expressions for $U^*(t)$ and $U^n(t)$, developed in Sec. 5.11, to generate useful approximations to time-dependent radiance fields. First or second order global approximations should suffice for many practical settings.

We note in passing that preliminary and informal numerical studies seem to indicate that the shapes (the directional structure) of N^n appear to be spherical (or very nearly so) when n is larger than some integer p which depends on the medium X and ρ. If this conjecture can be proved in general, (probably by means of the set up in 10.5) then an enormous advance in the practical utility of (13) can be made. This conjecture of the limiting shape of N^n as $n \to \infty$, bears a striking analog to the asymptotic radiance theorem studied elsewhere in this work (cf., e.g., Chapter 10). An important application would be to diffusion theory (see (78) of Sec. 6.6).

5.13 Light Storage Phenomena in Natural Optical Media

The applications of the natural mode of solution of radiative transfer problems in optical media discussed in this chapter will now be concluded with a definition and discussion of the light-storage phenomena in such media.

Everyday Examples of Light Storage

Those who have looked out of a window of an airplane as it descended into a sunbathed cloud layer may recall the sudden transition to a brilliant ambient field of light, and how the sensation of brightness in every direction increased to dazzling proportions as the airplane descended further into the upper regions of the cloud. This phenomenon is but one of many common examples of the storage of light by the mechanism of scattering. One can also see evidence of light storage on overcast nights on the outskirts of large cities: the cloud layer hovering low over the city is deeply and extensively illuminated from the street and building lights below. Flashes of lightning in storm clouds can light up an extensive cloud layer from horizon to horizon even though the actual volume taken up by the network of electrical discharges is a minute fraction of the illuminated volume. Lighthouses on densely fogged nights pour a well-defined beam of light into a surrounding fog with the result that the beam and the lighthouse are imbedded in a field of scattered light which, under suitable conditions, may be observed by approaching mariners far sooner than the light of the revolving beam. As one descends into a lake or the ocean on a sunny day, there is a shallow region near the surface in which the radiance measurably increases with increasing depth for various horizontal and upward-looking lines of sight.

These examples illustrate the phenomenon of the storage of light in scattering media. The sense of the work "storage" is used in its everyday sense: the accumulation or building up of radiant energy in the scattering material that surrounds the source of the energy. If one were to quickly extinguish the light source, the stored light would not immediately disappear with the extinction of the source; rather the scattered light stored in the earth's atmosphere would take on the order of a score of microseconds to be lost into space, or converted into longer wavelengths of radiation and other forms of energy. The decaying atmospheric light field is like the diminishing reverberation of organ notes in a spacious auditorium in which the acoustical energy is momentarily entrapped and redirected by the walls of the auditorium (cf. Sec. 5.6). In the case of light, the walls of the auditorium are replaced by multitudes of tiny scattering centers comprising clouds, fogs, or parts of the entire atmosphere, and the hydrosphere of the earth: the light impinges on the scattering centers and is redirected again and again by scattering.

Thus, the energy of a pencil of photons, which ordinarily traverses a given volume of empty space in one microsecond, could, in principle, be cycled and recycled within the confines of the volume for a period of several dozens of microseconds before it escapes or is transformed. Therefore, if a continuous steady beam of light is poured into such a volume, the steady state density of scattered light stored within the volume could be tens of times greater than the average density of the light ordinarily within the beam.

Do all these phenomena have a common simple description? Is there a small set of properties of the medium and of the source that, when isolated, can serve as the salient parameters in an analytical description of the stored light field? The answer is 'yes'; the natural mode of analysis of light fields plays an essential role in formulating the details of the answer.

In this section we embark on a preliminary attempt to describe the phenomenon of light storage in precisely defined terms. Once we have decided on an exact radiometric definition of the term "stored light energy," we go on to formulate a simple mathematical model of the light field in a scattering-absorbing medium which can describe how the stored light energy depends on the inherent optical properties of the medium, the geometry of the medium, and the properties of the light source.

It turns out that there are several ways in which we may formulate the description of "stored light energy." The form of the description depends on one's choice of the radiometric quantity used in the description. For example, we find that there is a description associated with the radiometric concept of radiance, another description with irradiance, another with radiant density, and still another with radiant energy.

In the present discussion we will limit our attention to the description of stored light energy exclusively by means of the concept of radiant energy. The resulting description is by far the most natural of all the various possibilities; it is, by a happy coincidence, also the most simple to deal with, and the easiest from which to draw examples.

In the event that more detailed descriptions of storage phenomena than those developed in the present study are ever required, such as n-ary radiance N^n or radiance N, recall that we have formulated the requisite time-dependent transport equations of these radiometric quantities in Sec. 5.2. Therefore, the work of this section should readily be extended to the radiance case by interested researchers. The investigation of the time-dependent radiant flux problem made in the preceding sections also supplements the results of the present study by providing detailed numerical and graphical illustrations (Figs. 5.13-5.24) of the solutions of the n-ary radiant energy equations, and related radiometric concepts, which play an important role in the storage capacity concept.

Storage Capacity

Let "U" represent the directly observable steady state radiant energy attained in an arbitrary medium X under arbitrary growth conditions; let "U^o" represent the amount of U consisting of *residual radiant energy* from the source (associated with photons which have not yet been scattered or absorbed subsequent to entry into X); and finally, let "U^*" represent the amount of U consisting of *scattered radiant energy* within the medium (associated with photons which have undergone at least one scattrring operation). The ratio U^*/U is then a measure of the relative amount of scattered radiant energy in the medium X. It is a number which lies between zero and one and will be referred to as the *storage capacity* of the medium X.

In the case of an infinite homogeneous medium whose steady state light field has been attained under standard growth conditions (Sec. 5.11), the storage capacity has a particularly simple representation in terms of the total volume scattering coefficient s, and the volume attenuation coefficient α of the medium:

$$\text{storage capacity} = \frac{U^*}{U} = \frac{s}{\alpha} = \rho \qquad (1)$$

where ρ is the scattering-attenuation ratio. In the case of nonhomogeneous or finite media, the storage capacity is a more complicated function of ρ and the geometry of the medium. (Examples of more general storage capacity formulas will be given below in (5) and (6).) But even in the present simple context, we gain important insight into storage phenomena in general: the storage capacity depends basically on the *relative* magnitudes of s and α. Thus if we consider two media, one in which $s = 0.01/\text{m}$, $\alpha = 0.02/\text{m}$, and another in which $s = 0.10/\text{m}$, $\alpha = 0.20/\text{m}$, we see that the former medium has an attentuation length of $1/\alpha = 50$ m while the latter while the latter medium is an order of magnitude more optically dense with an attenuation length of $1/\alpha = 5$ m. However, the scattering-attenuation ratio for each medium is $\rho = 0.5$. Thus, despite the great disparity in optical density of these media, their storage capacities have a common value, namely $U^*/U = 0.5$, indicating that in the steady state in each medium, the stored radiant energy (in scattered form) is 50% of the total observable energy within each medium.

Methods of Determining Storage Capacity

The problem of determining the storage capacity of an infinite or very extensive optical medium (one in which the boundaries play a negligible role) is readily solved using the results developed in the preceding sections on n-ary radiant energy. In particular, for homogeneous infinite media, the storage capacity reduces to a very simply obtained single number ρ, as shown above. The number ρ is readily determined in practice by a few local measurements. However, the infinite settings are occasionally inadequate models of real situations. In real media in terrestrial settings we usually dispense with computation programs and go directly to the medium (clouds, lakes, oceans) to perform measurements *in situ* over the given region. By following the definition of storage capacity to the letter, we need only try to measure the radiant energy U^* and U by measuring scalar irradiance at each point throughout the medium and find the quotient U^*/U. However, to probe the medium point by point is always laborious and occasionally impossible. A practicable scheme for measuring storage capacity of real media would be one in which all internal probings are obviated. We thus set up the following problem for study: Is there some way of determining $U^*(X)/U(X)$ for a medium X by limiting all radiometric measurements to the boundary of X? The answer is in the affirmative. We now present the details of a possible empirical procedure leading to the storage capacity of a natural optical medium.

The discussion begins with the steady state version of (24) of Sec. 5.8 applied to a homogeneous, bounded region X of some real optical medium. The incident radiant flux on X is arbitrarily disposed over the boundary and X is assumed to have no internal emission sources. Thus we begin with:

$$0 = -\alpha U^n(X) + sU^{n-1}(X) + \frac{1}{v}\bar{P}^n(X) \qquad (2)$$

for $n \geq 1$. Here $\bar{P}^n(X)$ is the net inward radiant n-ary flux across the boundary of X. The n-ary radiant flux is indexed relative to the incident radiant flux on the boundary of the optical medium in which X is located. Thus if the optical medium is the ocean and X is a cube 10 m on a side whose center is located 100 m below the surface, then the n-ary radiant flux in the cube is relative to the incident radiant flux on the surface of the ocean. Summing each side of (2) over all $n \geq 1$:

$$0 = -\alpha \sum_{n=1}^{\infty} U^n(X) + s \sum_{n=1}^{\infty} U^{n-1}(X) + \frac{1}{v} \sum_{n=1}^{\infty} \bar{P}^n(X)$$

Using the natural solution properties this becomes:

$$0 = -\alpha U^*(X) + sU(X) + \frac{1}{v}\bar{P}^*(X) \qquad (3)$$

where we have written:

$$"\bar{P}^*(X)" \quad \text{for} \quad \sum_{n=1}^{\infty} \bar{P}^n(X) \quad . \tag{4}$$

In accordance with our preceding remarks, we are interested in estimating the quantity $U^*(X)$ with the ultimate goal in mind of estimating the ratio $U^*(X)/U(X)$. But any such estimation must be couched in terms of *observable* or *simply calculable* quantities. $U^*(X)$ is not directly observable; and $U(X)$, while observable, is not simply calculable. (It requires a determination of observable radiant density $u(x)$ at each point x of X.) In casting about for easily observable and simply calculable quantities, the observable net flux $\bar{P}(X)$, the residual net flux $\bar{P}^o(X)$ and the residual energy $U^o(X)$ immediately come to mind. If we can obtain an expression for $U^*(X)/U(X)$ in terms of $\bar{P}(X)$, $\bar{P}^o(X)$ and $U^o(X)$, we will have obtained the best solution possible to the problem of empirically determining the storage capacity of a *finite* homogeneous medium.

It turns out that the characterization of $U^*(X)/U(X)$ in terms of $\bar{P}(X)$, $\bar{P}^o(X)$ and $U^o(X)$ is relatively easy to achieve. Starting with (3), and noting by (33) of Sec. 5.8 that we have:

$$\bar{P}(X) = \bar{P}^o(X) + \bar{P}^*(X),$$

we can recast (3) into the form:

$$-\frac{1}{v}\bar{P}(X) + \frac{1}{v}\bar{P}^o(X) = -aU^*(X) + sU^o(X) \quad .$$

We can then represent the nonobservable $U^*(X)$ in terms of observable and calculable quantities:

$$U^*(X) = \frac{s}{a}U^o(X) + \frac{1}{av}[\bar{P}(X) - \bar{P}^o(X)]$$

Hence

$$\boxed{\frac{U^*(X)}{U(X)} = 1 - \frac{a}{\alpha + \left[\dfrac{\bar{P}(X) - \bar{P}^o(X)}{vU^o(X)}\right]}} \tag{5}$$

Equation (5) gives the desired general formulation of the storage capacity of a *finite* homogeneous medium X in terms of the directly observable net inward flux $\bar{P}(X)$ over the boundary of X, the calculable net inward residual flux $\bar{P}^o(X)$ over the boundary of X, and the calculable residual energy content $U^o(X)$ of X. The volume absorption coefficient a and the volume attenuation coefficient α are the inherent optical properties of X which enter into the calculation and which are assumed known.

It should be remarked that equation (5) is an *exact* and computable formula for the storage capacity $U^*(X)/U(X)$ whenever X is any finite homogeneous medium with $a \geq 0$, irradiated by sources in an arbitrary manner and in which the resultant light field is in steady state. If X is infinite in all directions or very extensive, then it may be that $\overline{P}(X) = \overline{P}^o(X)$, and (5) reduces to (1). The condition $\overline{P}^o(X) = \overline{P}(X)$ means that $\overline{P}(X) = 0$, i.e., that there is no net scattered flux across the boundaries of X. This could happen when the boundaries are infinitely far removed, or when a small volume is deep inside an extensive medium.

Example

To illustrate how (5) is used in particular contexts, consider for example a horizontally extensive cloud stratum, or ocean layer with upper boundary on the surface, which is of finite geometric depth under a clear sunlit sky or clear moonlit sky. To fix ideas, consider the ocean layer. We agree that the principal source of flux is to be the sun or moon, as the case may be, with negligible auxiliary sources associated with the sky and ground (or lower layers in the case of the ocean). Suppose the sun cannot be seen through the given layer as one is looking up from below. It may be checked that the difference $\overline{P}(X) - \overline{P}^o(X)$ in (5) then reduces essentially to $-P^*(X,+)$, where $P^*(X,+)$ is the total net outward rate of flow of stored energy across the two boundaries of X. (The inward flow $P^*(X,-)$ is set to zero.) Suppose also that the outward rate of low from X over its lower boundary is small compared to that of its upper boundary (which is compatible with the assumptions above). Then:

$$U^o(X) = \frac{N^o \Omega A}{v\, \alpha \sec\theta} = \frac{P^o(X,-)}{v\alpha}$$

where N^o is the radiance of the sun or moon at the upper boundary of X, θ its angle from the zenith, Ω is its solid angle subtense, and A is the area of the upper boundary of the cloud. The second equality follows from the definition of inward residual flux $P^o(X,-)$ over the upper boundary of X. Hence (5) becomes

$$\boxed{\frac{U^*(X)}{U(X)} = \frac{\rho - R(X)}{1 - R(X)}} \quad , \tag{6}$$

where "$R(X)$" stands for $P^*(X,+)/P^o(X,-)$, the reflectance of X at its upper boundary, a directly measurable quantity.

As a simple numerical illustration of (6), suppose that we take the case of a part X of the ocean for which (6) holds and for which it is found that $\rho = 0.4$ and that $R(X) = 0.02$ for a given wavelength of light around the middle of the visible spectrum. Then the storage capacity U^*/U is:

$$\frac{0.4 - 0.02}{1 - 0.02} = \frac{0.38}{0.98} = 0.39$$

If some time later U^o is known to be a certain amount over the same layer, then, if "C" denotes the storage capcity, clearly:

$$\boxed{U = \frac{U^o}{1-C}} \qquad (7)$$

and hence the directly observable radiant energy in the layer is estimable from U^o and knowledge of C.

Equations (6) and (7) illustrate but two of the many practical formulas which may be deduced--under various hypotheses--from the exact formula (5). The preceding derivation will suffice to indicate the general outline of such procedures, and we leave the exploration of other possibilities to the interested reader.

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5.14 Operator-Theoretic Basis for the Natural Solution Procedure

We close the present chapter with an overview of the theoretical activities of the chapter. As in the earlier general discussions of the canonical equations (Sec. 4.7) the present discussion will perhaps not so much increase our ability to solve specific problems of applied radiative transfer as it will deepen insight into the essential structure of the natural solution procedure, and therefore radiative transfer theory. In particular the general results below will show how radiative transfer theory, via the integral form of the equation of transfer, is connected to those parts of the main stream of mathematical physics which share with the present field certain operator equations whose mode of solution coincides, on the abstract level, with the natural mode of solution studied in this chapter. The discussion is intended to be intuitive, as far as the material will allow.

Let L be a general (not necessarily linear) operator defined on a domain $\mathfrak{D}$ of functions such that Lf is in $\mathfrak{D}$ whenever f is in $\mathfrak{D}$. Thus L maps elements of $\mathfrak{D}$ into $\mathfrak{D}$. Next suppose $\mathfrak{D}$ has a "distance function" d defined on it such that if f and g are in $\mathfrak{D}$, then d(f,g) is a nonnegative real number with the properties:

(i) $d(f,g) = 0$ if and only if $f = g$

(ii) $d(f,g) = d(g,f)$

(iii) $d(f,h) \leq d(f,g) + d(g,h)$

The function d is called a *metric* for $\mathfrak{D}$, and as can be seen, it has the three main properties of ordinary distance relation of everyday life. We summarize all this by saying that the pair $(\mathfrak{D},d)$ is a *metric space*.

Now the connection between $(\mathfrak{D},d)$ and the radiative transfer setting of this chapter is quite easily made. Let X be an optical medium with initial radiance N^o and let S^1 be the operator in (5) of Sec. 5.7. Then write: