to motivating the method. Then follows a specialized development of the method using the functions which have given the method its name (Sec. 6.3) but which, in view of the exposition of Sec. 6.2, need no longer exclusively be used. An illustrative example of the spherical harmonic method is given in Sec. 6.4 for plane-parallel media. The discussion of the algebraic idea underlying the spherical harmonic method will be taken up again as a matter of course in Chapter 7 wherein we shall view the method from a more fundamental point of view, namely from the viewpoint of the generalized invariant imbedding relation (Sec. 7.10). In Sec. 6.5, we turn to the diffusion methods, developing them directly from the equation of transfer by imposing the characteristic assumptions of each theory into the equation. The solutions of some of the more famous models in the classical diffusion method are discussed in Sec. 6.6. In Sec. 6.7 the Milne model for infinite media with point sources is discussed, followed by some relatively recent results on a related problem on point source problems in semi-infinite media. The chapter is concluded in Sec. 6.8 by a brief bibliographic survey of other classical methods of solution comprising some of the stock in trade of current radiative transfer theory.

6.1 The Bases of the Spherical Harmonic Method

In this section we shall describe the physical and mathematical bases of the spherical harmonic method. We begin with a brief discussion of the motivation for factoring the radiance function values $N(x,\xi)$ into a sum of products of the form: $f(x)g(\xi)$. We then go on to show how this intuitively and physically natural decomposition is sanctioned and given a direct representation in terms of vector space theory. To accomplish this program, the mathematical prerequisites will entail no more than standard advanced calculus techniques.

Physical Motivations

The steady state radiance function is essentially a function of two variables: the spatial variable x and the directional variable ξ. When one examines the equation of transfer, in either its integrodifferential or integral forms, one is confronted with the complicating presence of the integral term--which represents an integration over the directional variable. If it weren't for that integral term, the equation of transfer would be a simple differential equation and the theory would long ago have been worked out and forgotten by mathematicians! When an investigator, new to the field of radiative transfer theory, encounters the equation of transfer, one of his more probable actions would be to see what would happen if the radiance function N is assumed to be the product of two functions f and g, such that:

$$N(x,\xi) = f(x)g(\xi) \quad . \tag{1}$$

Could the radiance function in some optical media be represented simply as such a product? It would be instructive to follow the consequences of this query, as it is at once one of the most natural and fruitful of questions to investigate in the task of solving transfer problems.

The immediate effect of such an assumption as (1) would be the reduction of the path function N_* to the form:

$$N_*(x,\xi) = \int_{\Xi} N(x,\xi')\ \sigma(x;\xi';\xi)\ d\Omega(\xi)$$

$$= f(x) \int_{\Xi} g(\xi')\ \sigma(x;\xi';\xi)\ d\Omega(\xi') \quad . \qquad (2)$$

It looks as if the assumption (1) is ineffective unless a similar assumption is made about the volume scattering function. Thus, in the spirit of (1), another assumption is made, now about σ: We assume that two functions c and p exist and are such that:

$$\sigma(x;\xi';\xi) = c(x)p(\xi';\xi) \qquad (3)$$

Using (3) in (2), the representation of $N_*(x,\xi)$ becomes:

$$N_*(x,\xi) = f(x)\ c(x) \int_{\Xi} g(\xi')\ p(\xi';\xi)\ d\Omega(\xi')$$

$$= f_*(x)\ g_*(\xi) \qquad (4)$$

where f_* and g_* are defined in the obvious way. Therefore, under the additional assumption (3), the path function N_* may, like N itself be represented as a product of two functions: one of x alone, the other of ξ alone.

The next step in the explorations would be to see if the equation of transfer becomes more tractable with (1) and (3) as starting points. Thus, starting with the equation of transfer:

$$\xi \cdot \nabla N(x,\xi) = \frac{dN(x,\xi)}{dr} = - \alpha(x,\xi)\ N(x,\xi) + N_*(x,\xi) \quad , \qquad (5)$$

and using (1) and (3), the equation becomes:

$$g(\xi)\ \frac{df(x)}{dr} = - \alpha(x,\xi)\ f(x)\ g(\xi) + f_*(x)\ g_*(\xi) \qquad (6)$$

Having split apart the spatial and directional components of σ, as shown in (3), it is physically reasonable (but not logically necessary) to do likewise with α. Succumbing for the moment to physical reasonability, so that the discussion can proceed, we assume $\alpha(x,\cdot)$ to be constant valued on Ξ for every x in X, and write simply "$\alpha(x)$" for this common value at x. Then (6) can be rearranged into the form:

$$\frac{dr(x)}{dr} = f(x)\left[c(x)\ g_*(\xi)/g(\xi) - \alpha(x)\right] \tag{7}$$

Two observations may now be made. First, the results of the accumulated assumptions, succinctly residing in (7), show that $f(x)$ is in principle determinable by a simple integration of the differential equation (7) along a path of sight provided the values of the parenthesized terms in (7) are known. The second observation is that the values of the parenthesized terms in (7) are known once the quotient $g_*(\xi)/g(\xi)$ is known. By an inspection of (7), it is clear that this quotient must be some number independent of ξ. Hence we write:

$$g_*(\xi)/g(\xi) = \lambda \quad , \tag{8}$$

which then in turn requires the function g to satisfy the integral equation of the form:

$$\lambda g(\xi) = \int_{\Xi} g(\xi')\ p(\xi';\xi)\ d\Omega(\xi') \tag{9}$$

The net result of the assumptions (1) and (3) are to reduce the problem of the solution of (5) into subproblems: the solution of an integral equation (9) for g, with an appropriate λ; and a solution of the simple ordinary differential equation (7) for f, using the λ obtained in process of finding g.

It appears therefore that up to this point a definite step has been made in the solution of (5) by adopting the assumptions (1) and (3). It seems worthwhile to follow this promising start and to attempt to carry the solution of (9) to completion. If this can be done for all physically reasonable assumptions on $p(\xi';\xi)$ in (3), then a general solution of the equation of transfer will have been found. Toward this end we will adopt for $p(\xi';\xi)$ the property of (*weak*) *isotropy*, i.e., the property that for every ξ' and ξ:

$$\int_{\Xi} p(\xi';\xi)\ d\Omega(\xi) = \int_{\Xi} p(\xi';\xi)\ d\Omega(\xi') \quad .$$

Since either integral will be independent of ξ, or ξ', we shall set its fixed value equal to 1. This puts the burden of the correct magnitude of σ on $c(x)$ in (3). In fact we now see that $c(x)$ is none other than the volume total scattering value at x in X because, by (3) of Sec. 4.2:

$$s(x;\xi') = \int_{\Xi} \sigma(x;\xi';\xi)\, d\Omega(\xi) = c(x) \int_{\Xi} p(\xi';\xi)\, d\Omega(\xi)$$

$$= c(x) \quad .$$

Hence $s(x;\xi')$ is independent of ξ', and we write "$s(x)$" for this common value at x. In this way we simultaneously normalize p and give c a physical interpretation.

A similar normalization can be made of g in (1) with the corresponding effect of giving f a convenient physical interpretation. Thus, requiring g to have the property:

$$\int_{\Xi} g(\xi)\, d\Omega(\xi) = 1$$

it follows from (1) that:

$$h(x) = \int_{\Xi} N(x,\xi)\, d\Omega(\xi) = f(x) \int_{\Xi} g(\xi)\, d\Omega(\xi) = f(x) \quad .$$

Hence f is in this case simply the scalar irradiance function h.

Returning now to the two reduced equations (7) and (9) we have from (7) and (8) that:

$$\frac{dh(x)}{dr} = h(x) \left(\lambda s(x) - \alpha(x) \right) \tag{10}$$

Furthermore, from (9), by integrating each side over Ξ, we find:

$$\lambda \int g(\xi)\, d\Omega(\xi) = \int_{\Xi} g(\xi') \left[\int_{\Xi} p(\xi';\xi)\, d\Omega(\xi) \right] d\Omega(\xi')$$

whence

$$\lambda = 1 \tag{11}$$

so that (10) reduces to:

$$\frac{dh(x)}{dr} = - a(x) h(x) \tag{12}$$

and (9) becomes:

$$g(\xi) = \int_{\Xi} g(\xi') \; p(\xi';\xi) \; d\Omega(\xi') \tag{13}$$

We now have reduced the problem of determining the radiance function N, under the assumptions (1) and (3), to the problem of a simple integration of (12) along a path with respect to path length r, and the solution of (13). The solution of equation (12) presents no difficulty, the general solution being:

$$h(x) = h(x_o) \exp \left\{ - \int_0^r a(x') \, dr' \right\} \tag{14}$$

when the integration is taken along a straight path $\mathscr{P}_r(x_o,\xi)$ of length r from point x_o to x. The intermediate point x' is the form $x_o + r'\xi$, $0 \leq r' \leq r$.

Finally, we turn to (13) and immediately observe that any constant function on Ξ, whose value for every ξ in Ξ is some arbitrary fixed value g_o, is a solution. It follows that, if g is any nonconstant solution of (13) then so will $g + g_o$ be a solution of (13). This nonuniqueness of solutions of (13) is a most undesirable state of affairs for a physical model of the light field. This means that, on physical grounds, we must generally reject the model constituted by equations (12) and (13). It follows further that we must reject either or both assumptions (1), (3) which gave rise to (12) and (13). Since (3) is quite tenable on physical grounds, it follows that we must generally* reject (1). In this way we have shown that the initial attempt to factor N into a product of a scalar irradiance function h and a directional function g is untenable on physical grounds. By repeating the essential steps of the arguments between (1) and (13) the same negative conclusion may be deduced for the case where N is represented as a finite sum of terms of the form $h_i g_i$.

The intuitive concept of factoring N into spatial and directional components in general media has thus been shown to be unsupportable on practical physical grounds. However, the factoring may be possible in certain geometrically and physically ideal media. Indeed, as we saw in Sec. 4.4, plane-parallel media with uniform volume scattering functions permit such a factoring of N. According to (9) of Sec. 4.4, we

---

*In particular, if a g can be found which satisfies (13), then some approximate models may be found by adjusting g_o empirically in $N(x,\xi) = h(x) \, (g(\xi) + g_o(\xi))$.

have

$$g(\xi) = \frac{\rho}{4\pi} \frac{1}{1 + \left(\frac{K}{\alpha}\right)\cos\theta} \tag{15}$$

where K is now determined by the requirement that the normalization property of g holds. Thus by adding two more assumptions to (1) and (3), namely that h(x) varied exponentially with a certain fixed exponential decay rate K, and that $\sigma(x;\xi';\xi)$ is independent of ξ' and ξ, a very special factorable radiance function is forthcoming.

The additional physical conditions of the required special exponential character of h and the uniform directional structure of σ are quite severe restrictions to impose on general media in order to obtain a factoring of N. However, as we shall see later [(40) of Sec. 6.6 and (3) of Sec. 7.10 and Sec. 10.5], it is a property of certain extensive homogeneous media that the radiance function N at great distances from the boundaries of such media comes arbitrarily close (for correspondingly great distances) to functions of the form hg, i.e., to factored form, in which there is a spatial factor h and a directional factor g.

The conclusions of the various arguments presented above may now be summarized.

(i) In general media X for which (3) holds, the assumption that there exists a function g on Ξ such that $N(x,\xi) = h(x)\,g(\xi)$ for every x in X and ξ in Ξ is generally untenable on physical grounds (the associated solutions are not unique). More generally, finite representations of the form

$$N(x,\xi) = \sum_{i=0}^{n} h_i(x) g_i(x), \quad n < \infty$$

are also untenable.

(ii) In some extensive, homogeneous media X, there exists a function g on Ξ such that $N(x,\xi) \to h(x)g(\xi)$ for every ξ in Ξ and x sufficiently far from the boundaries of X. By comparing the conclusions summarized in (i) and (ii), we see from (i) that on the one hand the original intuitive guess as to the factorability of N into the form gh was generally incorrect; by conclusion (ii), on the other hand, there is a small solid core of truth inherent in the intuitive guess. Furthermore, while finite representations of N in the form

$$\sum_{i=0}^{n} h_i g_i$$

are generally incorrect, these representations may possibly be so constructed that they increase in accuracy with an increase in the number of terms of the sum. In particular it would seem that by choosing sufficiently large numbers of terms for

$$\sum_{i=0}^{n} h_i g_i \quad ,$$

these approximations to N may be improved at all points of a medium X. Then at large distances from the boundaries of X there will, by (ii), be a single term $h_i g_i$ of

$$\sum_{i=0}^{n} h_i g_i$$

which will dominate the others and which will essentially represent N in those regions.

With these observations we have reached the last stage of the physical motivation for the abstract harmonic representation method. We thereby are led to consider *infinite* series of the form:

$$\sum_{i=0}^{\infty} h_i(x) g_i(\xi)$$

which, for given fixed x in X, represents the radiance distribution values $N(x,\xi)$ for every direction ξ in Ξ.

An Algebraic Setting for Radiance Distributions

The preceding discussion has motivated the representation of a radiance distribution $N(x,\cdot)$ at a fixed point x in an optical medium X by means of an infinite series of functions, in the form:

$$N(x,\xi) = \sum_{i=0}^{\infty} f_i(x) \ \phi_i(\xi) \tag{16}$$

This constitutes the first step in constructing the abstract harmonic representation of $N(x,\cdot)$.

The next step calls for the construction of an infinite family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions, each with Ξ as domain, and with the following properties. First, the ϕ_i's are generally allowed to be complex valued. This provides a great theoretical convenience and in no way forces N to be complex valued under specific physical conditions. Second, we require that the family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be *orthonormal*, i.e.,

$$\int_{\Xi} \phi_i(\xi) \ \overline{\phi}_j(\xi) \ d\Omega(\xi) = \delta_{ij} \tag{17}$$

where δ_{ij} is the Kronecker delta, i.e., δ_{ij} is zero whenever $i \neq j$, and one whenever $i = j$. This operation of integration and others similar to it will arise sufficiently often in the following discussion that it will be convenient to abbreviate it in general by writing:

$$"[\phi,\psi]" \quad \text{for} \quad \int_{\Xi} \phi(\xi)\, \overline{\psi}(\xi)\, d\Omega(\xi) \qquad (18)$$

where ϕ and ψ are any two functions on Ξ so that the integral of their product, as in (18), is defined. The bar over a function denotes complex conjugation. We call $[\phi,\psi]$ the *inner* (or scalar) *product* of ϕ and ψ.

The reason for the terminology "inner product" stems from the deep similarity of this inner product with the classical scalar product $x \cdot y$ of two vectors x and y in euclidean three space. The most striking similarities are paired off in the list below. Their proofs are immediate:

(i)

If α_1, α_2, α_3 are pairwise orthogonal unit vectors of E_3, then $\alpha_i \cdot \alpha_j = \delta_{ij}$

(i)

If $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is an orthonormal family of functions on Ξ, then $[\phi_i,\phi_j] = \delta_{ij}$

(ii)

If, for a vector ξ in Ξ there exist three numbers c_1, c_2, c_3 such that $\xi = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$, then $c_i = \xi \cdot \alpha_i$

(ii)

If, for a function g on Ξ, there exist n numbers c_0, c_1, c_2, ..., c_n, such that

$$g(\xi) = \sum_{j=0}^{n} c_j\phi_j(\xi),$$

then $c_i = [g,\phi_i]$

(iii)

$$x\cdot(y+z) = x\cdot y + x\cdot z$$
$$(x+y)\cdot z = x\cdot z + y\cdot z$$

(iii)

$$[f,g+h] = [f,g] + [f,h]$$
$$[f+g,h] = [f,h] + [g,h]$$

(iv)

$$(cx)\cdot y = c(x\cdot y) = x\cdot cy)$$

(iv)

$$[cf,g] = c[f,g]$$
$$[f,cg] = \bar{c}[f,g]$$

The physical motivations discussed above have led us to consider *infinite* series, so that the vector-spacelike property (ii) for inner product will be postulated to hold for infinite series. The specific form of the infinite version of (ii) we shall adopt is as follows (the mathematical regularity properties of integrability are omitted for simplicity of exposition):

Completeness property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$. *If F is a function on* Ξ, *and if for every* $j \geq 0$ *we write:*

$$"f_j" \quad \text{for} \quad [F,\phi_j]$$

then:

$$F(\xi) = \sum_{j=0}^{\infty} f_j \ \phi_j(\xi) \tag{19}$$

for every ξ *in* Ξ.

The algebraic setting for radiance distributions discussed in example 15 of Sec. 2.11, now may be used once again. In fact we can easily extend that setting for our present purposes. We therefore imagine all possible radiance distributions at a fixed point x in X and imagine further *all* their negatives and imaginaries (-N(x,•) is the *negative* of N(x,•), iN(x,•) where $i = \sqrt{-1}$, is the *imaginary* of N(x,•)) thrown in with them. The totality $\mathcal{N}(x)$ of these and all possible sums of them form a vector space in the general sense: Sums of members of $\mathcal{N}(x)$ are again in $\mathcal{N}(x)$; and multiplication of members of $\mathcal{N}(x)$ by complex numbers are again in $\mathcal{N}(x)$. The additional details of verification are simple and need not detain us here. The main fact to observe is that the set of all integrable radiance distributions at a point x in X can be imbedded in a vector space of functions on Ξ which includes an orthonormal set $\{\phi_0, \phi_1, \phi_2, \ldots\}$ such that the completeness property holds for $\{\phi_0, \phi_1, \phi_2, \ldots\}$. This is the algebraic setting for radiance distributions in which the abstract spherical harmonic method will be discussed.

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6.2 Abstract Spherical Harmonic Method

The motivation and prerequisites of the abstract spherical harmonic method having been dispatched in Sec. 6.1, we turn directly to the method itself, now applied to the general time-dependent equation of transfer with source term ((14) of Sec. 3.15):

$$\frac{1}{v}\frac{\partial N}{\partial t} + \xi \cdot \nabla N = -\alpha N + N_* + N_\eta \quad . \tag{1}$$

where N is defined on a general optical medium X which may be finite or infinite, generally inhomogeneous, but isotropic. We assume furthermore that there exists an orthonormal family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions on Ξ which has the completeness property.

The completeness property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ applied to the radiance distribution N(x,•) at x in X yields:

$$N(x,\xi,t) = \sum_{j=0}^{\infty} f_j(x,t) \ \phi_j(\xi) \tag{2}$$

where we have written:

$$\text{"}f_j(x,t)\text{" for } [N^*(x,\cdot,t), \phi_j] \tag{3}$$

Thus $f_j(x,t)$ is the scalar obtained by performing the integration: