

If some time later U^0 is known to be a certain amount over the same layer, then, if "C" denotes the storage capacity, clearly:

$$U = \frac{U^0}{1-C} \quad (7)$$

and hence the directly observable radiant energy in the layer is estimable from U^0 and knowledge of C.

Equations (6) and (7) illustrate but two of the many practical formulas which may be deduced--under various hypotheses--from the exact formula (5). The preceding derivation will suffice to indicate the general outline of such procedures, and we leave the exploration of other possibilities to the interested reader.

5.14 Operator-Theoretic Basis for the Natural Solution Procedure

We close the present chapter with an overview of the theoretical activities of the chapter. As in the earlier general discussions of the canonical equations (Sec. 4.7) the present discussion will perhaps not so much increase our ability to solve specific problems of applied radiative transfer as it will deepen insight into the essential structure of the natural solution procedure, and therefore radiative transfer theory. In particular the general results below will show how radiative transfer theory, via the integral form of the equation of transfer, is connected to those parts of the main stream of mathematical physics which share with the present field certain operator equations whose mode of solution coincides, on the abstract level, with the natural mode of solution studied in this chapter. The discussion is intended to be intuitive, as far as the material will allow.

Let L be a general (not necessarily linear) operator defined on a domain \mathcal{D} of functions such that Lf is in \mathcal{D} whenever f is in \mathcal{D} . Thus L maps elements of \mathcal{D} into \mathcal{D} . Next suppose \mathcal{D} has a "distance function" d defined on it such that if f and g are in \mathcal{D} , then $d(f,g)$ is a nonnegative real number with the properties:

- (i) $d(f,g) = 0$ if and only if $f = g$
- (ii) $d(f,g) = d(g,f)$
- (iii) $d(f,h) \leq d(f,g) + d(g,h)$

The function d is called a *metric* for \mathcal{D} , and as can be seen, it has the three main properties of ordinary distance relation of everyday life. We summarize all this by saying that the pair (\mathcal{D}, d) is a *metric space*.

Now the connection between (\mathcal{D}, d) and the radiative transfer setting of this chapter is quite easily made. Let X be an optical medium with initial radiance N^0 and let S^1 be the operator in (5) of Sec. 5.7. Then write:

$$"L" \text{ for } N^0 + (\cdot) S^1 \quad (1)$$

and we have an example of the operator L above, where \mathcal{D} is now the set of all radiance functions on X . Thus if N is a radiance function on X (i.e., N has the dimensions of radiance) then certainly

$$N^0 + NS^1$$

is again radiance function on X . We are not asserting at the moment that N is a solution of the equation of transfer, but merely making an observation that the function displayed above has the dimensions of radiance, and that is all at the moment that is required for admission into \mathcal{D} . Hence L as defined in (1) maps elements of \mathcal{D} back into \mathcal{D} .

Next we show that there is a very natural counterpart in radiative transfer theory to the abstract metric d for each fixed time t and bounded optical medium X . Let us write

$$"d(f,g)" \text{ for } \int_X \frac{1}{v(x)} \left[\int_E |f(x,\xi,t) - g(x,\xi,t)| d\Omega(\xi) \right] dV(x) \quad (2)$$

It is easy to verify that if $f = g$, then $d(f,g) = 0$, and that if $d(f,g) = 0$, then $f = g$ except on sets of directions ξ and points x of zero measure. This exception can be smoothed over by advanced technical devices,* and we henceforth can assume condition (i) for a metric to be satisfied. Next one can verify conditions (ii) and (iii) with ease and the verification is left to interested readers. We call the metric function d as defined in (2), the *radiometric*. By various standard techniques (e.g., averaging) (2) can readily be extended to unbounded media. An alternate choice of metric can also be made by writing

$$"d(f,g)" \text{ for } \sup_{x,\xi} |f(x,\xi,t) - g(x,\xi,t)| \quad (2a)$$

where

$$" \sup_{x,\xi} h(x,\xi) "$$

*In particular, this can be done by means of equivalence classes of functions, an equivalence class being the set of all radiance functions on a domain Y which differ from one another at most on subsets of Y of zero measure. Then we go on to work with equivalence sets of functions rather than individual functions. However, for the present we work directly with the radiance functions, with no essential loss of rigor.

means the supremum (the maximum) of the values of $h(x, \xi)$ as x, ξ vary over all permissible values in the domain of h . The function d in (2a) also satisfies all the properties (i) to (iii) of a metric. We shall call d in (2a) the *supremum metric*.

We summarize what has been done so far: The operator (1) associated with the integral equation of transfer of classical radiative transfer theory may be viewed as a special case of an abstract operator L on a metric space (\mathcal{D}, d) , the particular classical form of the operator being given in (1), with \mathcal{D} being the class of all radiance functions on X , and with d the radiometric as defined in (2) or the supremum metric as given in (2a). In what follows we allow \mathcal{D} to contain negative valued radiance functions as well as nonnegative valued radiance functions. Of course in physically meaningful applications we shall always work with the latter; however, for mathematical purposes it is convenient also to have the former.

We now come to a key property of the radiative transfer operator S^1 which can be abstracted from the setting of the present chapter and carried out far into the reaches of abstract operator theory, where its general utility can be more easily discerned. In Sec. 5.7 we showed that if N is an upper bound (or supremum) of a radiance function, then (cf. (7) of Sec. 5.7):

$$(NS^1)(x, \xi, t) \leq N\rho(1 - e^{-t/T_\alpha})$$

for every x in X , ξ in Ξ and t in $(0, \infty)$, where " T_α " stands for $1/\nu_\alpha$. From this we are led to deduce that for every pair f, g of radiance functions, and with the supremum metric (2a).

$$d(Lf, Lg) \leq c d(f, g) \quad (3)$$

where c is a number which depends only on t , ρ and T_α , i.e., where we have written:

$$"c" \text{ for } \rho(1 - e^{-t/T_\alpha})$$

In all normal optical media (i.e., for which $0 < \rho < 1$), we have $0 < c < 1$ whenever $t > 0$. The proof of (3) is immediate, using the definitions (1) and (2a). Whenever an operator L on a general metric space (\mathcal{D}, d) has property (3), we say that L is a *contraction mapping* or that it has the *contraction property*. Hence our particular classical radiative transfer operator L given in (1) is a contraction mapping, relative to (2a). The reader may show that (3) also holds under suitable conditions, relative to (2).

To summarize our findings so far: The operator L associated with the time-dependent integral equation of transfer may be viewed as a special case of a contraction mapping L on a metric space (\mathcal{D}, d) .

We now have developed enough abstract machinery to illustrate the essential activity of the natural solution procedure, on a very general level--a level which is in contact

with the general representations of widely different natural phenomena in modern physics. Let us choose any function $f^{(0)}$ in \mathcal{D} and write:

$$"f^{(1)}" \text{ for } Lf^{(0)}$$

Thus we operate on $f^{(0)}$ in \mathcal{D} with L to obtain $f^{(1)}$ in \mathcal{D} . We repeat this operation a finite number of times to obtain $f^{(n)}$ where we have written:

$$"f^{(n)}" \text{ for } Lf^{(n-1)}$$

In this way we obtain a sequence

$$\{f^{(0)}, f^{(1)}, \dots, f^{(n)}, \dots\}$$

of functions in \mathcal{D} . As in the case of Sec. 5.1, we can define iterates L^n of L so that (cf., e.g., (11) of Sec. 5.1):

$$f^{(n)} = L^n f^{(0)}$$

Before going on, the reader should verify that if we use L in (1), and N^0 for $f^{(1)}$, then $f^{(n)}$ is simply

$$\sum_{j=0}^n N^j,$$

i.e., the sum of the n -ary radiances up to order n .

Since L is a contraction mapping, we have, for $m \geq n$:

$$\begin{aligned} d(f^{(n)}, f^{(m)}) &= d(L^n f^{(0)}, L^m f^{(0)}) \\ &\leq c^n d(f^{(0)}, L^{m-n} f^{(0)}) \\ &\leq c^n \left\{ d(f^{(0)}, f^{(1)}) + d(f^{(1)}, f^{(2)}) + \dots + d(f^{(m-n-1)}, f^{(m-n)}) \right\} \\ &\leq c^n d(f^{(0)}, f^{(1)}) \left\{ 1 + c + c^2 + \dots + c^{m-n-1} \right\} \\ &\leq \frac{c^n d(f^{(0)}, f^{(1)})}{(1 - c)} \end{aligned}$$

(4)

Since c is less than 1, c^n is arbitrarily small for sufficiently large n . Thus the sequence

$$\{f^{(0)}, f^{(1)}, \dots, f^{(n)}, \dots\}$$

constructed above is a Cauchy sequence (in the sense of modern calculus). By establishing this feature of the sequence we have reached the penultimate step in our general discussion of the natural solution procedure.

The significance of the Cauchy sequence feature of

$$\{ f^{(0)}, f^{(1)}, \dots, f^{(n)}, \dots \}$$

is this: In all physically meaningful settings for the metric space (\mathcal{D}, d) , it is possible to arrange matters so that, whenever a sequence

$$\left\{ f^{(n)} \right\}$$

of elements in \mathcal{D} is a Cauchy sequence in the sense of (4), then that sequence has a limit in \mathcal{D} . In general, whenever a metric space (\mathcal{D}, d) has this property, we say that (\mathcal{D}, d) is complete. It is easy to show that all physically meaningful radiative transfer settings always can be represented by complete metric spaces (\mathcal{D}, d) . Let us assume therefore for the remainder of the discussion that (\mathcal{D}, d) is complete.

Taking up the thread of the argument at (4) we now can assert the existence of a limit function f to the sequence constructed above. Thus let us write:

$$"f" \text{ for } \lim_n f^{(n)} \quad (5)$$

We now show that f has two very important properties:

- (i) f satisfies the operator equation $f = Lf$
- (ii) f is the only function in \mathcal{D} for which (i) holds, i.e., if $g = Lg$ and $f = Lf$, then $f = g$.

Property (i) follows readily by noting that, by definition, $f^{(n)} = Lf^{(n-1)}$. Hence applying the limit operation to each side of this identity, the result follows by observing that L is a continuous mapping* (so that the limit operation can be pushed past L and made to act directly on $f^{(n-1)}$). Property (ii) follows from (i) and the contraction property of L :

$$d(f, g) = d(Lf, Lg) \leq c d(f, g)$$

From this (since $c < 1$) we must have $d(f, g) = 0$, so that $f = g$.

Let us now make the final summary of what has been done so far in this section: The natural mode of solution in radiative transfer theory has been found to take its place as a special case of a very general operator technique in modern functional analysis. This technique is based on the following theorem (cf., e.g., [140]):

Theorem (Principle of Contraction Mappings). Every contraction mapping L on a complete metric space (\mathcal{D}, d) generates one and only one solution of the equation $f = Lf$.

*A point which is readily established in functional analysis texts (cf., e.g., [140]).

The classical radiative transfer setting entities are paired off with the abstract setting entities of the preceding theorem as follows:

<u>In Radiative Transfer Theory</u>	<u>In the Theorem</u>
a) Set \mathcal{D} of all radiance functions on an optical medium X	\mathcal{D}
b) The radiometric d , as in (2) or (2a)	d
c) The operator L , as in (1)	L

We will make one final remark on the existence of the solution f of the general operator equation $f = Lf$. This is the observation that the solution f defined in (5) is independent of the initial function $f^{(0)}$ starting the chain of iterations $L^n f^{(0)}$. This fact becomes clear, at least logically, by noting the uniqueness property (ii) above. For if $f^{(0)}$ and $g^{(0)}$ are two distinct initial functions, then construction of their iteration sequences yields f and g such that property (i) holds for each.

5.15 Bibliographic Notes for Chapter 5

The natural mode of solution of the equation of transfer studied in this chapter, as noted in the introduction, plays a unique, fundamental role in radiative transfer theory. The formal power of the method and its intuitive simplicity cannot be overemphasized. For some historical notes on the natural mode of solution, see Secs. 26 and 42 of Ref. [251]. For recent modifications of the iterative concept of solutions of functional equations, especially for numerical purposes, see [171].

The development of the natural solution, as presented in Secs. 5.1 and 5.4, follows in the main that given in Ref. [251]. The canonical representation of primary radiance in (8) or (9) of Sec. 5.3 is occasionally referred to as "Seeliger's formula," and is to be conceptually distinguished from the more useful and accurate representation of $N_{\frac{1}{2}}$ given in (5) of Sec. 4.4. The only common feature of the two radiance representations is that they both fall within the purview of the basic canonical formula (4) of Sec. 4.7.

The discussion of the "optical ringing problem" in Secs. 5.7 and 5.8 is based on the natural-solution approach to the time-dependent radiative transfer problem, and is designed to be more precise than simple time-dependent *classical* diffusion theory (Sec. 6.6). The approach outlined in these sections is drawn from the results in Ref. [211]. A related approach to the optical ringing problem from the point of view of temporal metric spaces was tentatively explored in the series of reports [236]. Further approaches to time-dependent radiative transfer problem are possible via the higher-order diffusion equations. See Table 1 of Sec. 6.5. The truncated natural-solution inequalities in Sec. 5.7 are based on [239]. Further inequalities in this circle of ideas may be found in Ref. [67].