$$

Dimensionless Forms for $U(t;\alpha)$, $U(t;a)$

From (9) of Sec. 5.10 and the conversion rules we obtain:

$$\boxed{\frac{d\tilde{U}(\tau;\alpha)}{d\tau} = \tilde{U}^{o}(\tau)} \tag{22}$$

whence, under standard growth conditions:

$$\boxed{\tilde{U}(\tau;\alpha) = (\tau-1) + e^{-\tau}} \tag{23}$$

This agrees with the special case (14) of the representation of $\tilde{U}^{*}(\tau)$ (alias $\tilde{U}(\tau;s)$), i.e., under the special case where $s = \alpha$. Finally, from (16) of Sec. 5.10:

$$\boxed{\frac{d\tilde{U}(\tau;a)}{d\tau} = (1-\rho)\ \tilde{U}(\tau)} \tag{24}$$

whence, under standard growth conditions:

$$\boxed{\tilde{U}(\tau;a) = \tau - \tilde{U}(\tau)}\ . \tag{25}$$

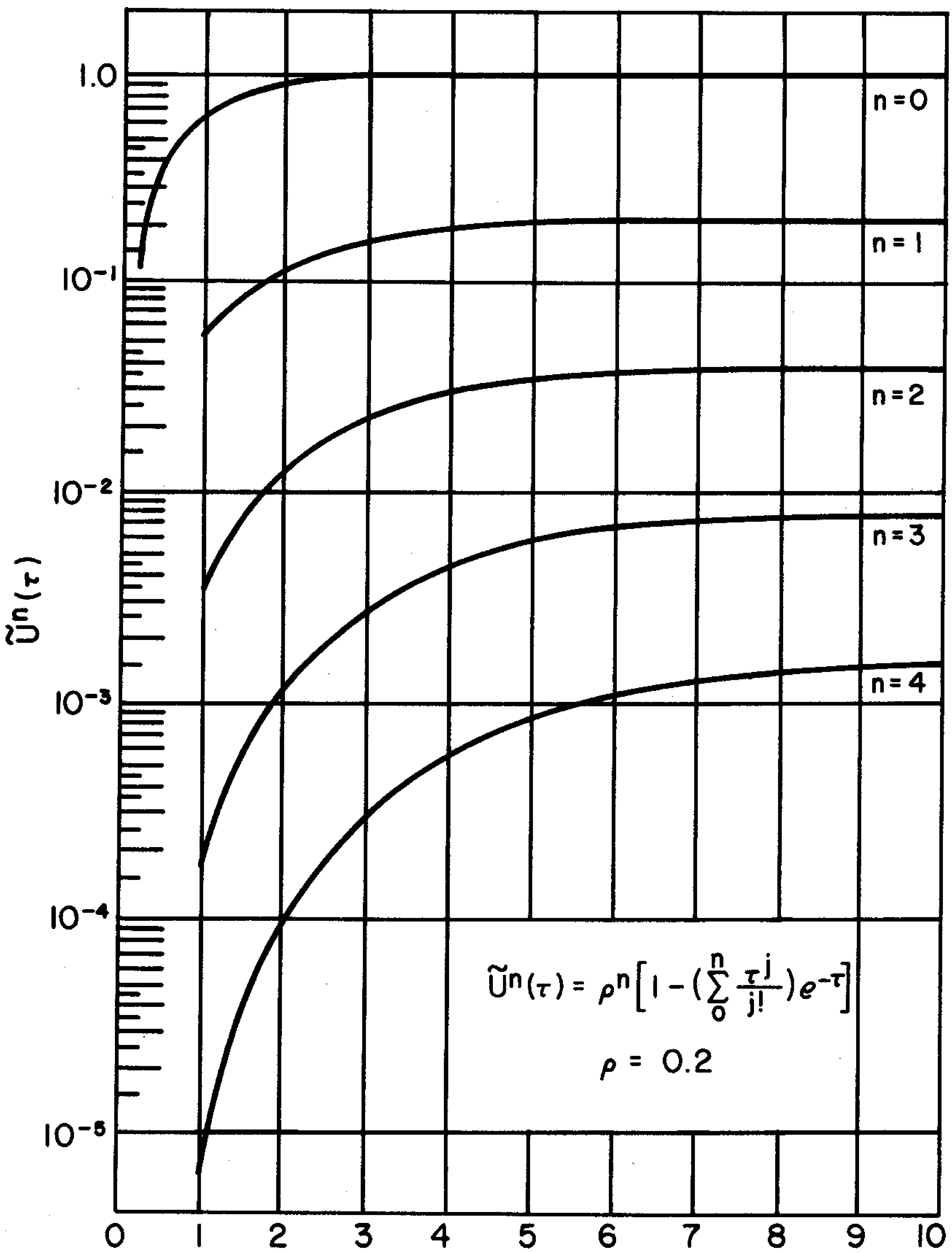

FIG. 5.13 A plot of $\tilde{U}^n(\tau)$ versus τ for n = 0, 1, 2, 3, 4 in an optical medium which has ρ = 0.2 (see (8) of Sec. 5.11).

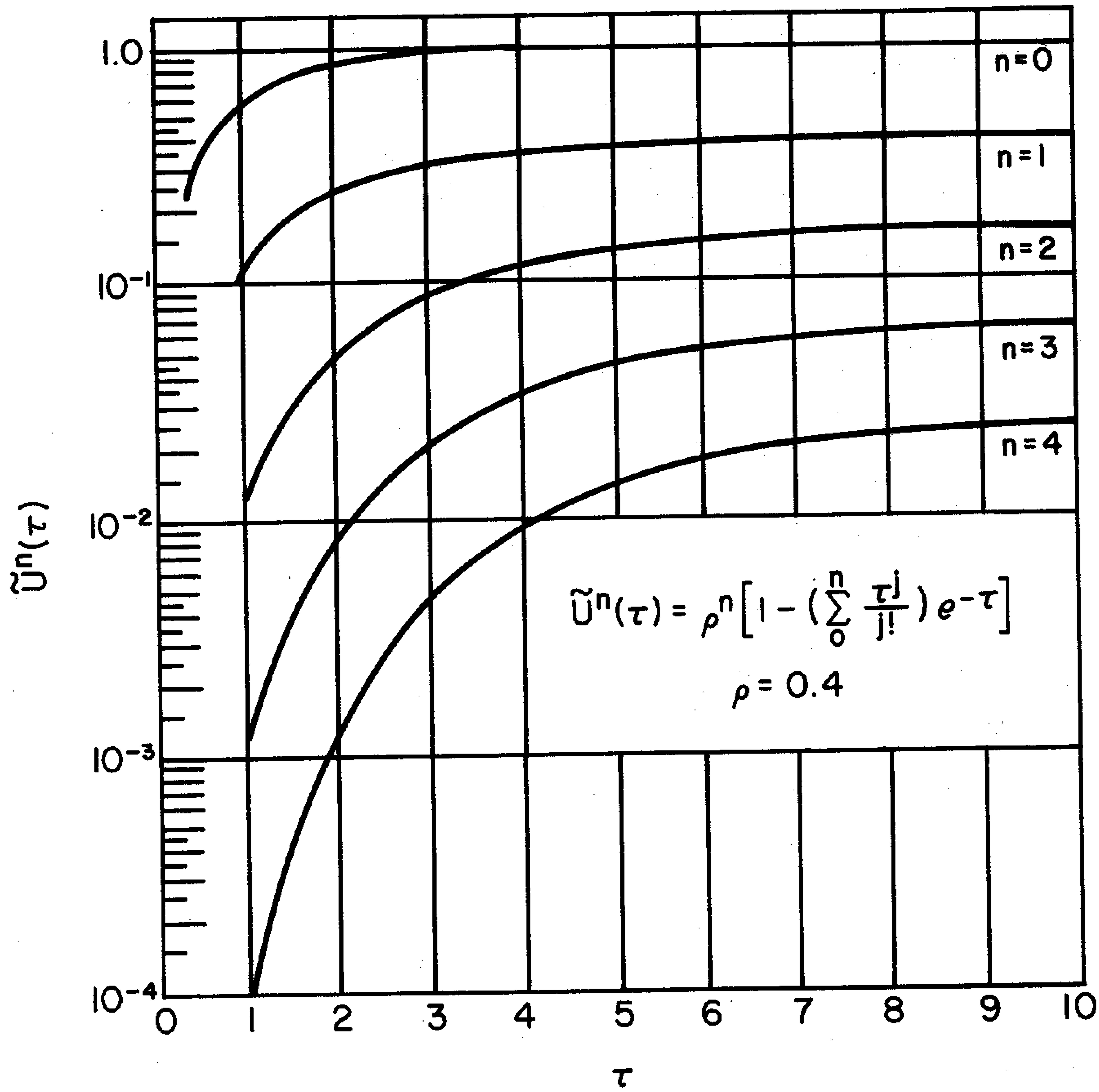

FIG. 5.14 A plot of $\tilde{U}^n(\tau)$ versus τ for n = 0, 1, 2, 3, 4 in an optical medium which has ρ = 0.4 (see (8) of Sec. 5.11). Note that the vertical spread of the curves is decreasing, and that the steady state values of $\tilde{U}^n(\tau)$ crowd closer together for higher ρ values.

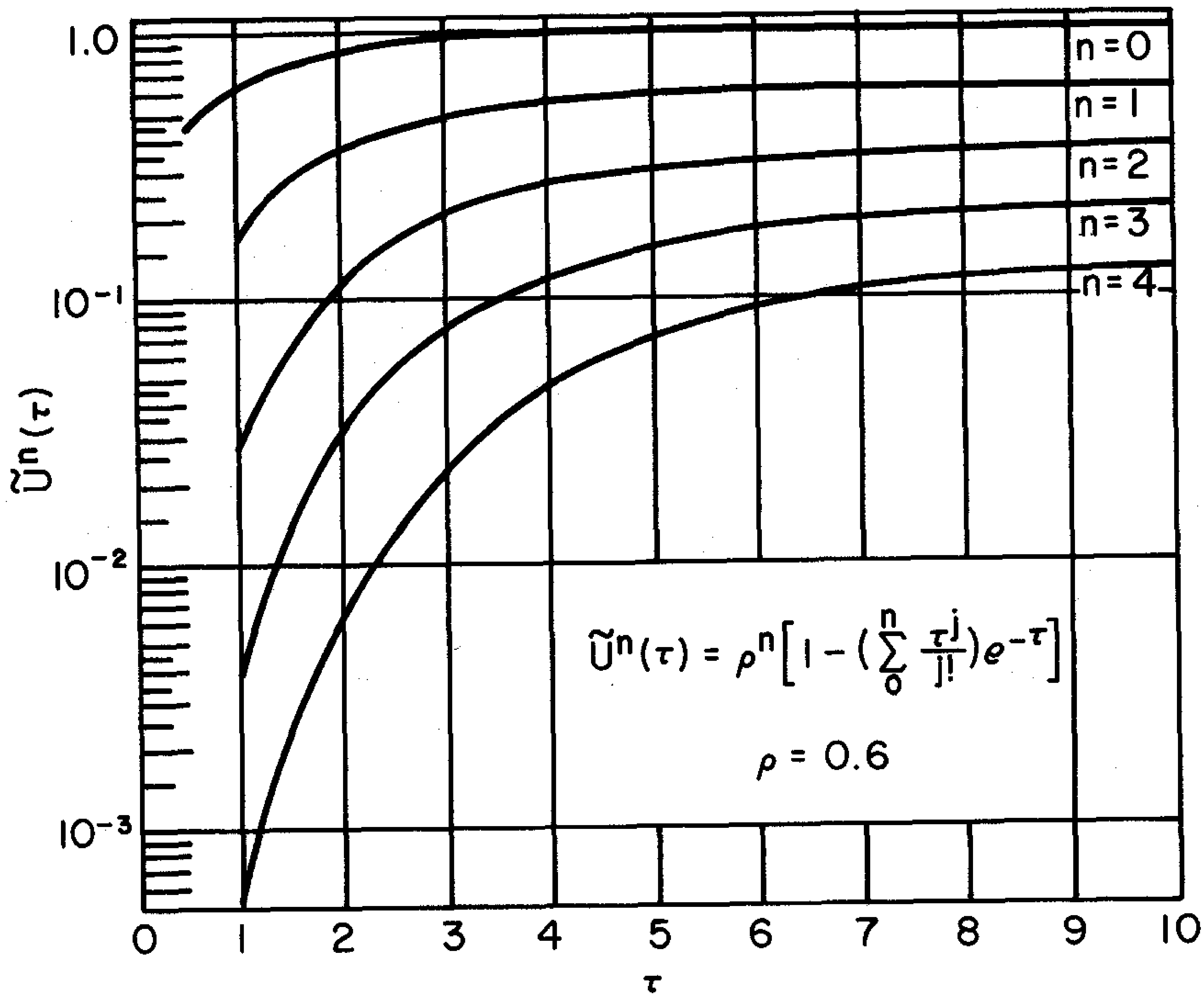

FIG. 5.15 Continuation of Figures 5.13, 5.14.

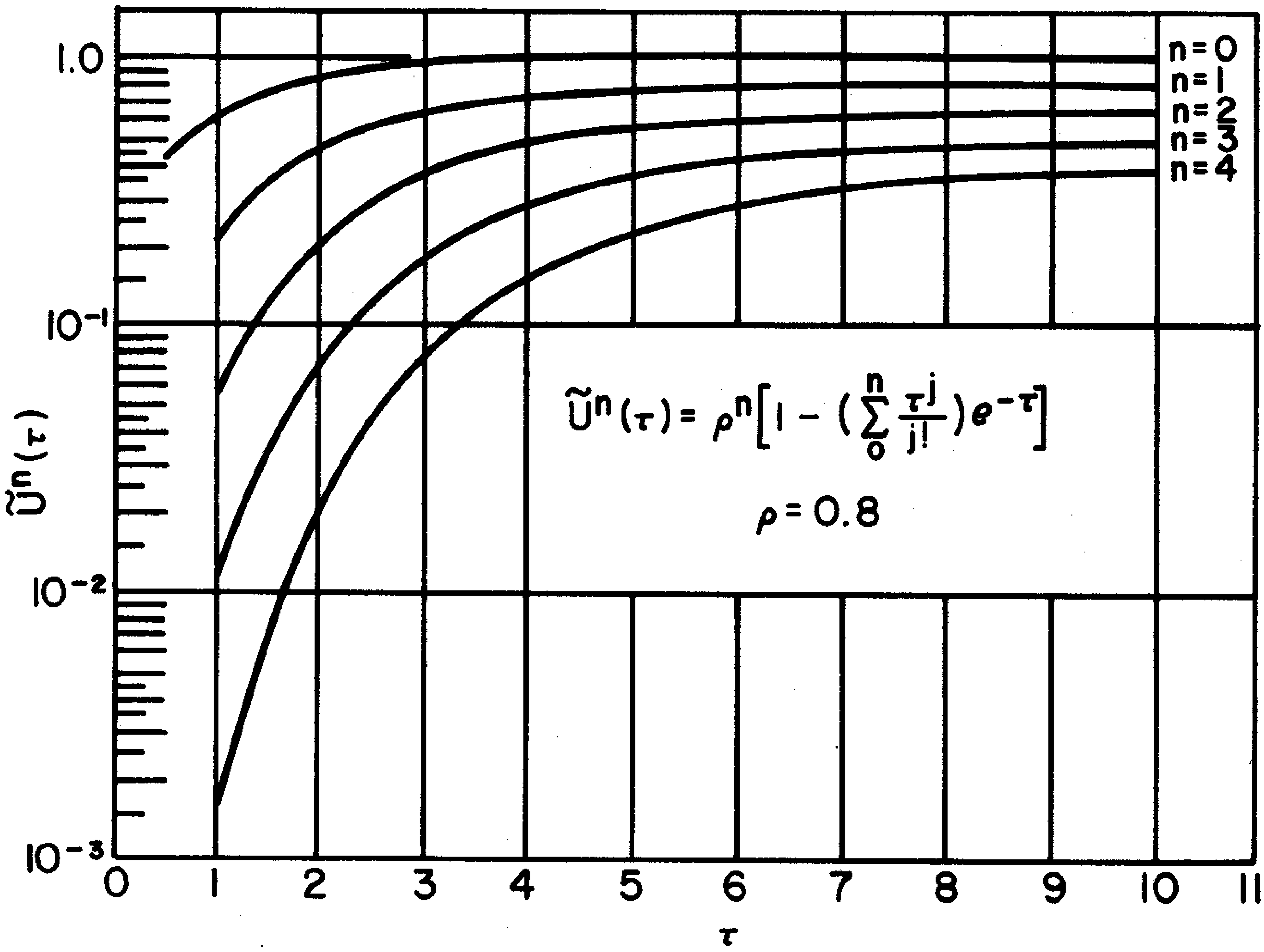

FIG. 5.16 Continuation of Figs. 5.13 through 5.15.

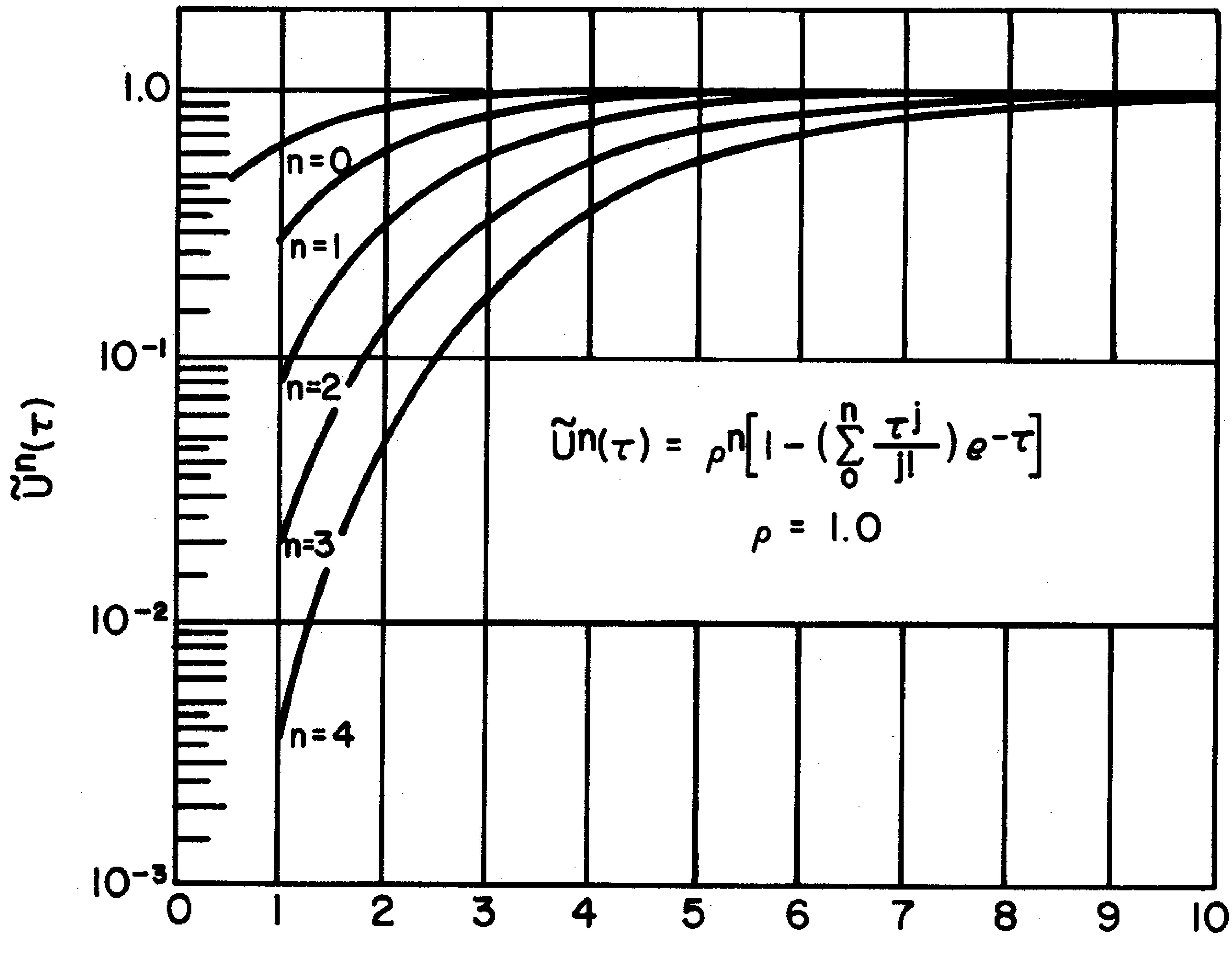

FIG. 5.17 Conclusion of Figs. 5.13 through 5.16.

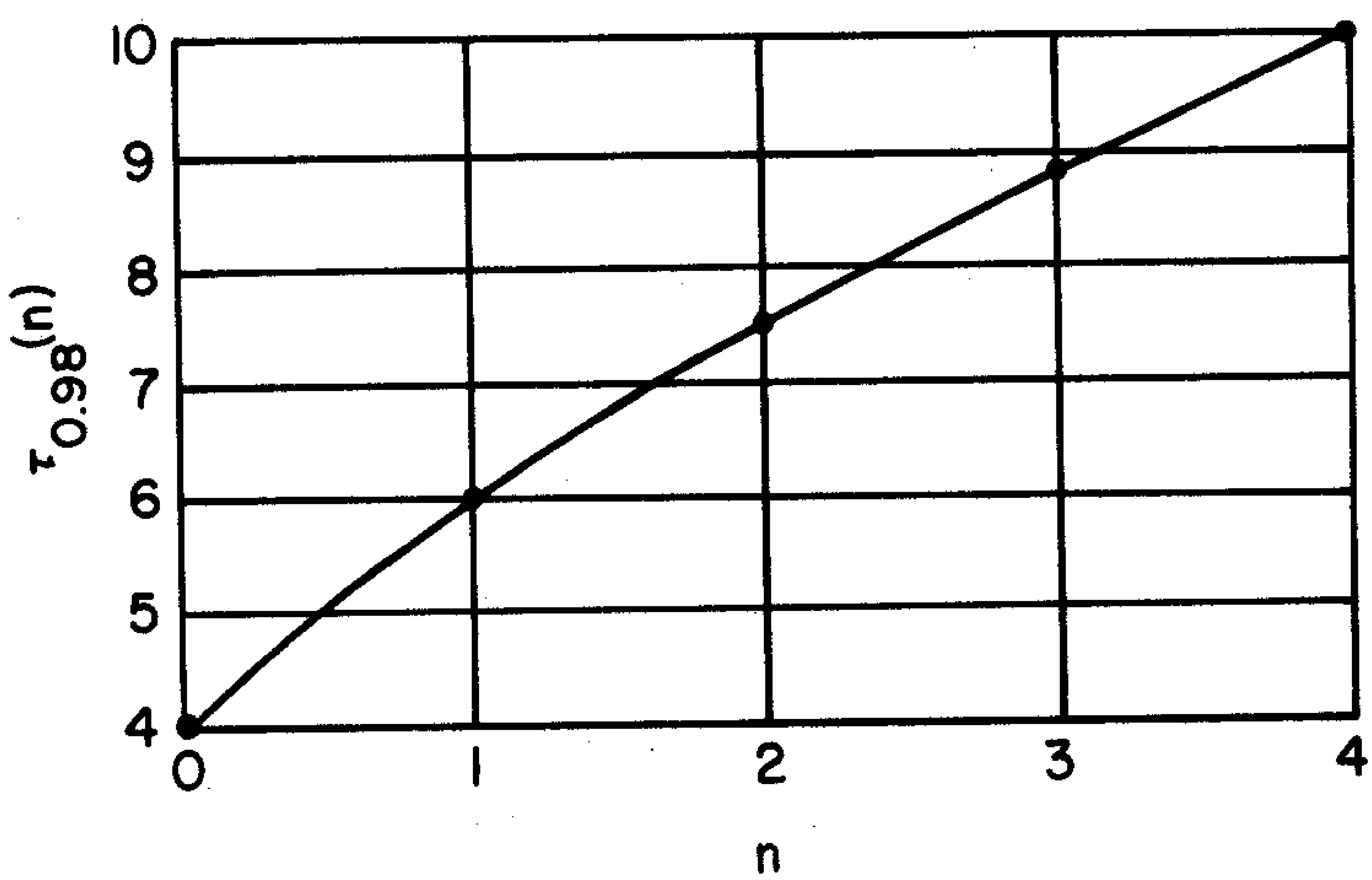

FIG. 5.18 A plot of time constants for $\tilde{U}^n(T)$, $n = 0,1,2,3,4$ in which $c = 0.98$. (See (27) of Sec. 5.11.)

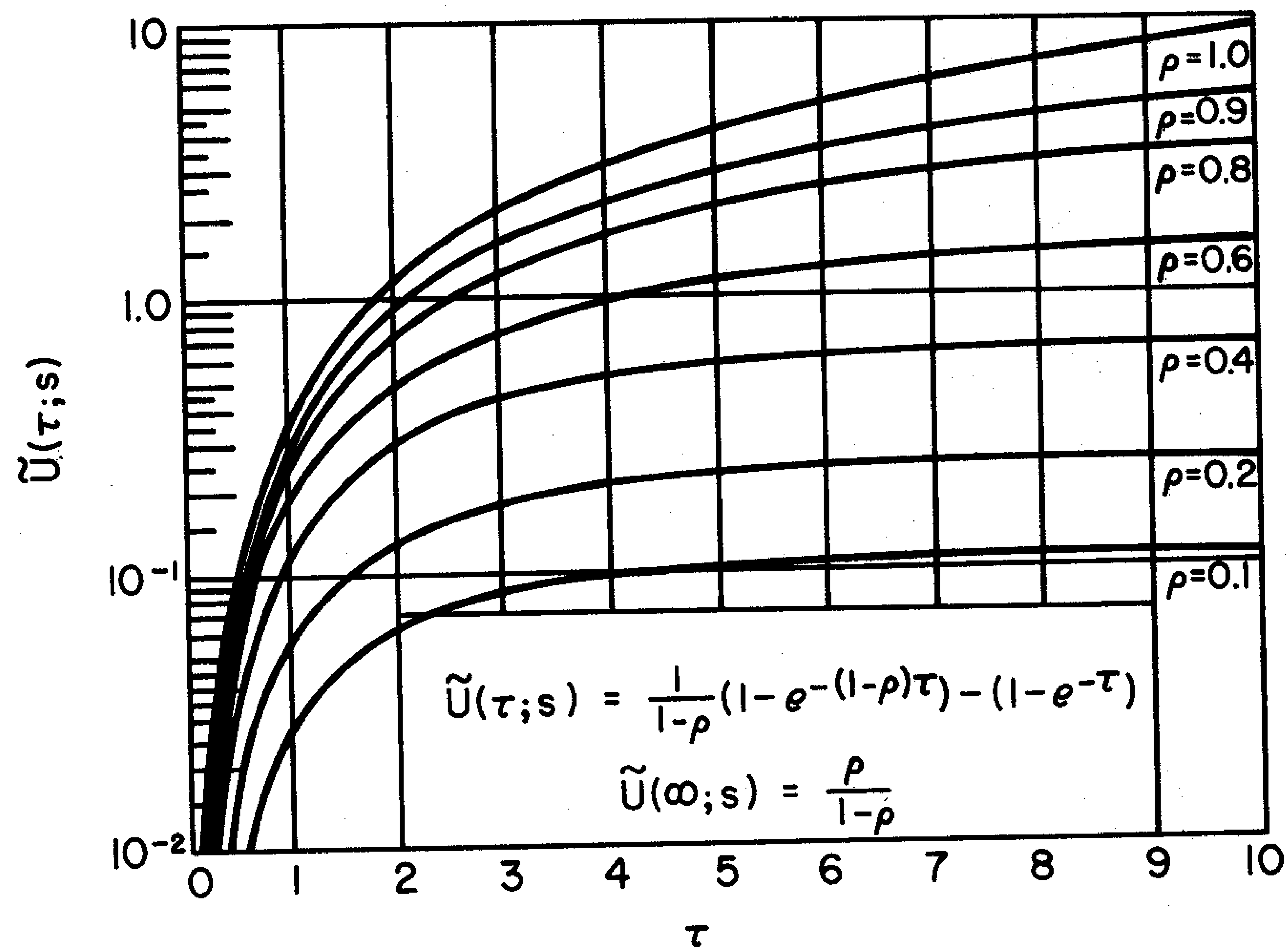

FIG. 5.19 Plots of $\tilde{U}(\tau;s)$ ($=\tilde{U}^*(\tau)$) versus relative time τ. Each curve represents a different scattering attenuation ratio ρ. $\tilde{U}(\tau;s)$ is the dimensionless form of $U(t;s)$, and this latter quantity is the total amount of scattered radiant energy in the optical medium at time t after the steady source has been turned on. $U(t;s)$ is the sum of all n-ary radiant energy components $U^n(t)$, n = 1,2,3,.... Some of the latter quantities are plotted in Figs. 5.13 through 5.17, in dimensionless form. Each curve in the present figure, except for $\rho = 1$, levels off to approach the asymptote $\rho/(1-\rho)$. (See (15) of Sec. 5.11.)

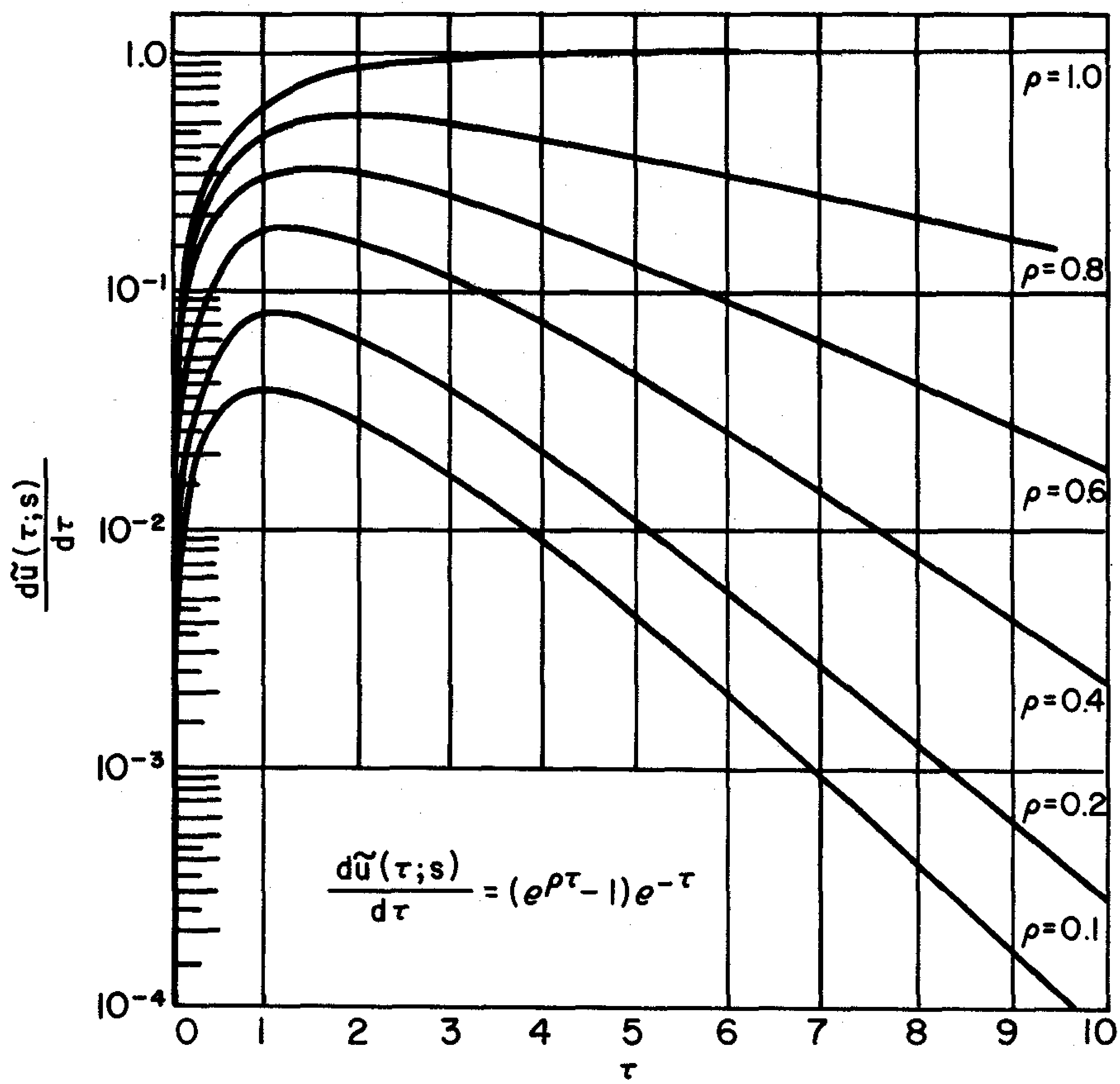

FIG. 5.20 Showing the evolution, in time, of the scattered radiant energy (see (17) of Sec. 5.11).

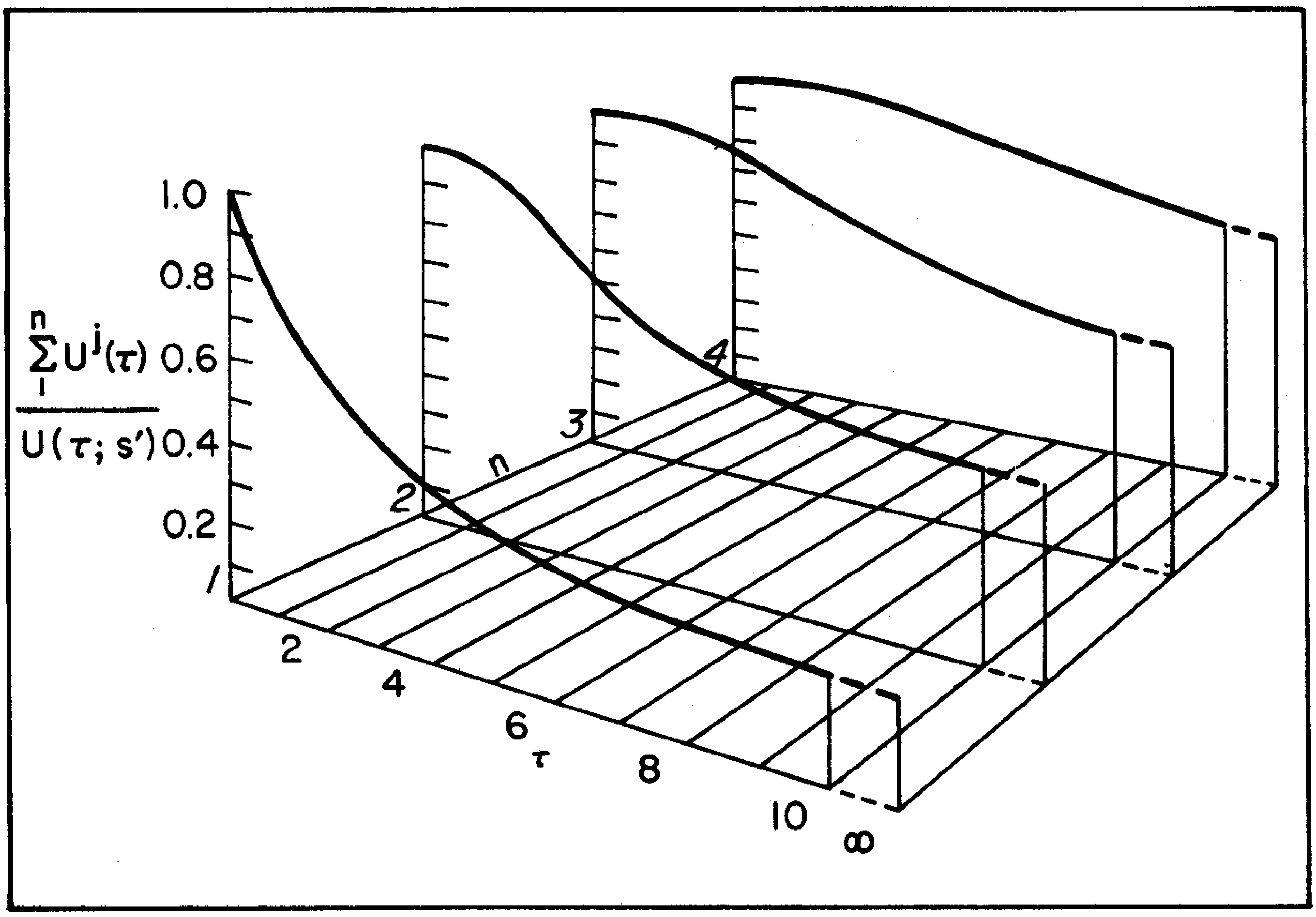

FIG. 5.21 A plot showing the relative magnitude of the sum of the first n scattering orders

$$\sum_{j=1}^{n} U^j(\tau)$$

of radiant energy at time τ as compared to the total amount U(τ;s) of scattered radiant energy at the same time. The plot is for a space with scattering-attenuation ratio ρ = 0.8. Observe that for fixed n, the ratio is monotonic *decreasing* with time τ. For fixed time τ, the ratio *increases* with increasing scattering order. As an example, let n = 3, and τ = 5. Then the ratio of $U^j(\tau)$ to U(τ;s) is 0.8; for τ = 10, the ratio is 0.6; and in the limit, as $\tau \to \infty$, the ratio is 0.48. Hence, at steady state the amount of radiant energy having been scattered, once, twice, or three times is 48 percent (= $1 - \rho^n$) of all that has been scattered in general (see Fig. 5.22).

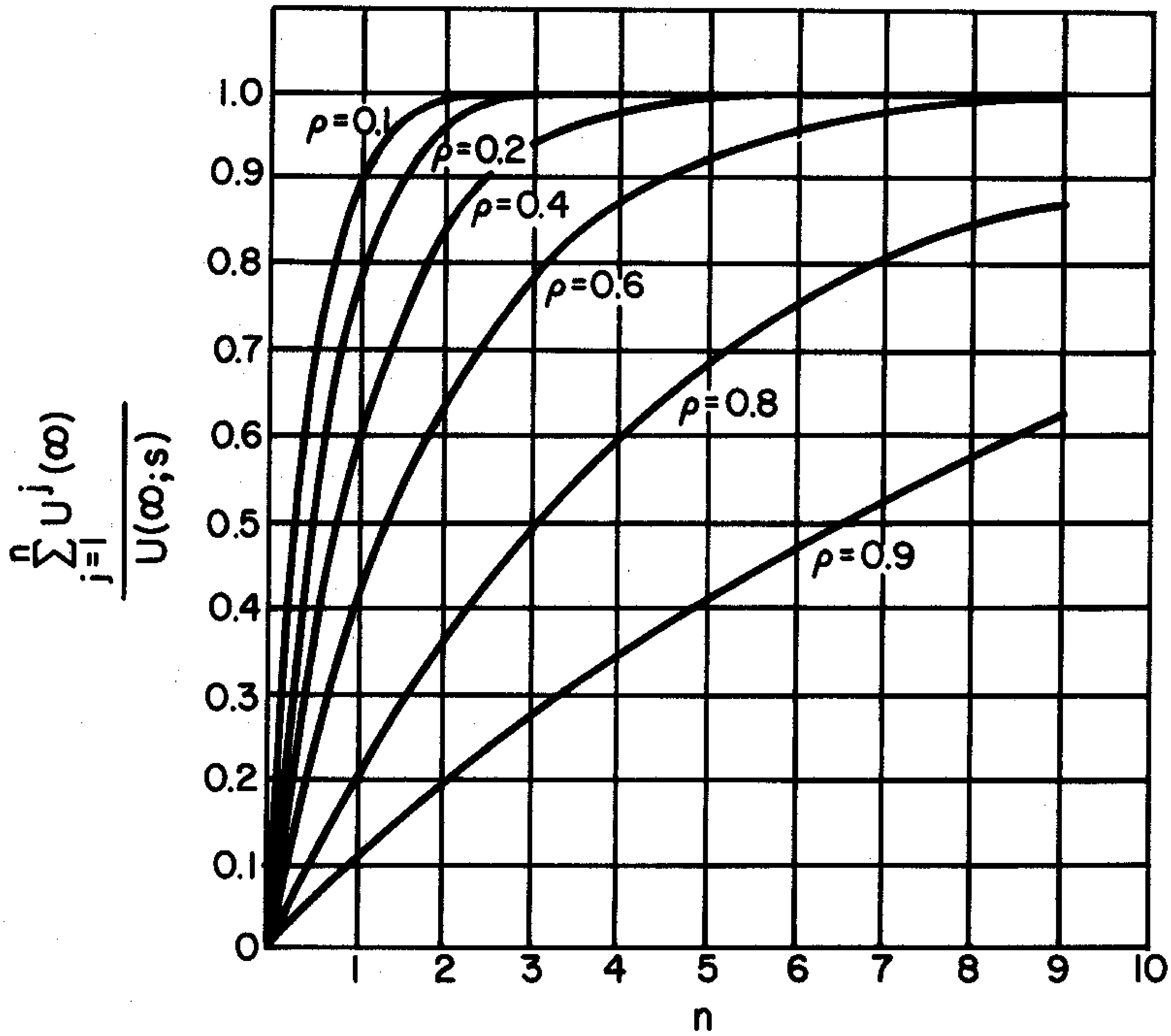

FIG. 5.22 The limiting values, for $\tau = \infty$, of the ratios in Fig. 5.21.

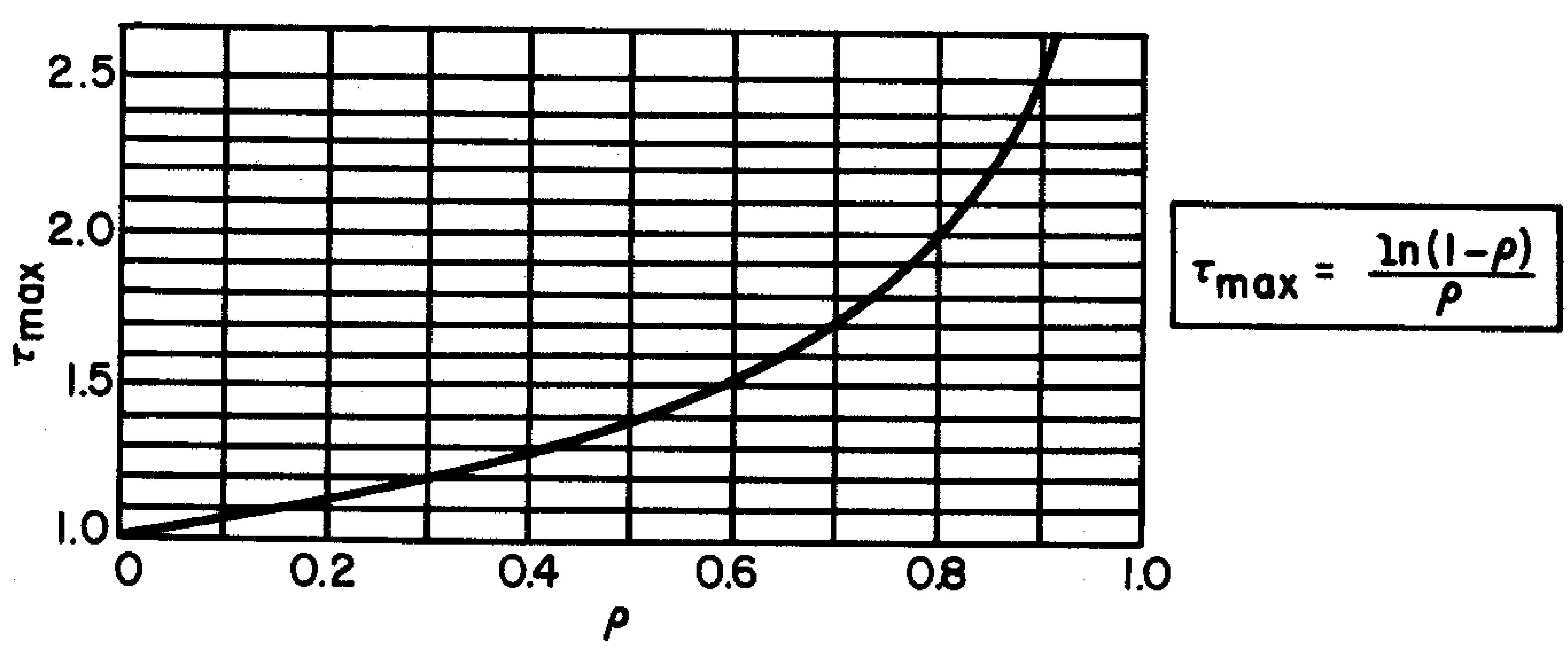

FIG. 5.23 The relative times for the occurrences of the maxima in Fig. 5.20, plotted as a function of ρ. For example, the curve labeled "ρ = 0.08" in Fig. 5.20 has its maximum at about τ = 2.

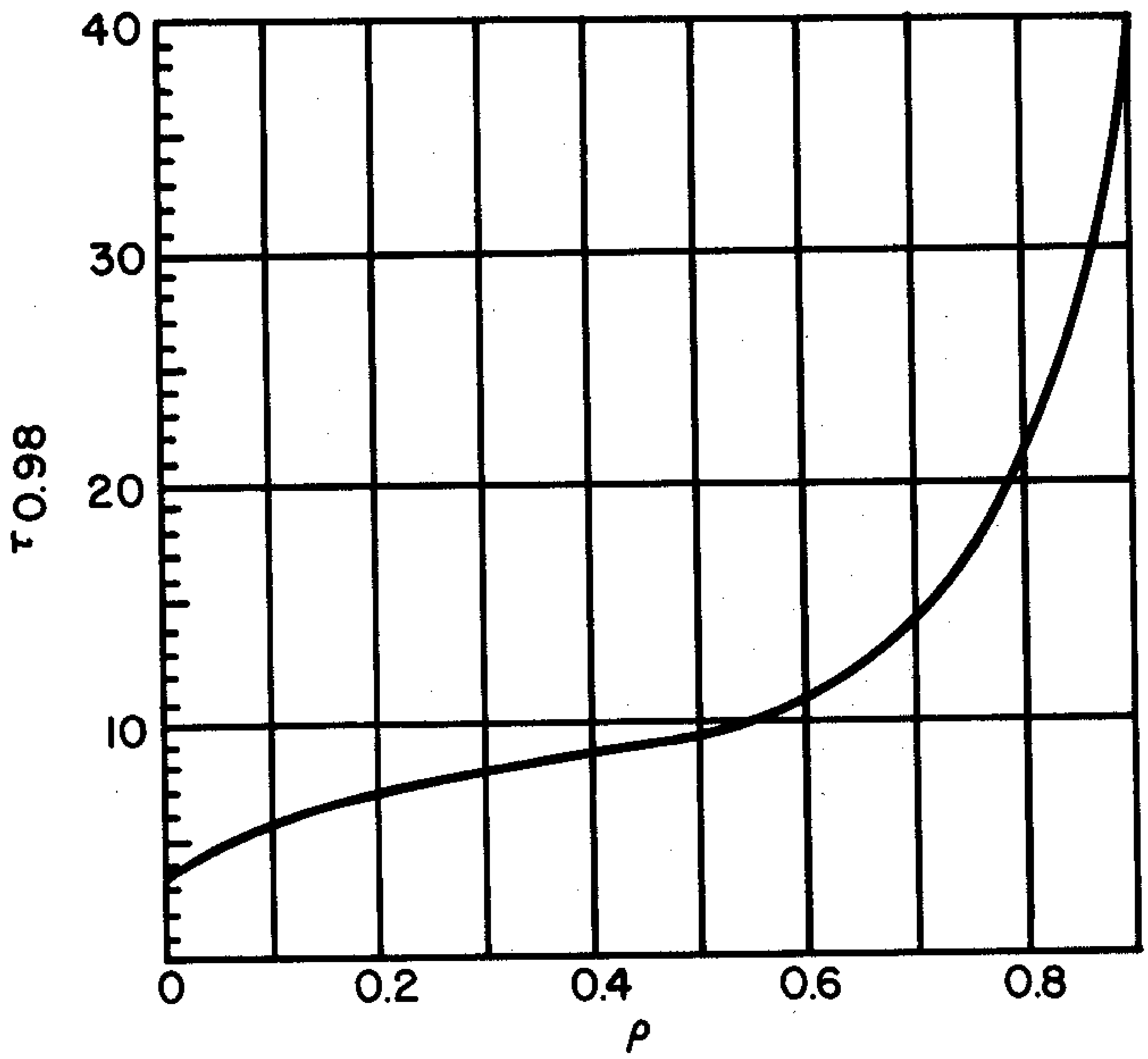

FIG. 5.24 The time constant $\tau_{0.98}$ as a function of scattering-attenuation ratio ρ. (See (26) of Sec. 5.11.)

A Discussion of Time Constants

Time-dependent natural phenomena may be broadly classed into two main groups: those that are periodic and those that are not periodic over a given time interval. Periodic phenomena can in turn be characterized in part by means of their *periods*, i.e., the smallest intervals of time over which they exhibit a basic cycle of behavior. Nonperiodic phenomena on the other hand have very many ways of being nonperiodic, and there is no simple single number which suggests itself as a suitable measure of such general nonperiodicities. Of the great variety of nonperiodic phenomena, however, there are those which appear to eventually tend with increasing time toward a well-defined limit. These *nonperiodic limiting* phenomena can then be characterized in a manner analogous to the periodic phenomena, i.e., by means of single numbers which suitably measure such simple nonperiodicities. One useful means is the concept of the *time constant* of such phenomena. The time constant, broadly speaking, is that interval of time over which the nonperiodic limiting phenomenon evolves from some standard initial state until it arrives just within a prescribed "distance" of its limit state.

Time-dependent light fields in natural optical media are generally phenomena of the *nonperiodic limiting* type discussed above. Therefore the notion of a time constant characterization of such phenomena seems worthwhile exploring. In the discussion that follows we shall examine some possible candidates for time constants of transient light fields in natural optical media. One major fact that will emerge from the discussion is that there is a large number of possible candidates for time constants, each valuable in the context in which it is found and used. Thus it will turn out that, in the long run, no one single time constant will suffice for the description of every instance in the great variety of time-dependent radiant energy fields encountered in the various natural media (oceans, lakes, atmosphere). *The best choice of time constant that can be made will vary jointly with the type of radiometric concept used (radiance, irradiance, or any of the variety of radiant energies discussed so far) and the space in which the light field is evolving.*

To illustrate the thesis just stated, consider once again the residual radiant energy $U^o(t)$ discussed in Sec. 5.8, now in comparison with the directly observable radiant energy $U(t)$. We saw in Sec. 5.8 the exact analogy that held between a simple resistance-capacitance DC circuit and an infinite homogeneous optical medium in which $U^o(t)$ was evolving. This analogy suggested that the candidate for the time constant associated with U^o in the medium was T_α. Comparing the form of $U^o(t)$ with that of $U(t)$ as given in (14) of Sec. 5.10, we see that in the same medium, but now with reference to $U(t)$, the most obvious candidate for the time constant is T_a. Thus by switching from $U^o(t)$ to $U(t)$ the appropriate choice for time constant correspondingly goes from T_α to T_a.

As another illustration of the thesis of this discussion, consider the scattered radiant energy $U^*(t)$ $(=U(t;s))$ as given in (12) of Sec. 5.10 and its dimensionless graphical representation in Fig. 5.19. The steady state value of $\tilde{U}^*(\tau)$ is $\rho/(1-\rho)$ in normal spaces, i.e., spaces in which $0 < \rho < 1$. Figure 5.19 shows how $\tilde{U}^*(\infty)$ approaches this value asymptotically for selected values of ρ. For example, if $\rho = 0.4$ then $\tilde{U}^*(\infty) = 0.4/(1-0.4) = 0.67$. This value has been attained (at least visually, according to the graph) at about eight relative time units. More generally, in a given space with $0 < \rho < 1$, let c be any number such that $0 < c < 1$. Then we require that value τ_c of τ such that:

$$\frac{c\rho}{1-\rho} = \tilde{U}^*(\tau_c)$$

$$= \frac{1}{1-\rho}\left(1-e^{-(1-\rho)\tau_c}\right) - \left(1-e^{-\tau_c}\right) \qquad (26)$$

For every ρ, $0 < \rho < 1$, the number τ_c always exists since $\tilde{U}^*(\tau)$ is continuous and increases monotonically toward its limit, and so eventually takes on the value $c\rho/(1-\rho)$ for $0 < c < 1$. A graph of τ_c for $c = 0.98$ is given in Fig. 5.24 as a function of ρ. For example, for $\rho = 0.4$, $\tau_c = 8$, and so we return to the visual estimate given above. The graph of Fig. 5.24 shows generally that the greater the scattering attenuation ratio, the greater $\tau_{0.98}$--this much could be guessed on intuitive grounds--however, the exact quantitative manner of the increase in $\tau_{0.98}$ is interesting to observe. The numbers τ_c, therefore, can serve as time constants for scattered radiant energy after a choice of c is made.

The time-dependent structure of the scattered radiant energy $\tilde{U}^*(\tau)$ has an additional feature to that of asymptoticity which may serve to be a workable basis for the definition of a time constant. A study of the rate of growth of $\tilde{U}^*(\tau)$ in Sec. 5.11 showed that the derivative of the rate of growth starts out positive, becomes zero at relative time $-\ln(1-\rho)/\rho$, and then remains negative for all subsequent relative times in very given normal medium (cf. (18) of sec. 5.11). This suggests that τ_{max}, the relative time of the maximum rate of growth, is a possible candidate for a time constant for a given medium, for it defines a distinguishable point of inflection on the growth curve of $\tilde{U}^*(\tau)$. Figure 5.23 depicts τ_{max} as a function of ρ for a selected range of normal spaces. The point to observe here is that we need not always base time constant definitions on the feature of asymptoticity of a nonperiodic phenomenon. Well-defined maxima or minima or points of inflection of growth curves may also serve as adequate bases for time constants.

It is interesting to observe how the notion of a time constant can be extended to each of the n-ary radiant energy fields U^n, $n > 0$. The best candidate for the time constant varies with the scattering order n. Thus, suppose c is any number such that $0 < c < 1$. Let $\tau_c(n)$ be that relative time for which:

$$c\tilde{U}^n(\infty) = \tilde{U}^n(\tau_c(n)) = \rho^n \; [1-F_n(\tau_c(n))]$$

holds. That is we require $\tau_c(n)$ such that:

$$1-c = F_n(\tau_c(n)) \quad . \tag{27}$$

As in the case of (26), $\tau_c(n)$ exists for every $n \geq 1$ and c such that $0 < c < 1$. The basis for this conclusion is property 2 of $U^n(t)$, stated in Sec. 5.10, which implies that $\tilde{U}^n(\tau)$ increases monotonically and continuously to its limit. Figure 5.18 depicts a plot of $\tau_c(n)$ for $c = 0.98$ and $n = 0, 1, 2, 3, 4$.

Still one more variation in the concept of time constant follows from the observation that the curves of $\tilde{U}^n(\tau)$ have inflection points at relative times $\tau = n$. Thus setting:

$$\frac{d^2\tilde{U}^n(\tau)}{d\tau^2} = 0 \quad ,$$

implies

$$\tau = n \tag{28}$$

Hence, as in the case of $\tilde{U}^*(\tau)$, we can use the inflection points as identifiable characteristics of the growth curves of U^n. Observe how the time constants suggested by (28) are independent of ρ, and hence the medium, and depend only on n; yet the similar type of time constant for the sum $\tilde{U}^*$ of t the n-ary fields $\tilde{U}^n$ indeed depends on ρ.

With these illustrations we rest our case concerning the nonexistence of a single universally applicable time constant for characterizing transient light fields in extensive optical media. Perhaps, if a single time constant were demanded which could be pressed into use more often than all the other time constants discussed in the present chapter, then we might tentatively suggest T_α for consideration. For T_α appears quite often in the energy context and most critically in the radiance context of (10) of Sec. 5.7. Furthermore, T_α is based on the one inherent optical property (namely α) of optical media which is the most thoroughly documented and which is the most readily measured member of the basic trio α, σ, a.

Finally, we observe that all our preceding deliberations concerned unbounded media--or very extensive media in which their boundaries played a negligible role. For a discussion of the theory of time constants in bounded media in which the sensitivity of radiometer instruments also plays a role the reader may consult the papers in parts IV, V of [236]. These references are part of a set of five reports in which the main discussion centers on the study of the general *metric properties* of time dependent light fields. The theory of the time constant found in [236] is one of the several applications of the general metric theory developed in the series.

<< Table of Contents << Title Page >> Next Page >>

5.12 Global Approximations of General Radiance Fields

In this and the following section some of the theory of time-dependent n-ary radiant energy fields will be applied to two general problems of radiative transfer theory. In the present section attention will be directed to the problem of finding relatively simple approximations of time dependent and steady state radiance fields in optical media. In particular it will be shown how the n-ary radiant energy fields may be used to obtain approximations of the observable radiance field such that the approximations are *exact on a global level* over the given medium.

The precise meaning of this phrase will become clear during the course of the constructions of the approximations, to which we now turn. Unless specifically stated otherwise, all constructions will take place on a general optical medium X with arbitrary source conditions.

We begin with the observation that the operator formula

$$N^n = N^1 S^{n-1} \quad ,$$

based on the theory of Sec. 5.1, suggests the following simple approximation, where we write:

$$"N_g^n" \quad \text{for} \quad \frac{U^n}{U^1} N^1 \qquad (1)$$

Here U^n, $N \geq 1$, is the n-ary radiant energy in X, and N^1 is the primary radiance function in X. N_g^n is called the *global approximation* of N^n for $n \geq 1$.