then:

$$F(\xi) = \sum_{j=0}^{\infty} f_j \phi_j(\xi) \tag{19}$$

for every ξ *in* Ξ.

The algebraic setting for radiance distributions discussed in example 15 of Sec. 2.11, now may be used once again. In fact we can easily extend that setting for our present purposes. We therefore imagine all possible radiance distributions at a fixed point x in X and imagine further *all* their negatives and imaginaries ($-N(x,\cdot)$ is the *negative* of $N(x,\cdot)$, $iN(x,\cdot)$ where $i = \sqrt{-1}$, is the *imaginary* of $N(x,\cdot)$) thrown in with them. The totality $\mathcal{N}(x)$ of these and all possible sums of them form a vector space in the general sense: Sums of members of $\mathcal{N}(x)$ are again in $\mathcal{N}(x)$; and multiplication of members of $\mathcal{N}(x)$ by complex numbers are again in $\mathcal{N}(x)$. The additional details of verification are simple and need not detain us here. The main fact to observe is that the set of all integrable radiance distributions at a point x in X can be imbedded in a vector space of functions on Ξ which includes an orthonormal set $\{\phi_0, \phi_1, \phi_2, \ldots\}$ such that the completeness property holds for $\{\phi_0, \phi_1, \phi_2, \ldots\}$. This is the algebraic setting for radiance distributions in which the abstract spherical harmonic method will be discussed.

6.2 Abstract Spherical Harmonic Method

The motivation and prerequisites of the abstract spherical harmonic method having been dispatched in Sec. 6.1, we turn directly to the method itself, now applied to the general time-dependent equation of transfer with source term ((14) of Sec. 3.15):

$$\frac{1}{v}\frac{\partial N}{\partial t} + \xi \cdot \nabla N = -\alpha N + N_* + N_\eta \quad . \tag{1}$$

where N is defined on a general optical medium X which may be finite or infinite, generally inhomogeneous, but isotropic. We assume furthermore that there exists an orthonormal family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions on Ξ which has the completeness property.

The completeness property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ applied to the radiance distribution $N(x,\cdot)$ at x in X yields:

$$N(x,\xi,t) = \sum_{j=0}^{\infty} f_j(x,t)\, \phi_j(\xi) \tag{2}$$

where we have written:

$$\text{"}f_j(x,t)\text{"} \quad \text{for} \quad [N^*(x,\cdot,t), \phi_j] \tag{3}$$

Thus $f_j(x,t)$ is the scalar obtained by performing the integration:

$$\int_{\Xi} N(x,\xi,t)\ \overline{\phi}_j(\xi)\ d\Omega(\xi) \quad .$$

In a similar manner we obtain:

$$N_\eta(x,\xi,t) = \sum_{j=0}^{\infty} f_{\eta,j}(x,t)\ \phi_j(\xi) \tag{4}$$

as the representation of the emission funcion N_η, where we have written:

$$\text{"}f_{\eta,j}(x,t)\text{"} \quad \text{for} \quad [N_\eta(x,\cdot,t),\ \phi_j] \quad . \tag{5}$$

The representation of the volume scattering function σ is next. Since σ uses two directional variables, we use the completeness property twice. First we obtain:

$$\sigma(x;\xi';\xi;t) = \sum_{j=0}^{\infty} \sigma_j(x;\xi';t)\ \phi_j(\xi) \tag{6}$$

where we have written:

$$\text{"}\sigma_j(x;\xi';t)\text{"} \quad \text{for} \quad [\sigma(x;\xi';\cdot;t),\ \phi_j] \tag{7}$$

Next we obtain:

$$\sigma_j(x;\xi';t) = \sum_{k=0}^{\infty} \sigma_{jk}(x,t)\ \overline{\phi}_k(\xi') \tag{8}$$

where we have written:

$$\text{"}\sigma_{jk}(x;t)\text{"} \quad \text{for} \quad [\sigma_j(x;\cdot;t),\ \overline{\phi}_k] \tag{9}$$

Combining these representations, we have:

$$\sigma(x;\xi';\xi;t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \sigma_{jk}(x;t)\ \overline{\phi}_k(\xi')\ \phi_j(\xi) \tag{10}$$

The reason for introducing the conjugates of the ϕ_k into (10) will become clear shortly.

Now the whole purpose of the spherical harmonic method, as we have seen in Sec. 6.1, is to effectively separate the spatial variables from the directional variables in the equation of transfer so that the latter may be contained in a system of simple, directly integrable differential equations involving spatial variables only. We now apply the abstract harmonic representations of N, N_η, and σ to the equation of transfer (1), and effect such a separation of variables. On the right side of (1) we have N_η already represented. Then for the term N* (the summations all go from 0 to ∞):

$$N_*(x,\xi,t) = \int_\Xi \left(\sum_i f_i(x,t)\phi_i(\xi')\right)\left(\sum_j\sum_k \sigma_{jk}(x;t)\overline{\phi}_k(\xi')\phi_j(\xi)\right) d\Omega(\xi')$$

$$= \sum_i f_i(x,t)\left[\int_\Xi \phi_i(\xi')\left(\sum_j\sum_k \sigma_{jk}(x;t)\overline{\phi}_k(\xi')\phi_j(\xi)\right) d\Omega(\xi')\right]$$

$$= \sum_i f_i(x,t)\left[\sum_j\sum_k \sigma_{jk}(x;t)\phi_j(\xi)\int_\Xi \phi_i(\xi')\overline{\phi}_k(\xi')\, d\Omega(\xi')\right]$$

$$= \sum_i f_i(x,t)\left[\sum_j\sum_k \sigma_{jk}(x;t)\phi_j(\xi)\ \delta_{ik}\right]$$

$$= \sum_i f_i(x,t)\left[\sum_j \sigma_{ji}(x;t)\phi_j(\xi)\right]$$

$$= \sum_j \left[\sum_i f_i(x,t)\sigma_{ji}(x;t)\right]\phi_j(\xi) \tag{11}$$

Since the medium X is assumed isotropic, the volume attenuation function values $\alpha(x;\xi)$ are independent of ξ, and so α need not be represented by a series of the complete family $\{\phi_0, \phi_1, \phi_2, \ldots\}$. By means of (4), (10), and (11) we can therefore represent the right side of (1) in the form:

$$\sum_{j=0}^{\infty}\left[-\alpha(x)f_j(x,t) + \sum_{i=0}^{\infty} f_i(x,t)\sigma_{ji}(x;t) + f_{\eta,j}(x,t)\right]\phi_j(\xi) \tag{12}$$

Attention is now directed to the left side of (1). The time derivative term is directly treated to yield:

$$\sum_{j=0}^{\infty} \frac{1}{v}\frac{\partial f_j(x,t)}{\partial t}\phi_j(\xi) \quad . \tag{13}$$

The spatial derivative term becomes:

$$\xi \cdot \nabla N(x,\xi,t) = \xi \cdot \nabla \left(\sum_{j=0}^{\infty} f_j(x,t)\ \phi_j(\xi) \right)$$

$$= \sum_{j=0}^{\infty} \left[\xi \cdot \nabla f_j(x,t) \right] \phi_j(\xi) \qquad (14)$$

Combining (12), (13), and (14) according to (1), we have:

$$\sum_{j=0} \left[\frac{1}{v} \frac{\partial f_j(x,t)}{\partial t} + \xi \cdot \nabla f_j(x,t) + \alpha(x)\ f_j(x,t) - \sum_{i=0} f_i(x,t) \sigma_{ji}(x;t) - f_{\eta,j}(x,t) \right] \phi_j(\xi) = 0 \qquad (15)$$

If it weren't for the spatial derivative term the contents of the square bracket would have been free of the variable ξ, and a system of equations would have been obtained by setting each bracketed jth term to zero. At any rate we can eliminate the presence of ξ by an integration over Ξ. The orthonormality property of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is available for use in this task. Thus multiplying each side of (15) by $\overline{\phi}_k(\xi)$ and integrating over Ξ, the orthonormality property immediately yields

$$\frac{1}{v} \frac{\partial f_k(x,t)}{\partial t} + \sum_{j=0}^{\infty} \int_{\Xi} \xi \cdot \nabla f_j(x,t)\ \phi_j(\xi)\ \overline{\phi}_k(\xi)\ d\Omega(\xi)$$

$$= - \alpha(x) f_k(x,t) + \sum_{j=0}^{\infty} f_i(x;t)\ \sigma_{jk}(x;t) + f_{\eta,k}(x,t) \qquad (16)$$

If we now write:

$$\text{"}D_{jk}\text{"} \quad \text{for} \quad \int_{\Xi} \xi \cdot \nabla(\ )\ \phi_j(\xi)\ \overline{\phi}_k(\xi)\ d\Omega(\xi) \quad , \qquad (17)$$

then we obtain, at last, the spherical harmonic analysis of (1):

$$\boxed{\frac{1}{v} \cdot \frac{\partial f_k}{\partial t} + \sum_{j=0}^{\infty} f_j D_{jk} = - \alpha\ f_k + \sum_{j=0}^{\infty} f_j\ \sigma_{jk} + f_{\eta,k} \qquad k = 0, 1, 2, \ldots} \qquad (18)$$

This is the requisite abstract spherical harmonic system of partial differential equations for the family $\{f_0, f_1, f_2, \ldots\}$ of functions, the *abstract harmonic coefficient functions* of the radiance distribution $N(x,\cdot)$. Knowledge of these f_j

allows construction of $N(x,\cdot)$ according to (2). The heart of the abstract harmonic method of solving the equation of transfer thus resides in (18).

Finite Forms of the Abstract Harmonic Equations

An inspection of the system (18) of abstract harmonic equations governing the harmonic coefficient functions f_k shows two infinite series involved in the system. The presence of these infinite series could occasionally negate the practical utility of the system, for example in numerical solution work. It is interesting to observe, however, that these infinite series may be rigorously removed and replaced by finite sums under the combined action of two very general conditions, one physical, the other mathematical. The mathematical condition simplifies the differential operator series; the physical condition simplifies the scattering term series. We shall now briefly indicate the nature of these conditions.

We shall say that the family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ of functions on Ξ has the *finite recurrence property* of degree ν if for every element ξ' in Ξ and every ϕ_j in the family, there exist ν constants A_{jk} and ν elements $\phi_{\alpha_1}, \ldots, \phi_{\alpha_\nu}$ of $\{\phi_0, \phi_1, \phi_2, \ldots\}$ such that

$$\xi \cdot \xi'\phi_j(\xi) = \sum_{k=1}^{\nu} A_{jk}\, \phi_{\alpha_k}(\xi) \tag{19}$$

holds for every ξ in Ξ. The motivation for this property arises in an attempt to simplify the form of the operators D_{jk} and to reduce to a finite series the infinite series involving them in (18). For example, in an orthogonal, three-dimensional coordinate frame in which $x = (x_1, x_2, x_3)$, we have:

$$\nabla = \mathbf{i}\,\frac{\partial}{\partial x_1} + \mathbf{j}\,\frac{\partial}{\partial x_2} + \mathbf{k}\,\frac{\partial}{\partial x_3}$$

We use this form in (17) to obtain the representation

$$D_{jk} = a_{jk}\,\frac{\partial}{\partial x_1} + b_{jk}\,\frac{\partial}{\partial x_2} + c_{jk}\,\frac{\partial}{\partial x_3} \tag{20}$$

where we have written:

$$\text{"}a_{jk}\text{" for } \int_{\Xi} \xi \cdot \mathbf{i}\, \phi_j(\xi)\, \overline{\phi}_k(\xi)\, d\Omega(\xi) \tag{21}$$

$$\text{"}b_{jk}\text{" for } \int_{\Xi} \xi \cdot \mathbf{j}\, \phi_j(\xi)\, \overline{\phi}_k(\xi)\, d\Omega(\xi) \tag{22}$$

$$"c_{jk}" \quad \text{for} \quad \int_{\Xi} \xi \cdot \mathbf{k}\, \phi_j(\xi)\, \overline{\Phi}_k(\xi)\, d\Omega(\xi) \qquad (23)$$

By postulating a finite recurrence property of degree ν for $\{\phi_0, \phi_1, \phi_2, \ldots\}$, if follows that $a_{jk} = 0$ whenever the indices k and j differ by a sufficiently large amount: indeed $a_{jk} = 0$ for all but at most ν terms. Similarly with b_{jk} and c_{jk}. This means that for fixed k $D_{jk} = 0$ whenever j is sufficiently large, and so the number of terms on the left of (18) become finite in the present case. It turns out that any orthonormal family obtained from suitable nth order ordinary differential equations (a rich source of orthonormal families by means of Sturm-Liouville theory) will possess a finite recurrence property of degree ν.

Finally, the physical condition which simplifies the abstract harmonic equations is that of isotropy of the medium. In the present case the isotropy reduces the general functional dependence of σ on the independent variables ξ' and ξ to the special dependence of σ on the scalar product $\xi' \cdot \xi$ of the directions. This simplified structure of σ in turn manifests itself in a simplification of the representation (10) to the form:

$$\sigma(x;\xi';\xi;t) = \sum_{j=0}^{\infty} \sigma_j(x;t)\, \overline{\Phi}_j(\xi')\, \phi_j(\xi) \quad . \qquad (24)$$

We shall not go into the derivation details of this relation in the present abstract case. It suffices to note that this form can be obtained when the members of the orthonormal family $\{\phi_0, \phi_1, \phi_2, \ldots\}$ obey a general type of *addition theorem* often valid for functions arising in Sturm-Liouville theory. Examples of addition theorems for such functions, are, e.g., in [318]. (See (12) and (15) of Sec. 6.3.)

The simplifying effect of (24) becomes evident when we recalculate $N_*(x,\xi,t)$ after the manner of (11):

$$N_*(x,\xi,t) = \int_{\Xi} \left(\sum_i f_i(x,t)\, \phi_i(\xi')\right)\left(\sum_j \sigma_j(x;t)\overline{\Phi}_j(\xi')\, \phi_j(\xi)\right) d\Omega(\xi')$$

$$= \sum_i f_i(x,t) \left[\int_{\Xi} \phi_i(\xi')\left(\sum_j \phi_j(x;t)\overline{\Phi}_j(\xi')\phi_j(\xi)\right)\right] d\Omega(\xi')$$

$$= \sum_i f_i(x,t)\left[\sum_j \sigma_j(x;t)\phi_j(\xi)\int_{\Xi} \phi_i(\xi')\overline{\Phi}_j(\xi')\right] d\Omega(\xi')$$

$$= \sum_i f_i(x,t) \left[\sum_j \sigma_j(x;t)\ \phi_j(\xi)\ \delta_{ij} \right]$$

$$= \sum_i f_i(x,t)\ \sigma_i(x;t)\ \phi_i(\xi) \tag{25}$$

By combining the preceding two conditions, the total effect on (18) is a complete finitization of each equation in the system of equations, thereby rendering them more effective for numerical computations. We may summarize these constructions as follows:

Let X be an arbitrary isotropic, inhomogeneous optical medium with internal emission radiance function N_η and general time-dependent radiance field N as governed by the equation of transfer (1). Let $\{\phi_0, \phi_1, \phi_2, \ldots\}$ be an orthonormal family of functions defined on the unit sphere Ξ such that: the family (a) possesses the completeness property (see (19) of Sec. 6.1); (b) possesses the finite recurrence property (1C); (c) satisfies an addition theorem (24). Then each member of the general abstract harmonic system of partial differential equations (18) reduces to the following finite form: For some positive integer ν:

$$\frac{1}{v}\frac{\partial f_k}{\partial t} + \sum_{j=0}^{\nu} f_j D_{jk} = -\alpha f_k + f_k \sigma_k + f_{\eta,k} \qquad k = 0,1,2,\ldots \tag{26}$$

6.3 Classical Spherical Harmonic Method: General Media

The general theory of the abstract harmonic method developed in the preceding section will now be illustrated for the classical case in which the orthonormal family is constructed from familes of associated Legendre functions of the first kind and circular (trigonometric) functions. The optical medium X will be generally inhomogeneous and isotropic, with time varying inherent optical properties, and given internal sources.

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The Orthonormal Family

We begin by observing that the classical spherical harmonic method customarily uses the ordered pair (μ,ϕ) of numbers to specify a point ξ in Ξ, where we have written "μ" for cos θ, and where (θ,ϕ) are the two angles customarily used to specify ξ in Ξ (see Sec. 2.4 and also example 14 of Sec. 2.11 for an earlier use of μ in conjunction with Legendre polynomials). The range of the variable μ is thus the interval [-1,1], and the range of ϕ, $[0,2\pi]$. Every ξ in Ξ determines a unique (θ,ϕ), that is a unique μ in [-1,1] and a unique ϕ in $[0,2\pi]$.