CHAPTER 6

CLASSICAL SOLUTIONS OF THE EQUATION OF TRANSFER

6.0 Introduction

In this chapter we shall conduct an exposition of the two most important classical modes of solution of the equation of transfer used in practice besides the canonical and the natural modes discussed in the preceding two chapters. These classical modes are the powerful spherical harmonic method, and the mathematically interesting diffusion method. The spherical harmonic method is classical in the sense that it dates back to Eddington and Jeans [120], two of the pioneers of radiative transfer theory. The spherical harmonic method represents radiance functions in terms of sums of products of two factors: one factor being purely spatial, the other purely directional, an intuitively natural representation for functions defined on the phase space $X \times \Xi$. On the other hand, there are two main theories of diffusion: the classical and the exact theories. The classical diffusion method is based on Fick's law and views photons in optical media as swarms of particles diffusing with great speed, but generally in the manner of classical diffusion processes, such as heat conduction and Brownian motion. The exact diffusion method, which in its essential modern form dates back to the work of Hopf [111], transcends in accuracy the classical diffusion method but is less general in applicability than the spherical harmonic method, in that it applies strictly only to general transport media whose volume scattering function values $\sigma(x;\xi';\xi)$ are independent of the directions ξ' and ξ. However, the relatively great tractability of the equation of transfer resulting from the introduction of this simplification has led to many interesting and fairly detailed exact solutions of the transfer equation, some of which are quite valuable in practice. For this reason we include in our present discussions a brief exposition of the two main diffusion methods. Together, the spherical harmonic method and the diffusion methods form useful adjuncts to the basic natural mode of solution and the canonical mode of solution discussed earlier in this work.

The plan of the chapter is as follows: We begin with the spherical harmonic method. To show the extraordinarily wide scope and power of the method and also its inherent simplicity we derive it in much more general settings than is customary, and from an abstract algebraic point of view. This will be done in Sec. 6.2, after a preliminary section devoted to motivating the method. Then follows a specialized development of the method using the functions which have given the method its name (Sec. 6.3) but which, in view of the exposition of Sec. 6.2, need no longer exclusively be used. An illustrative example of the spherical harmonic method is given in Sec. 6.4 for plane-parallel media. The discussion of the algebraic idea underlying the spherical harmonic method will be taken up again as a matter of course in Chapter 7 wherein we shall view the method from a more fundamental point of view, namely from the viewpoint of the generalized invariant imbedding relation (Sec. 7.10). In Sec. 6.5, we turn to the diffusion methods, developing them directly from the equation of transfer by imposing the characteristic assumptions of each theory into the equation. The solutions of some of the more famous models in the classical diffusion method are discussed in Sec. 6.6. In Sec. 6.7 the Milne model for infinite media with point sources is discussed, followed by some relatively recent results on a related problem on point source problems in semi-infinite media. The chapter is concluded in Sec. 6.8 by a brief bibliographic survey of other classical methods of solution comprising some of the stock in trade of current radiative transfer theory.

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6.1 The Bases of the Spherical Harmonic Method

In this section we shall describe the physical and mathematical bases of the spherical harmonic method. We begin with a brief discussion of the motivation for factoring the radiance function values $N(x,\xi)$ into a sum of products of the form: $f(x)g(\xi)$. We then go on to show how this intuitively and physically natural decomposition is sanctioned and given a direct representation in terms of vector space theory. To accomplish this program, the mathematical prerequisites will entail no more than standard advanced calculus techniques.

Physical Motivations

The steady state radiance function is essentially a function of two variables: the spatial variable x and the directional variable ξ. When one examines the equation of transfer, in either its integrodifferential or integral forms, one is confronted with the complicating presence of the integral term--which represents an integration over the directional variable. If it weren't for that integral term, the equation of transfer would be a simple differential equation and the theory would long ago have been worked out and forgotten by mathematicians! When an investigator, new to the field of radiative transfer theory, encounters the equation of transfer, one of his more probable actions would be to see what would happen if the radiance function N is assumed to be the product of two functions f and g, such that:

$$N(x,\xi) = f(x)g(\xi) \quad . \tag{1}$$

Could the radiance function in some optical media be represented simply as such a product? It would be instructive to follow the consequences of this query, as it is at once one